13–5

Law of Cosines

What YOU'LL LEARN

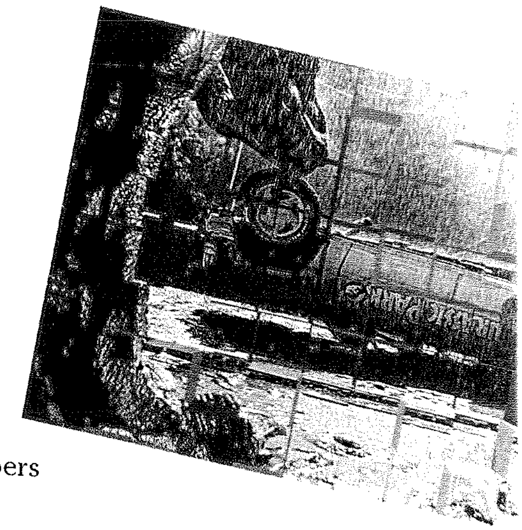
- To solve triangles by using the law of cosines.

Why IT'S IMPORTANT

You can use the law of cosines to solve problems involving paleontology and emergency medicine.

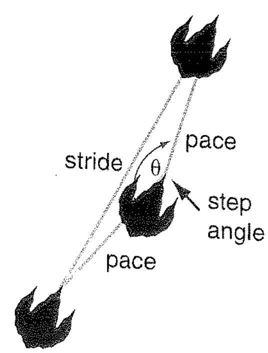
Real World APPLICATION

Paleontology



If Jurassic Park were a real place, it would be easy for scientists to study how dinosaurs move from place to place. But, since it's not, scientists are left to study the footprints made by dinosaurs millions of years ago. At dinosaur digs, anthropologists use *locomotor parameters,* which are numbers associated with physical motion.

The figure at the right shows footprints of a carnivorous dinosaur taken from the Glen Rose formation in Texas. The *pace* is the distance from the left footprint to the right footprint, and vice versa. The *stride* is the distance from left footprint to the next left footprint or the right footprint to the next right footprint. If an animal walks in such a way that the footprints are directly in line, the stride will be twice the pace. But usually, the footprints show a "zig-zag" pattern that can be described numerically by the *step angle,* θ. An efficient walker has a step angle that approaches 180°, meaning that the animal minimizes zig-zag motion while maximizing forward motion.

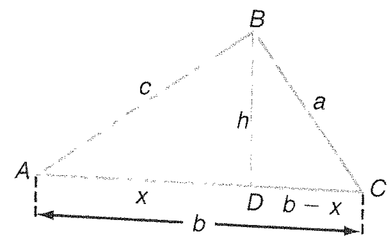


INTERNET CONNECTION

Data Update For more information on dinosaur digs, visit: **www.algebra2.glencoe.com**

Anthropologists use trigonometry to determine the step angle. However, problems such as this, in which you know the measures of the sides of a triangle, cannot be solved using the law of sines. You can solve problems such as this by using the **law of cosines.**

To derive the law of cosines, consider ΔABC with height h units and sides with lengths a units, b units, and c units. Suppose segment AD is x units long. Then segment DC is $b - x$ units long. What relationship exists between a, b, c, and A?



$a^2 = (b - x)^2 + h^2$ *Use the Pythagorean theorem for ΔDBC.*

$a^2 = b^2 - 2bx + x^2 + h^2$ *Expand $(b - x)^2$.*

$a^2 = b^2 - 2bx + c^2$ *In ΔADB, $c^2 = x^2 + h^2$.*

$a^2 = b^2 - 2b(c \cos A) + c^2$ $\cos A = \frac{x}{c}$, so $x = c \cos A$.

$a^2 = b^2 + c^2 - 2bc \cos A$

The measure a is now defined in terms of the measures of the other two sides and angle A. You can find two other formulas relating the lengths of sides to the cosine of B and C in a similar way. All three formulas can be summarized as follows.

