47. If $\cos\theta = \frac{2}{3}$, find all possible values of $\sin\theta$.

48. If $\sec\theta = -3$, find all possible values of $\sin\theta$ and $\cos\theta$.

49. If $\cos\theta = 0$, find all possible values of $\sin\theta$ and $\tan\theta$.

Suppose θ is an angle in standard position with the given conditions. State the quadrant or quadrants in which the terminal side of θ lies.

50. $\sin\theta > 0$

51. $\sin\theta > 0, \cos\theta < 0$

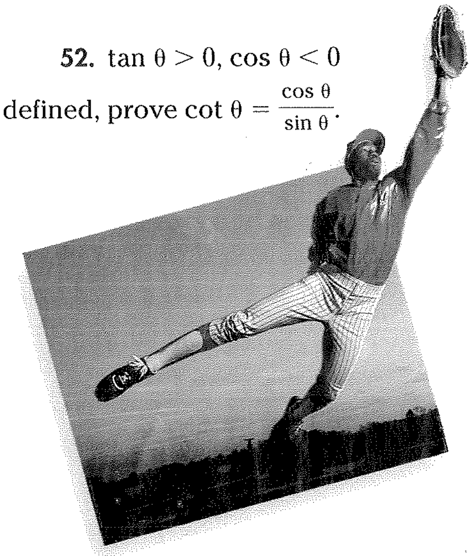
52. $\tan\theta > 0, \cos\theta < 0$

Critical Thinking

53. If θ is any angle for which the functions are defined, prove $\cot\theta = \frac{\cos\theta}{\sin\theta}$.

Applications and Problem Solving

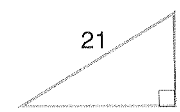
Real World



54. **Baseball** The formula $R = \frac{V_0^2 \sin 2\theta}{g}$ gives the distance of a baseball that is hit at an initial velocity of V_0 feet per second at an angle of θ with the ground. The variable g represents the acceleration due to gravity, which is 32 feet per second^2.
   a. If the ball was hit with an initial velocity of 100 feet per second at an angle of 45°, how far was it hit?
   b. Which angle will result in the greatest distance? Explain your reasoning.

55. **Basketball** The maximum height that a basketball reaches after being shot is given by the formula $H = \frac{V_0^2 \sin^2\theta}{2g}$, where V_0 represents the initial velocity, θ represents the degree measure of the angle which the path of the basketball makes with the ground, and g represents the acceleration due to gravity, 32 feet per second^2. Find the maximum height reached by a free-throw if it is shot with an initial velocity of 25 feet per second at an angle of 65°.

Mixed Review



56. Change $\frac{5\pi}{6}$ radians to degrees. (Lesson 13–2)

57. Solve ΔABC shown at the right if $c = 21$ and $b = 18$. (Lesson 13–1)

58. **Probability** In homeroom, 3 of the 16 girls have red hair, and 2 of the 15 boys have red hair. What is the probability of selecting a boy or a red-haired person as homeroom representative to student council? (Lesson 12–6)

59. Find the sum of the infinite geometric series $\frac{4}{3} - \frac{2}{3} + \frac{1}{3} - \frac{1}{6} + \cdots$. (Lesson 11–5)

60. Name the next four terms of the arithmetic sequence 21, 15, 9, ... (Lesson 11–1)

61. **SAT Practice** In a jar there are 14 black marbles, 6 red marbles, and 4 white marbles. How many white marbles need to be added to the jar in order to double the probability of selecting a white marble?
   A 2 B 4 C 6 D 8 E 24

62. Simplify $\frac{w + 12}{4w - 16} - \frac{w + 4}{2w - 8}$. (Lesson 9–4)

63. Write the polynomial function of least degree with integral coefficients whose zeros are 6 and $4 - 2i$. (Lesson 8–4)

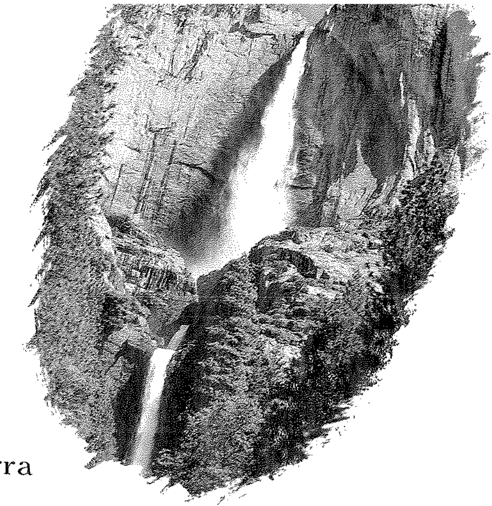
64. Find the slope-intercept form of the equation that passes through (−2, 5) and (3, 1) (Lesson 2–4)

65. Solve $-1.6m + 5 = -7.8$. (Lesson 1–4)

For Extra Practice, see page 907.

