13-1 An Introduction to Trigonometry

What YOU'LL LEARN

- To find values of trigonometric functions for acute angles, and
- to solve problems involving right triangles.

Why IT'S IMPORTANT

You can use trigonometry to solve problems involving surveying and literature.

GLOBAL CONNECTIONS

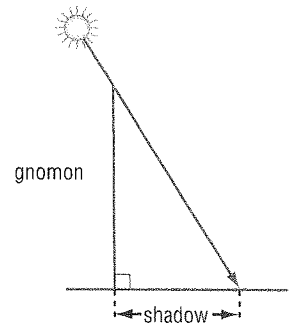
The gnomon dates from 3500 B.C. and the first Egyptian sundials from 1500 B.C. Next came the hemispherical sundial from Babylon about 300 B.C. More complex sundials like the hemicyclium, which uses conic sections, were created later by the Greeks.

CONNECTION
Literature

"Where was the sun? Over the oak.
Where was the shadow? Under the elm."

With these clues, Sherlock Holmes set out to solve a mystery in *The Musgrave Ritual.* Unfortunately, the elm had been struck by lightning ten years earlier, so Holmes was not able to measure the shadow directly. *You will solve the mystery in Exercise 50.*

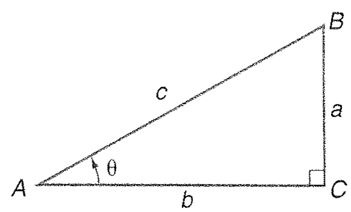
Sherlock Holmes was not the first to use "shadow reckoning." The study of **trigonometry** probably began when early astronomers used the length of a shadow cast by a stick, called a *gnomon,* to determine the time of day. Egyptians relied on sundials as early as 1500 B.C., and there is evidence that astronomers in China, Mesopotamia, and India also understood that as the sun rises in the sky, it casts a unique shadow. In other words, the shadow is a function of the time of day.



The word *trigonometry* was derived from two Greek words—*trigon* meaning triangle and *metra* meaning measurement. So, trigonometry began as the study of the relationships between the angles and sides of a right triangle.

Consider right triangle ABC shown below in which the measure of the acute angle A is identified with the Greek letter *theta,* θ.

The hypotenuse of the triangle is side $\overline{AB}$.
Its length is c units.
The leg opposite $\angle A$ is $\overline{BC}$.
Its length is a units.
The leg adjacent to $\angle A$ is $\overline{AC}$.
Its length is b units.



Using these sides, you can define six **trigonometric functions—sine, cosine, tangent, secant, cosecant,** and **cotangent,** which are abbreviated as sin, cos, tan, sec, csc, and cot, respectively.

Trigonometric Functions

If θ is the measure of one acute angle in a right triangle, a is the measure of the leg opposite θ, b is the measure of the leg adjacent to θ, and c is the measure of the hypotenuse, then the following are true.

sine $\theta = \frac{a}{c}$	**cosine** $\theta = \frac{b}{c}$	**tangent** $\theta = \frac{a}{b}$
cosecant $\theta = \frac{c}{a}$	**secant** $\theta = \frac{c}{b}$	**cotangent** $\theta = \frac{b}{a}$

Notice that the sine, cosine, and tangent functions are reciprocals of the cosecant, secant, and cotangent functions, respectively.

The following ratios may help you remember the sin, cos, and tan functions.

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

SOH-CAH-TOA is a mnemonic device for remembering the first letter of each word in the ratios. For example, SOH refers to sin-opposite-hypotenuse.

Focus On
PREREQUISITE SKILLS

To be successful in this chapter, you'll need to understand these concepts and be able to apply them. Refer to the lesson in parentheses if you need more review before beginning the chapter.

Name the quadrant in which a point is located. (Lesson 2–1)

Name the quadrant in which each point is located.

1. $A(-5, -4)$ **2.** $A(1, -2)$ **3.** $A(-4, 6)$ **4.** $A(2, 2)$

Determine the inverse of a function. (Lesson 8–8)

Find the inverse of each function. Then graph the function and its inverse.