Law of Cosines	**Let $\triangle ABC$ be any triangle with a, b, and c representing the measures of sides, and opposite angles with measurement A, B, and C, respectively. Then the following equations are true.** $$a^2 = b^2 + c^2 - 2bc\cos A$$ $$b^2 = a^2 + c^2 - 2ac\cos B$$ $$c^2 = a^2 + b^2 - 2ab\cos C$$

You can apply the law of cosines to a triangle if you know

- the measures of three sides, or
- the measures of two sides and the included angle.

Example 1 **Find c if $a = 15$, $b = 18$, and $C = 34°$.**

You are given the measure of two sides and the included angle.

$c^2 = a^2 + b^2 - 2ab\cos C$

$c^2 = 15^2 + 18^2 - 2(15)(18)\cos 34°$

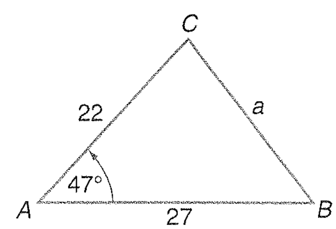
$c^2 \approx 101.32$

$c \approx 10.07$

Example 2 **Solve each triangle. Round to the nearest tenth.**

a. $A = 47°$, $c = 27$, $b = 22$

You are given the measures of two sides and the included angle. First, determine a by using the law of cosines.



$a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 22^2 + 27^2 - 2(22)(27)\cos 47°$

$a^2 \approx 402.8$

$a \approx 20.1$

Next, use the law of sines to determine the measure of an angle.

$\frac{\sin A}{a} = \frac{\sin B}{b}$

$\frac{\sin 47°}{20.1} \approx \frac{\sin B}{22}$ *$A = 47°$, $a \approx 20.1$, $b = 22$*

$\sin B \approx \frac{22 \sin 47°}{20.1}$ *$\sin 47° \approx 0.7314$*

$\sin B \approx 0.8005$

$B \approx 53.2°$

Now, determine the measure of the third angle, $\angle C$.

$47° + 53.2° + C \approx 180$

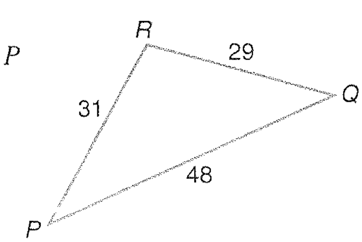
$C \approx 79.8°$

Therefore, $a \approx 20.1$, $B \approx 53.2°$, and $C \approx 79.8°$.

b. $p = 29$, $q = 31$, $r = 48$

You are given the measures of three sides.

Use the law of cosines to find the measure of an angle.



$p^2 = q^2 + r^2 - 2qr\cos P$

$29^2 = 31^2 + 48^2 - 2(31)(48)\cos P$

$2(31)(48)\cos P = 31^2 + 48^2 - 29^2$

$\cos P = \frac{31^2 + 48^2 - 29^2}{2(31)(48)}$

$\cos P \approx 0.8145$

$P \approx 35.5°$

Use the law of sines to determine the measure of another angle.

$$\frac{\sin P}{p} = \frac{\sin Q}{q}$$
$$\frac{\sin 35.5°}{29} = \frac{\sin Q}{31}$$
$$\sin Q \approx \frac{31 \sin 35.5°}{29}$$
$$\sin Q \approx 0.6208$$
$$Q \approx 38.4°$$

Now find the measure of the third angle.

$$35.5° + 38.4° + R \approx 180°$$
$$R \approx 106.1°$$

Therefore, $P \approx 35.5°$, $Q \approx 38.4°$, and $R \approx 106.1°$.

Example 3

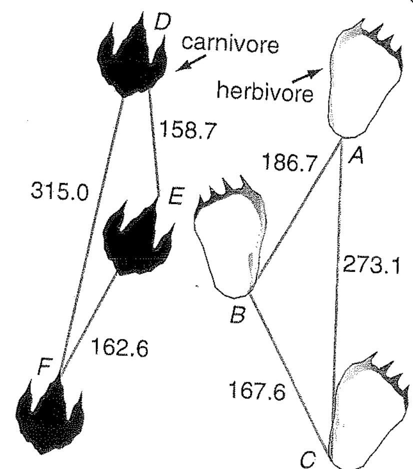
Real World APPLICATION

Paleontology

At the Glen Rose formation in Texas, an anthropologist measured the pace and stride of footprints made by a bipedal (two-footed), carnivorous (meat-eating) dinosaur and the hindfeet of a herbivorous (plant-eating) dinosaur. The data are shown at the right.