13–4

Law of Sines



What YOU'LL LEARN

- To solve triangles by using the law of sines, and
- to examine solutions.

Why IT'S IMPORTANT

You can use the law of sines to solve problems involving forestry and aviation.

Top Five List

States with the most visitors per year, in millions

1. California; 6.2
2. New York; 5.4
3. Texas; 5.0
4. Florida; 3.9
5. Hawaii; 2.2

Real World APPLICATION

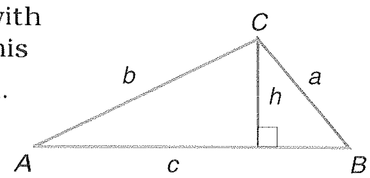
Forestry

Yosemite National Park, located in California's Sierra Nevada mountains, is home to beautiful meadows, spectacular waterfalls, and jagged mountains. More than 30 kinds of trees and more than 1300 species of plants can be found in the park.

When drought conditions exist in the park, the Park Service often imposes restrictions on open fires. Suppose two forest rangers, 10 miles apart from each other on a straight service road, both sight an illegal campfire away from the road. Using their radios to communicate with each other, they determine that the fire is between them. The first ranger's line of sight to the fire makes an angle of 34° with the road, and the second ranger's line of sight to the fire makes a 67° angle with the road. How far is the fire from each ranger? What is the shortest distance from the road to the fire? *This problem will be solved in Example 3.*

In Lesson 13–1, you solved problems that involved acute angles of right triangles. It is also possible to use trigonometric functions to solve triangles that do not necessarily contain a right angle. You can even use trigonometric functions to find the area of triangles.

Consider ΔABC with height h units and sides with lengths a units, b units, and c units. The area of this triangle is $\frac{1}{2}ch$. Note that $\sin A = \frac{h}{b}$ or $h = b \sin A$. By combining these equations, you can find a new formula for the area of the triangle.



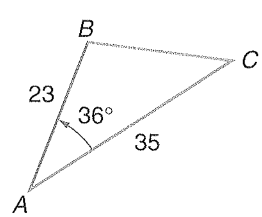
$$\text{Area} = \frac{1}{2}ch$$
$$= \frac{1}{2}c(b \sin A) \quad h = b \sin A$$

You can find two other formulas for the area of the triangle in a similar way.

$$\text{Area} = \frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C$$

These formulas allow you to find the area of any triangle when you know the measures of two sides and the included angle.

Example 1 **Find the area of ΔABC if $b = 35$, $c = 23$, and $A = 36°$.**



$$\text{Area} = \frac{1}{2}bc \sin A$$
$$= \frac{1}{2}(35)(23)\sin 36° \quad \sin 36° \approx 0.5878$$
$$\approx 236.59$$

To the nearest whole unit, the area is 237 square units.

All of the area formulas above represent the area of the same triangle. So, the following must be true.

$$\frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C$$

The **law of sines** is obtained by dividing each of the expressions above by $\frac{1}{2}abc$.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Law of Sines

> **Let $\triangle ABC$ be any triangle with a, b, and c representing the measures of sides opposite angles with measurements A, B, and C, respectively. Then,**
>
> $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

You can apply the law of sines to a triangle if you know

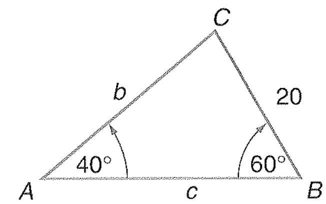
- the measures of two angles and the measure of any side, or
- the measures of two sides and the angle opposite one of the sides.

Example 2

Use the law of sines to solve the triangle below. Round measures of sides to the nearest tenth.

You are given the measures of two angles and a side.
First, find the measure of the third angle, $\angle C$.

$$40° + 60° + C = 180°$$
$$C = 80°$$



Now, use the law of sines to find b and c.