5. $f(x) = x - 2$

6. $f(x) = \frac{x-4}{3}$

7. $f(x) = x^2 + 3$

8. $f(x) = -2x + 1$

Find the composition of functions. (Lesson 8–7)

Find $g[h(x)]$ and $h[g(x)]$.

9. $h(x) = x + 2$
$g(x) = 4x - 6$

10. $h(x) = -x + 7$
$g(x) = -x + 1$

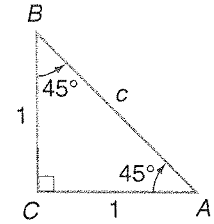
11. $h(x) = 2x - 1$
$g(x) = x^2$

12. $h(x) = -3x + 2$
$g(x) = -5x$

Focus On
READING SKILLS

In this chapter, you will learn about ***trigonometric functions***. Trigonometry is the study of triangle measurements. Notice that the words *trigonometry* and *triangle* both begin with the prefix "tri," which means three. In Lesson 13–1, you will learn about the six trigonometric functions—sine, cosine, tangent, secant, cosecant, cotangent. The ratios actually form three pairs which are reciprocals of each other. Sine and cosecant and reciprocals, cosine and secant are reciprocals, and, as you may have guessed from their names, tangent and cotangent are reciprocals.

To find the trigonometric values for a 45° angle, use an isosceles right triangle, ΔABC. Let the length of each leg be 1 unit. Use the Pythagorean theorem to find the length of the hypotenuse.



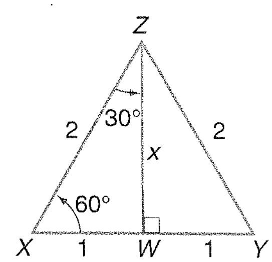
$$1^2 + 1^2 = c^2$$
$$2 = c^2$$
$$\sqrt{2} = c$$

The length of the hypotenuse is $\sqrt{2}$ units.

Therefore, the sine, cosine, and tangent values for 45° are as follows.

$\sin 45° = \frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$ $\quad$ $\cos 45° = \frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$ $\quad$ $\tan 45° = \frac{1}{1}$ or 1

To find the trigonometric values for a 30° angle, use an equilateral triangle, ΔXYZ. Let the length of each side be 2 units. The altitude $\overline{ZW}$ separates ΔXYZ into two 30°–60° right triangles. Since $\overline{ZW}$ is the perpendicular bisector of $\overline{XY}$, the length of $\overline{XW}$ is 1 unit. Find the length of altitude $\overline{ZW}$.



$$x^2 + 1^2 = 2^2 \qquad \textit{Pythagorean theorem}$$
$$x^2 = 3$$
$$x = \sqrt{3}$$

The length of altitude $\overline{ZW}$ is $\sqrt{3}$ units.

Therefore, the sine, cosine, and tangent values for 30° are as follows.

$\sin 30° = \frac{1}{2}$ $\quad$ $\cos 30° = \frac{\sqrt{3}}{2}$ $\quad$ $\tan 30° = \frac{1}{\sqrt{3}}$ or $\frac{\sqrt{3}}{3}$

To find the sine, cosine, and tangent for 60°, use ΔXWZ shown above.

$\sin 60° = \frac{\sqrt{3}}{2}$ $\quad$ $\cos 60° = \frac{1}{2}$ $\quad$ $\tan 60° = \frac{\sqrt{3}}{1}$ or $\sqrt{3}$

> **TECHNOLOGY Tips**
>
> Consult the user's guide for your calculator to see if you press SIN first or the angle measure first.

Before hand-held calculators became accessible, students had to rely on "trig tables" to find the values of trigonometric functions for angles other than 30°, 45°, and 60°. Today, you can use the SIN, COS, and TAN keys on your calculator to find these values.

EXPLORATION — CALCULATORS

In this Exploration, you will use a calculator to investigate the behavior of the sine and cosine functions for different values of θ. *Be sure your calculator is in degree mode.*

Your Turn

a. Choose five values of θ between 0° and 90° and evaluate sin θ for each of them. Write your answers as ordered pairs (θ, sin θ).