a. Find the step angle for each dinosaur.

b. What can you tell about the motion of each dinosaur from its step angle?



a. Find the step angle for the herbivore, $\angle B$. *$b = 273.1$, $a = 167.6$, $c = 186.7$*

$$b^2 = a^2 + c^2 - 2ac \cos B \quad \textit{Use the law of cosines.}$$
$$(273.1)^2 = (167.6)^2 + (186.7)^2 - 2(167.6)(186.7)\cos B$$
$$2(167.6)(186.7)\cos B = (167.6)^2 + (186.7)^2 - (273.1)^2$$
$$\cos B = \frac{(167.6)^2 + (186.7)^2 - (273.1)^2}{2(167.6)(186.7)}$$
$$\cos B \approx -0.1859$$
$$B \approx 100.7°$$

The step angle for the herbivore is 100.7°.

Find the step angle for the carnivore, $\angle E$. *$e = 315.0$, $d = 162.6$, $f = 158.7$*

$$e^2 = d^2 + f^2 - 2df \cos E \quad \textit{Use the law of cosines.}$$
$$(315.0)^2 = (162.6)^2 + (158.7)^2 - 2(162.6)(158.7)\cos E$$
$$2(162.6)(158.7)\cos E = (162.6)^2 + (158.7)^2 - (315.0)^2$$
$$\cos E = \frac{(162.6)^2 + (158.7)^2 - (315.0)^2}{2(162.6)(158.7)}$$
$$\cos E \approx -0.9223$$
$$E \approx 157.3°$$

The step angle for the carnivore is 157.3°.

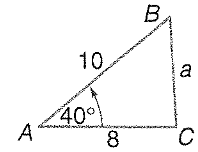
b. Since the step angle for the carnivore is closer to 180°, it appears as though the carnivore made more forward progress with each step than the sauropod. *Why do you suppose a step angle close to 180° was important for a carnivore?*

CHECK FOR UNDERSTANDING

Communicating Mathematics

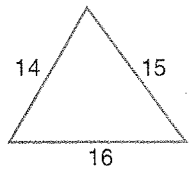
Study the lesson. Then complete the following.

1. **Describe** a set of conditions for which the law of cosines can be used.
2. **State** which form of the law of cosines you would use to find a in the triangle at the right.

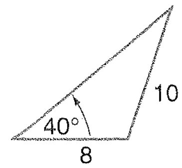


3. **Choose** the triangles that should be solved by beginning with the law of cosines.

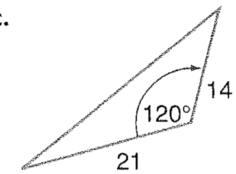
a.



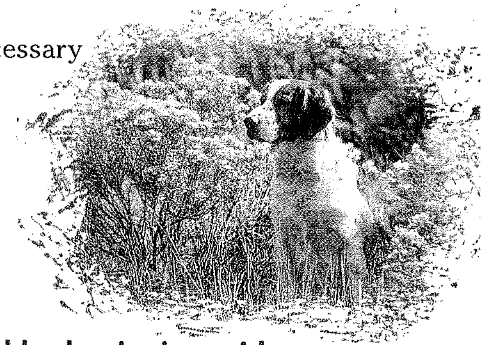
b.



c.



4. **Explain** why you cannot use the law of sines to solve a triangle if you are given $A = 80°$, $b = 20$, and $c = 55$.
5. **Make a chart** that summarizes the conditions necessary to use the law of sines and law of cosines.

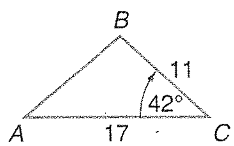


MODELING MATHEMATICS

6. Collect data from your classmates or members of the track team and determine their step angles. Compare and contrast the step angles when walking versus running. Collect data from your classmates' pets and compare and contrast the step angles of different kinds of pets.