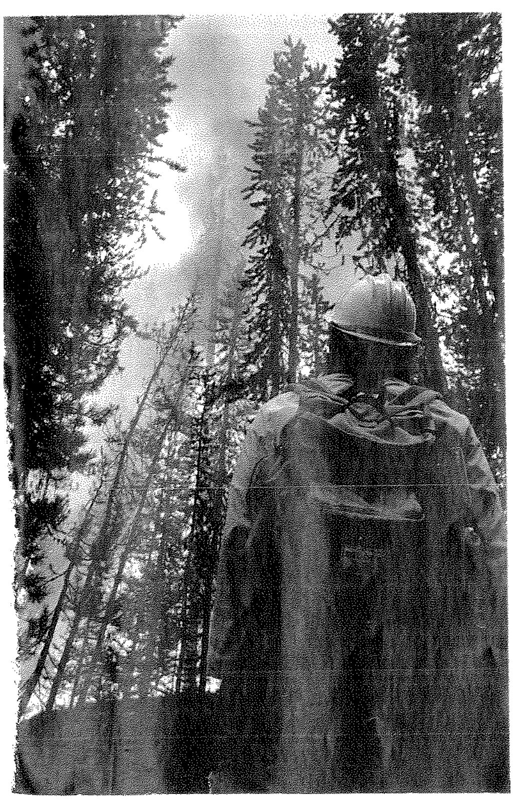
$\frac{\sin A}{a} = \frac{\sin B}{b}$	$\frac{\sin A}{a} = \frac{\sin C}{c}$
$\frac{\sin 40°}{20} = \frac{\sin 60°}{b}$	$\frac{\sin 40°}{20} = \frac{\sin 80°}{c}$
$b = \frac{20 \sin 60°}{\sin 40°}$	$c = \frac{20 \sin 80°}{\sin 40°}$
$b \approx 26.9$	$c = 30.6$

Therefore, $b \approx 26.9$, $c \approx 30.6$, and $C = 80°$.

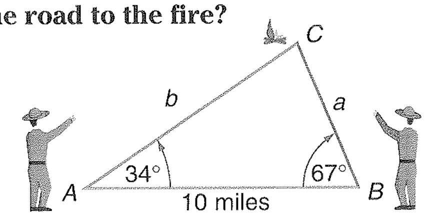
Example 3

Real World APPLICATION

Forestry



Refer to the application at the beginning of the lesson.
a. How far is the fire from each ranger?
b. What is the shortest distance from the road to the fire?



First, draw a diagram. You are given the measure of two angles and a side. Find the measure of the third angle, angle C.

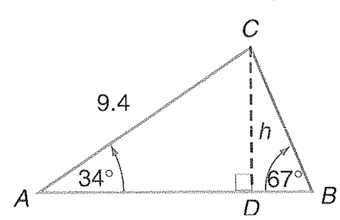
$$34° + 67° + C = 180°$$
$$C = 79°$$

a. Use the law of sines to find a and b, the distance from each ranger to the fire.

$\frac{\sin 79°}{10} = \frac{\sin 34°}{a}$	$\frac{\sin 79°}{10} = \frac{\sin 67°}{b}$
$a \approx \frac{10 \sin 34°}{\sin 79°}$	$b \approx \frac{10 \sin 67°}{\sin 79°}$
$a \approx 5.7$	$b \approx 9.4$

Therefore, the fire is 9.4 miles from Ranger A and 5.7 miles from Ranger B.

b. The shortest distance from the road to the fire is from point D, which is on the perpendicular from C to the road. Let h represent the measure of segment CD.



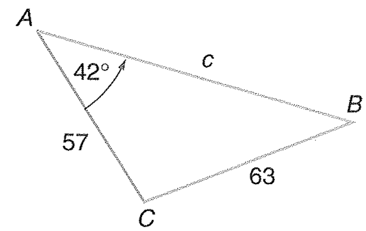
$$\sin 34° = \frac{h}{9.4}$$
$$h \approx 5.3$$

The shortest distance from the road to the fire is about 5.3 miles.

Example 4 **Use the law of sines to solve the triangle below. Round to the nearest tenth.**

You are given the measures of two sides and the angle opposite one of them. First, find the measure of the angle opposite the other given side, $\angle B$.

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$
$$\frac{\sin 42^\circ}{63} = \frac{\sin B}{57}$$
$$\sin B = \frac{57 \sin 42^\circ}{63}$$
$$\sin B \approx 0.6054$$
$$B \approx 37.3^\circ$$



Next, find the measure of the third angle, $\angle C$.

$$37.3^\circ + 42^\circ + C = 180^\circ$$
$$C \approx 100.7^\circ$$

Then, find the measure of c.

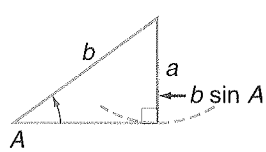
$$\frac{\sin A}{a} = \frac{\sin C}{c}$$
$$\frac{\sin 42^\circ}{63} = \frac{\sin 100.7^\circ}{c}$$
$$c = \frac{63 \sin 100.7^\circ}{\sin 42^\circ}$$
$$c \approx 92.5$$

Therefore, $C \approx 100.7^\circ$, $B \approx 37.3^\circ$, and $c \approx 92.5$ units.