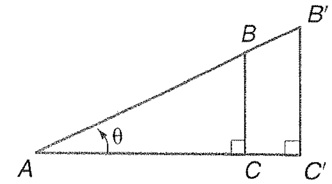
b. Graph the ordered pairs on a coordinate plane.

c. Use your graph to estimate the value of sin 0° and sin 90°. Check your answer with a calculator.

d. Explain how the value of sin θ changes as θ increases from 0° to 90°.

e. Repeat steps a–d using the cosine function.

The domain of each of these trigonometric functions is the set of all acute angles θ of a right triangle. The values of the functions depend only on the measure of θ and not on the size of the right triangle. For example, consider sin θ in the figure at the right.



Using $\triangle ABC$	*Using* $\triangle AB'C'$
$\sin\theta = \frac{BC}{AB}$	$\sin\theta = \frac{B'C'}{AB'}$

Notice that the triangles are similar because they are two right triangles that share a common angle, θ. Since they are similar, the ratios of corresponding sides are equal. That is, $\frac{BC}{AB} = \frac{B'C'}{AB'}$. Therefore, you will find the same value for sin θ regardless of which triangle you use.

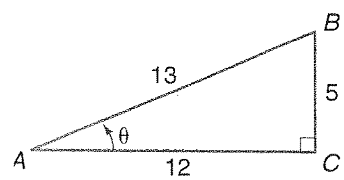
Example 1 **Find the values of the six trigonometric functions for angle θ. Round each value to four decimal places.**

For angle θ, the hypotenuse is $\overline{AB}$, the opposite leg is $\overline{BC}$, and the adjacent leg is $\overline{AC}$.

$\sin\theta = \frac{5}{13}$ or 0.3846 $\qquad \csc\theta = \frac{13}{5}$ or 2.6000

$\cos\theta = \frac{12}{13}$ or 0.9231 $\qquad \sec\theta = \frac{13}{12}$ or 1.0833

$\tan\theta = \frac{5}{12}$ or 0.4167 $\qquad \cot\theta = \frac{12}{5} = 2.4000$



Example 2 **Find tan *A* when cos *A* = $\frac{2}{3}$. Round to four decimal places.**

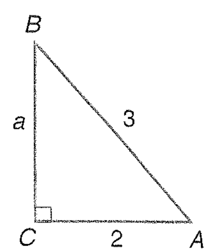
For convenience of notation, we refer to the angle with vertex A as angle A (∠A) and use A to stand for its measurement.

Since $\cos A = \frac{2}{3}$, the measure of the leg adjacent to ∠*A* is 2, and the measure of the hypotenuse is 3. Draw and label a right triangle. Then use the Pythagorean theorem to find the measure of the leg opposite ∠*A*. $\qquad$ *$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$*

$a^2 + 2^2 = 3^2$

$a^2 = 5$

$a = \sqrt{5}$



Now, find tan *A*.

$\tan A = \frac{\sqrt{5}}{2}$ $\qquad$ *$\tan A = \frac{\text{opposite}}{\text{adjacent}}$*

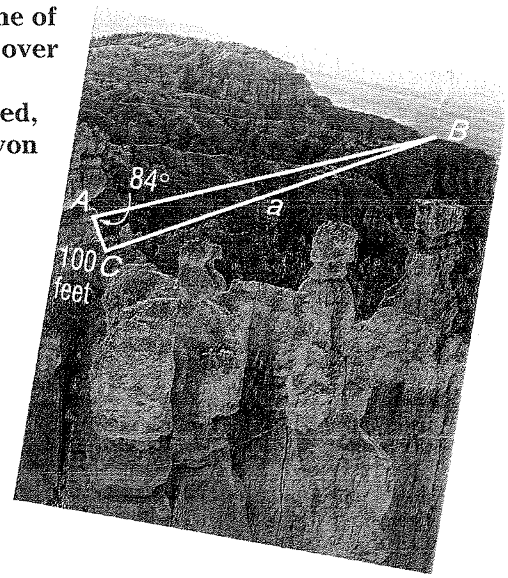
≈ 1.1180

Angles that measure 30°, 45°, and 60° occur frequently in trigonometry. You can find the values of the trigonometric functions for these angles by using the special characteristics of a 45°–45° right triangle and a 30°–60° right triangle.