Determine whether each triangle can be solved by beginning with the law of sines or law of cosines. Then solve each triangle.

7.



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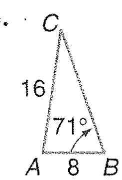


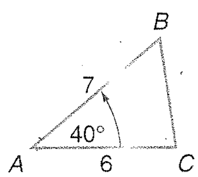
9. $a = 12, c = 15, A = 34°$
10. $a = 15, b = 18, c = 19$
11. The sides of a triangle are 6.8 cm, 8.4 cm, and 4.9 cm long. Find the measure of the smallest angle.

EXERCISES

Guided Practice

Determine whether each triangle can be solved by beginning with the law of sines or law of cosines. Then solve each triangle.

12.



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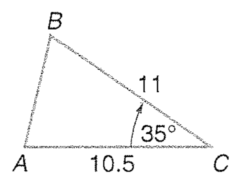

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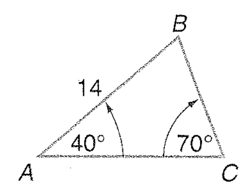

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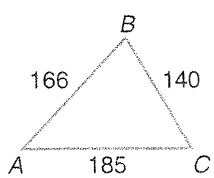

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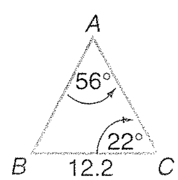
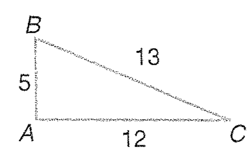
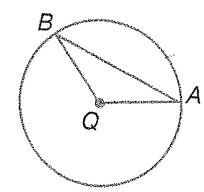


17.



18. $A = 35°, b = 16, c = 19$
19. $a = 20, c = 24, B = 47°$
20. $a = 21.5, b = 13, C = 38.3°$
21. $A = 40°, B = 59°, c = 14$
22. $a = 51, c = 61, B = 19°$
23. $a = 13.7, A = 25°, B = 78°$
24. $a = 15, b = 25, c = 40$
25. $a = 345, b = 648, c = 442$
26. $c = 10.3, a = 21.5, b = 16.7$
27. $A = 28°, b = 5, c = 4.9$
28. $A = 29°, b = 7.6, c = 14.1$
29. $a = 8, b = 24, c = 18$

30. The sides of a parallelogram measure 55 cm and 71 cm. Find the length of each diagonal if the larger angle measures 106°.



31. Circle Q at the right has a radius of 15 cm. Two radii $\overline{QA}$ and $\overline{QB}$ form an angle of 123°. Find the length of chord AB.

32. The sides of a triangle are 50 meters, 70 meters, and 85 meters. Find the measure of the angle opposite the shortest side.

Programming

33. The graphing calculator program at the right finds the measure of a side of a triangle using the law of cosines. A is the measure of the missing side, B and C are the measures of the second and third sides, and θ is the measure of the angle opposite side A.

```
PROGRAM: Cosine
:Degree
:Disp "INPUT B"
:Input B
:Disp "INPUT C"
:Input C
:Disp "INPUT θ"
:Input θ
:√(B²+C²−2BCcosθ)→A
:Disp "A="
:Disp A
```

Use the program to find the measure of the missing side.

a. $B = 2, C = 4, \theta = 78°$
b. $B = 9, C = 19, \theta = 45°$
c. $B = 5.4, C = 6.9, \theta = 95°$

Critical Thinking

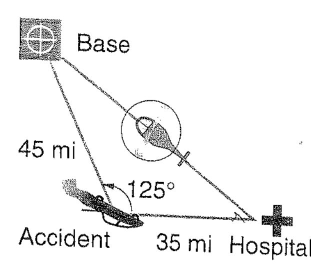
34. Explain how the Pythagorean theorem is a special case of the law of cosines.