Example 5 — **Real World APPLICATION** — **Surveying**

Utah's Bryce Canyon National Park contains some of the most colorful rock formations on Earth. For over 60 million years, water and ice have worn the canyon rocks into unusual shapes in shades of red, copper, pink, and cream. Some parts of the canyon are 1000 feet deep.

To find the distance across Bryce Canyon at a particular point, a surveyor sets up a transit at point *C* and sights a rock formation across the canyon at point *B*. Then the surveyor turns the transit 90° and sights point *A* that is 100 feet away. Using the transit at point *A*, the surveyor determines that the measure of ∠*A* is 84°. Find the distance across the canyon from *B* to *C*.



Let a represent the distance, in feet, from B to C.

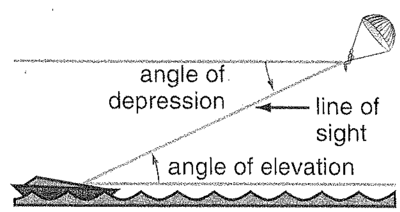
$$\tan 84° = \frac{a}{100} \quad \textit{tan} = \frac{\textit{opposite}}{\textit{adjacent}}$$

$$9.5144 \approx \frac{a}{100}$$

$$a \approx 951.44$$

The distance across Bryce Canyon at this point is about 951 feet.

Some applications of trigonometry use an **angle of elevation** or **angle of depression.** In the figure at the right, the angle formed by the line of sight from the boat and a horizontal line is called an angle of elevation. The angle formed by the line of sight from the parasail and a horizontal line is called an angle of depression.

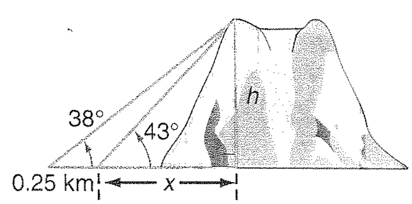


The line of sight is a transversal intersecting the two horizontal lines. The angle of elevation and the angle of depression are alternate interior angles. Since the horizontal lines are parallel, the angle of elevation and the angle of depression are congruent.

Example 6 — **Real World APPLICATION** — **Geology**

A geologist measured a 43° angle of elevation to the top of a volcano crater. After moving 0.25 kilometers farther away, the angle of elevation was 38°. How high is the top of the volcano crater?

The figure at the right shows two right triangles that share a common height. Let h represent the height of the crater in kilometers. Let x represent the side adjacent to the angle whose measure is 43°. Write a system of equations in two variables.



$$\tan 43° = \frac{h}{x} \qquad \tan 38° = \frac{h}{0.25 + x}$$

$$h = x \tan 43° \qquad h = (0.25 + x) \tan 38°$$

First, solve for x.

$$x \tan 43° = (0.25 + x) \tan 38° \quad \textit{Substitution}$$

$$x \tan 43° = 0.25 \tan 38° + x \tan 38° \quad \textit{Distributive property}$$

$$x \tan 43° - x \tan 38° = 0.25 \tan 38° \quad \textit{Subtract x tan 38° from each side}$$

$$x(\tan 43° - \tan 38°) = 0.25 \tan 38° \quad \text{[illegible]}$$

You can also use the inverse capabilities of a calculator to find the measure of an angle when you know one of its trigonometric ratios. *You will learn more about inverses in Lesson 13–7.*

Example 3 **Find x if $\sin x = 0.7590$. Round to the nearest degree.**

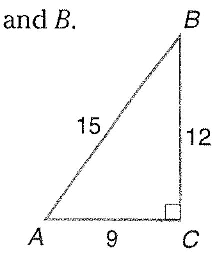
ENTER: 0.7590 [SIN⁻¹] 49.37611923 *The [SIN⁻¹] key may be a second function on your calculator.*

Therefore, x is approximately 49°.

If you know the measure of any two sides of a right triangle or the measure of one side and one acute angle, you can determine the measures of all the sides and angles of the triangle. This process of finding the missing measures is known as **solving a triangle.**

Example 4 **Solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.**

a. $\triangle ABC$



You know the measures of the sides. You need to find A and B.

Find A. $\sin A = \frac{12}{15}$ *$\sin A = \frac{\text{opposite}}{\text{hypotenuse}}$*

$\sin A = 0.8$

Use a calculator and the $\sin^{-1}$ function to find the angle whose sine is 0.8.

0.8 [SIN⁻¹] 53.130102

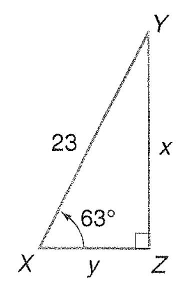
To the nearest degree, $A \approx 53°$.

Find B. $53° + B \approx 90°$ *Angles A and B are complementary.*

$B \approx 37°$

Therefore, $A \approx 53°$ and $B \approx 37°$.

b. $\triangle XYZ$



You know the measure of the hypotenuse and one acute angle. You need to find x, y, and Y.

Find x and y.

$\sin 63° = \frac{x}{23}$	$\cos 63° = \frac{y}{23}$
$0.8910 \approx \frac{x}{23}$	$0.4540 \approx \frac{y}{23}$
$x \approx 20.5$	$y \approx 10.4$

Find Y. $63° + Y = 90°$

$Y = 27°$

Therefore, $Y = 27°$, $x \approx 20.5$, and $y \approx 10.4$.

Trigonometry has many practical applications in the real world. Among the most important is the ability to find distances or lengths that cannot be measured directly.

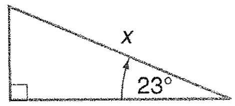
EXERCISES

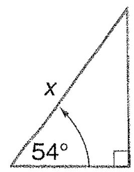
Practice

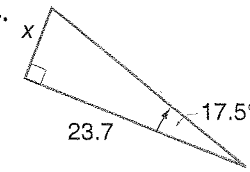
Suppose θ is an acute angle of a right triangle. For each function, find the values of the remaining five trigonometric functions of θ. Round to four decimal places.

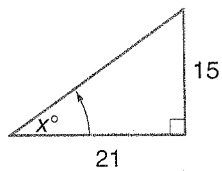
16. $\tan \theta = \frac{12}{5}$
17. $\cos \theta = \frac{1}{4}$
18. $\cot \theta = 2$
19. $\csc \theta = \frac{5}{2}$
20. $\sec \theta = 3$
21. $\sin \theta = 0.5$

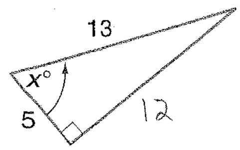
Write an equation involving sin, cos, or tan that can be used to find x. Then solve the equation. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

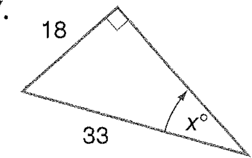
22. 

23. 

24. 

25. 

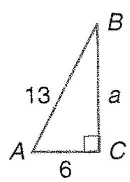
26. 

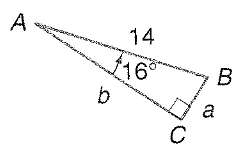
27. 

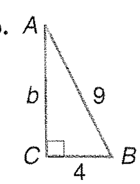
Find the value of x. Round to the nearest degree.

28. $\tan x = 0.5923$
29. $\cos x = 0.5269$
30. $\tan x = 0.2126$
31. $\sin x = 0.9998$
32. $\cos x = 0.9998$
33. $\sin x = 0.5000$

Solve each right triangle. Assume that *C* represents the right angle and *c* is the hypotenuse. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

34. 

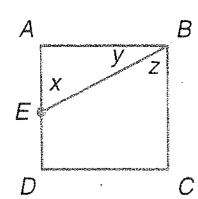
35. 

36. 

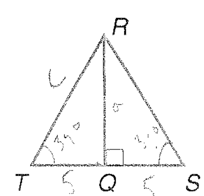
37. $B = 18°, a = \sqrt{15}$
38. $A = 56°, c = 16$
39. $A = 45°, c = 7\sqrt{2}$
40. $c = 25, A = 15°$
41. $B = 30°, b = 11$
42. $\tan A = \frac{7}{8}, a = 7$
43. $a = 7, A = 27°$
44. $\tan B = \frac{8}{6}, b = 8$
45. $\sin A = \frac{1}{3}, a = 5$

INTEGRATION
Geometry

46. In square *ABCD* at the right, the midpoint of side $\overline{AD}$ is *E*. Find the values of *x*, *y*, and *z* to the nearest tenth of a degree.