Applications and Problem Solving

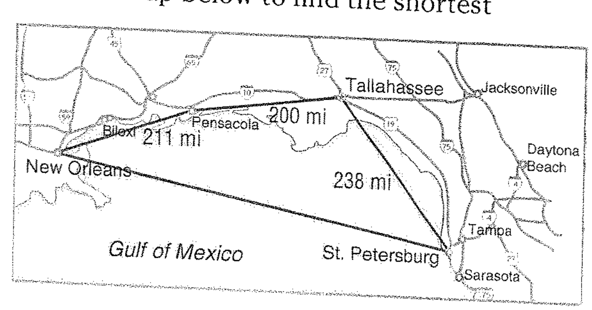
Real World

35. **Surveying** Two sides of a triangular plot of land have lengths of 400 feet and 600 feet. The measure of the angle between those sides is 46.3°. Find the perimeter and area of the plot.

36. **Emergency Medicine** A medical rescue helicopter has flown 45 miles from its home base to pick up an accident victim and 35 miles from there to the hospital. The angle between the two legs of the trip was 125°. The pilot needs to know how far he is now from his home base so he can decide whether to refuel before returning. How far is the hospital from the helicopter's base?

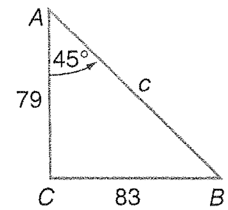


37. **Geography** Use the information in the map below to find the shortest distance from St. Petersburg, Florida, to New Orleans, Louisiana. The angle at Tallahassee, Florida, measures 125°, and the angle at Pensacola, Florida, measures 166°. (*Hint:* First find the distance from St. Petersburg to Pensacola.)



Mixed Review

38. Use the law of sines to solve the triangle at the right. (Lesson 13–4)



39. Change $-45°$ to radians. (Lesson 13–2)

40. How many 7-letter patterns can be formed from the letters of the word BENZENE? (Lesson 12–2)

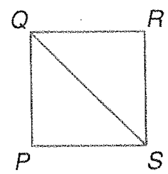
41. Graph $f(x) = \frac{x}{x-1}$. (Lesson 9–1)

42. **Geometry** The volume of a rectangular solid is 72 cubic units. The width is twice the height and the length is 7 units more than the height. Find the dimensions of the solid. (Lesson 8–5)

43. Solve $z^2 + 4z = 96$ by completing the square. (Lesson 6–3)

44. Simplify $(3 + \sqrt{2})(\sqrt{10} + \sqrt{5})$. (Lesson 5–6)

45. **SAT Practice Quantitative Comparison**



PQRS is a square.

Column A	Column B
2	$\frac{\text{length of } \overline{QS}}{\text{length of } \overline{RS}}$

A if the quantity in Column A is greater
B if the quantity in Column B is greater
C if the two quantities are equal
D if the relationship cannot be determined from the information given

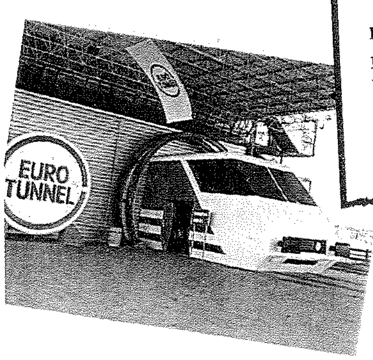
For Extra Practice, see page 907.

Mathematics and SOCIETY

The Channel Tunnel

The excerpt below appeared in an article in *Popular Science* in May, 1994.

THIS IS THE CHANNEL TUNNEL. DREAMED of by Napoleon, futilely attempted 100 years ago, and built at a cost greater than most countries' gross national product, the 31-mile underwater rail link between England and continental Europe is finally in place. By 1996, the system will transport some 8 million car passengers and 4.5 million bus passengers each year . . . The owner, Eurotunnel, hopes to entice a large chunk of the tourist traffic that currently uses hovercrafts and ferries to cross the choppy waters of the English Channel . . . Commonly referred to as "The Chunnel," the Channel Tunnel is actually a complex of three parallel passageways, dipping as far as 148 feet below the seabed of the English Channel. ■



1. The two long sections of tunnel were dug toward each other from the British and French sides until they finally met and were linked deep below the English Channel. What types of measurements were needed to ensure that the two sections would meet properly?
2. What types of hazards might Chunnel travelers be exposed to? How might the Chunnel designers have protected travelers against them?