47. Isosceles triangle *RST* below at the right has base $\overline{TS}$ measuring 10 centimeters and base angles each measuring 39°. Find the length of the altitude $\overline{QR}$.



Critical Thinking

48. Describe a set of given conditions for which it would be impossible to solve a right triangle.
49. Explain why the sine and cosine of an acute angle are never greater than 1 but the tangent of an acute angle may be greater than 1.

$$x = \frac{0.25 \tan 38^\circ}{\tan 43^\circ - \tan 38^\circ}$$
$$x \approx 1.29$$

Now, find h.
$$h = x \tan 43^\circ$$
$$h \approx 1.29(0.9325)$$
$$h \approx 1.20$$

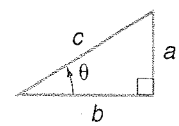
Therefore, the volcano crater is about 1.20 kilometers high.

CHECK FOR UNDERSTANDING

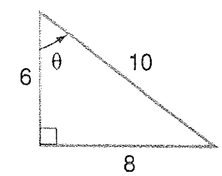
Communicating Mathematics

Study the lesson. Then complete the following.

1. **Identify** the hypotenuse, the leg adjacent to θ, and the leg opposite θ in the triangle at the right.



2. **Define** the word *trigonometry.*
3. **Evaluate** the six trigonometric functions of θ in the triangle at the right. Round your answers to four decimal places.



4. **Draw and label** a figure that shows an angle of depression from a person on a Ferris wheel who is looking at a friend waiting in line.

MATH JOURNAL

5. Write a paragraph explaining how you can use trigonometry to find the height of a flagpole. Include a drawing with your explanation.

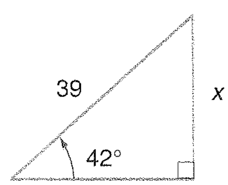
Guided Practice

Suppose θ is an acute angle of a right triangle. For each function, find the values of the remaining five trigonometric functions of θ. Round to four decimal places.

6. $\sin \theta = \frac{\sqrt{3}}{2}$
7. $\tan \theta = 2$

Write an equation involving sin, cos, or tan that can be used to find x. Then solve the equation. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

8.



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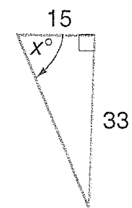
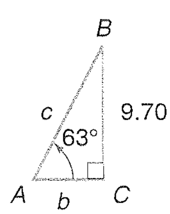


Find the value of x. Round to the nearest degree.

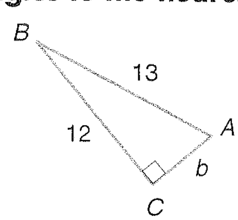
10. $\sin x = 0.7364$
11. $\cos x = 0.9912$

Solve each right triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

12.



13.



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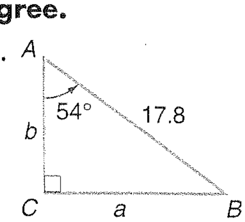
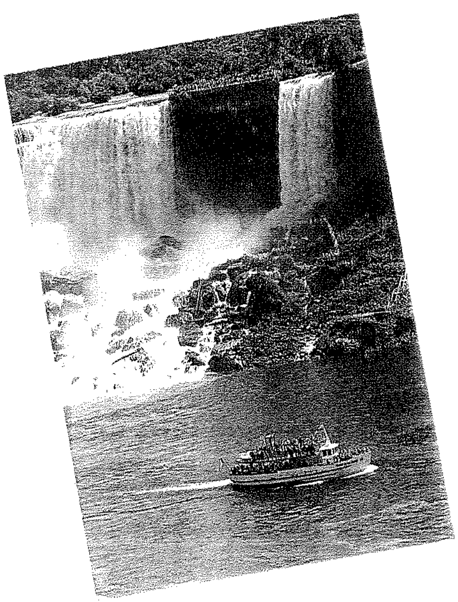




15. **Traveling** In a sightseeing boat near the base of the Horseshoe Falls at Niagara Falls, a passenger estimates the angle of elevation to the top of the falls to be 30°. If the Horseshoe Falls are 173 feet high, what is the distance from the boat to the base of the falls?

13-2

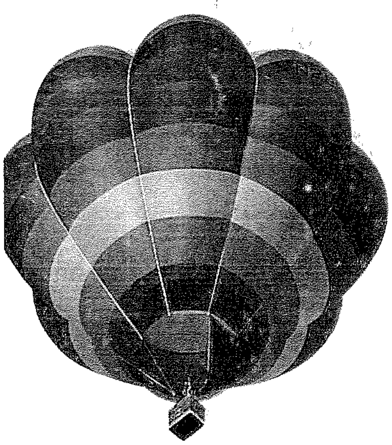
Angles and Angle Measure

What YOU'LL LEARN

- To change radian measure to degree measure and vice versa, and
- to identify coterminal angles.

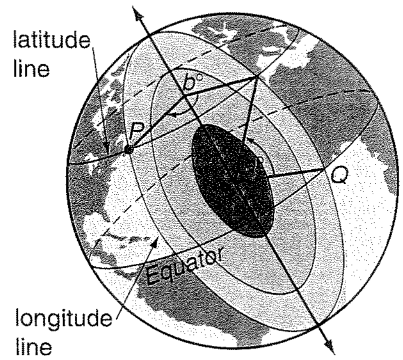
Why IT'S IMPORTANT

You can use angles to solve problems involving astronomy and geography.

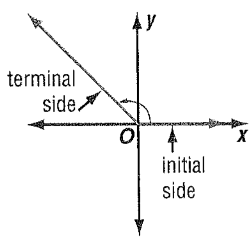


CONNECTION
Geography

When French novelist Jules Verne wrote *Around the World in Eighty Days* in 1873, traveling that great distance in such a short time was unheard of. Do you think you could travel westward around the world in just one day? How fast would you have to travel to accomplish this? *A problem like this will be solved in Exercise 50.*

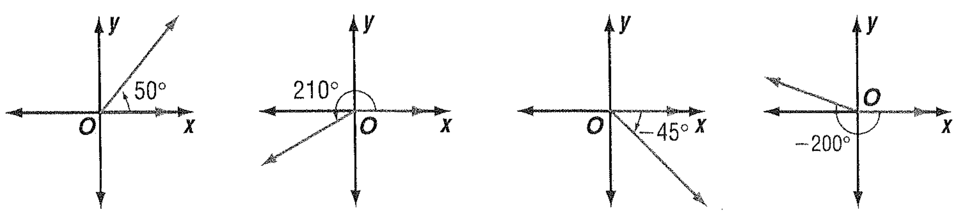


The answers to these questions depend on your position on Earth. Cartographers use a grid that contains circles through the poles, called *longitude* lines, and circles parallel to the equator, called *latitude* lines. In the figure at the right, point P is located by traveling north from point Q on the equator through a central angle of $a°$ to a circle of latitude, and then west along that circle through an angle of $b°$.

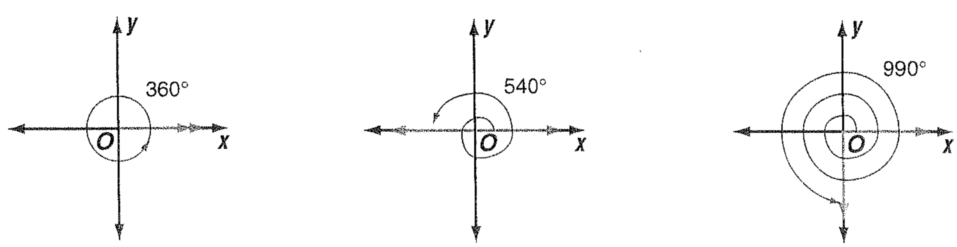


On a coordinate plane, an angle may be generated by the rotation of two rays that share a fixed endpoint at the origin. One ray, called the **initial side** of the angle, is fixed along the positive x-axis. The other ray, called the **terminal side** of the angle, can rotate about the center. An angle positioned so that its vertex is at the origin and its initial side is along the positive x-axis is said to be in **standard position.**

The measure of an angle is determined by the amount of rotation from the initial side to the terminal side. If the rotation is in a counterclockwise direction, the measure of the angle is positive. If the rotation is in a clockwise direction, the measure of the angle is negative.



When terminal sides rotate, they may sometimes make one or more revolutions. An angle whose terminal side has made exactly one revolution has a measure of 360°.



Applications and Problem Solving

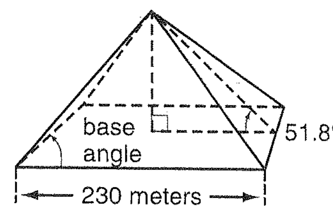
Real World

50. **Literature** Refer to the application at the beginning of the lesson. Sherlock Holmes needed to find the length of the shadow cast by the elm. He was able to determine that the elm tree was 64 feet tall before it was struck by lightning. At the appropriate time, Holmes used a 6-foot rod to cast a shadow. Its shadow was 9 feet. What was the length of the shadow of the elm?

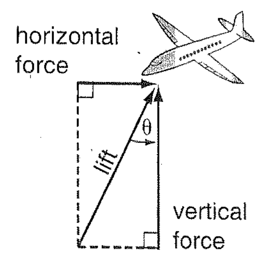
51. **Broadcasting** Dolores and Bill are standing 100 feet apart and in a straight line with the WWV television tower. The angle of elevation from Bill to the tower is 30° and the angle of elevation from Dolores is 20°. Find the height of the television tower to the nearest foot.

52. **Skiing** The Aerial run in Snowbird, Utah, is 8395 feet long. Its vertical drop is 2900 feet. If the slope were constant, estimate the angle of elevation that the run makes with the horizontal.

53. **History** The Great Pyramid in Egypt has a square base, 230 meters on each side. The triangular faces of the pyramid make an angle of 51.8° with the base. Suppose you want to make a model of the pyramid for your history project. What measure should you use for the base angle of each triangle?



54. **Aviation** When an airplane is flying, the air pressure creates a force called the *lift*, which is perpendicular to the wings. If a plane banks for a turn, this lift is separated into a horizontal and vertical force. The horizontal force is what is responsible for the turn, and the measure of the vertical force is the plane's weight.



   a. Suppose a plane weighs 500,000 pounds. Find the measure of the lift and horizontal component for a banking angle, θ, of 20°.

   b. If the maximum lift that the wings can sustain is 650,000 pounds, what is the maximum banking angle?

Mixed Review

55. **Statistics** Determine whether a sample of people attending a concert is a random sample for a survey of people's favorite performer. (Lesson 12–8)

56. Four of every 7 pitches thrown by Elias Ramos are strikes. What is the probability that 4 of the next 5 pitches will be strikes? (Lesson 12–7)

57. **Probability** State the probability of an event occurring given that the odds of the event are $\frac{5}{1}$. (Lesson 12–4)

58. Expand the binomial $(r + s)^6$. (Lesson 11–8)

59. Find the nth term of a geometric sequence in which $a_1 = 4$, $n = 3$, and $r = 5$. (Lesson 11–3)

60. Solve $\log_4(x + 2) + \log_4(x - 4) = 2$. (Lesson 10–3)

61. Simplify $\left(x^{\sqrt{2}}\right)^{\sqrt{8}}$. (Lesson 10–1)

62. **SAT Practice** If $6x^2 + 7x = 90$ and $6x^2 - 7y = 20$, then what is the value of $12x + 12y$?
    A 20 B 70 C 110 D 120 E 140

63. **Chemistry** A chemist performed an experiment that yields 1.8×10^{24} molecules of ethanol. The mole is the standard unit of measure in chemistry. There are 6.02×10^{23} molecules in a mole. How many moles of ethanol did the experiment yield? (Lesson 5–1)

64. What property of real numbers is demonstrated by $x(a + b) = xa + xb$? (Lesson 1–2)

For Extra Practice, see page 906.