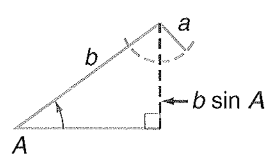
When solving a triangle, you must analyze the data to determine whether there is a solution or not. For example, if you are given the measures of two angles and a side, as in Examples 2 and 3, the triangle has a unique solution. However, if you are given the measures of two sides and the angle opposite one of them is given, a single solution may not exist. One of the following will be true.

- No triangle exists, and there is no solution.
- Exactly one triangle exists, and there is one unique solution.
- Two triangles exist, and there are two solutions.

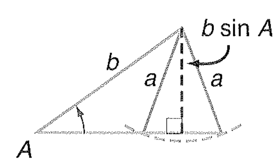
Suppose you are given a, b, and A. First, consider the case where $A < 90^\circ$. If $a < b$, there are three possibilities.



one solution
$a = b \sin A$

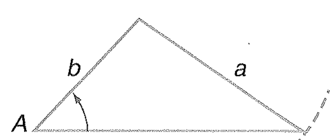


no solution
$a < b \sin A$

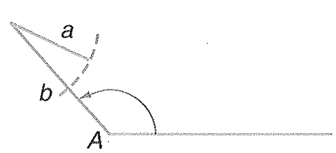


two solutions
$b > a > b \sin A$

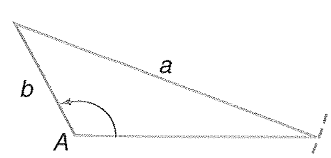
If $a > b$, there is one unique solution.



Consider the case where $A \geq 90^\circ$. There are two possibilities.



no solution, $a \leq b$



one solution, $a > b$

In Example 4, you were given the measures of two sides and the angle opposite one of them. When you solved the triangle, there was one unique solution. The following examples show triangles in which there are no solutions and two solutions.

Example 5

PROBLEM SOLVING

Examine the Solution

Solve each triangle described below.

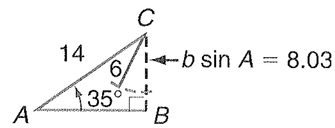
a. $A = 35°$, $b = 14$, **and** $a = 6$

Angle A is less than 90°. Find $b \sin A$ and compare with a.

$b \sin A = 14 \sin 35°$ *$b \sin A$ is the minimum distance from C to $\overline{AB}$.*

$= 14(0.5736)$

≈ 8.03



Since $6 < 8.03$, there is no solution.

b. $A = 42°$, $a = 13$, **and** $b = 16$.

$b \sin A = 16 \sin 42°$

$\approx 16(0.6691)$

≈ 10.71

Since $42° < 90°$ and $10.71 < 13 < 16$, there are two solutions. The two triangles to be solved are ΔABC and ΔADC.

When two solutions exist, it is called the ambiguous case. Why?

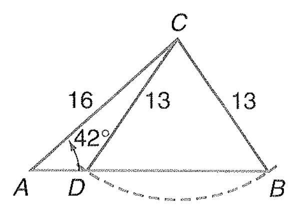
Solve ΔABC.

First, use the law of sines to find B.

$\frac{\sin 42°}{13} = \frac{\sin B}{16}$ *$\frac{\sin A}{a} = \frac{\sin B}{b}$*

$\sin B = \frac{16 \sin 42°}{13}$

$B \approx 55.4°$



Find $\angle ACB$.

$42° + 55.4° + \angle ACB \approx 180°$

$\angle ACB \approx 82.6°$

Find c.

$\frac{\sin 42°}{13} = \frac{\sin 82.6°}{c}$

$c \approx \frac{13 \sin 82.6°}{\sin 42°}$

$c \approx 19.3$

Therefore, $B \approx 55.4°$, $\angle ACB \approx 82.6°$, and $c \approx 19.3$.

Solve ΔADC.

First, find $\angle ADC$.

ΔDBC is isosceles, so the measures of the base angles are equal. Therefore, since $\angle B \approx 55.4°$, $\angle CDB \approx 55.4°$.

$\angle ADC$ is supplementary to $\angle CDB$, so $\angle ADC \approx 124.6°$.

Find $\angle ACD$.

$42° + 124.6° + \angle ACD \approx 180°$

$\angle ACD \approx 13.4°$

Then, use the law of sines to find the measure of segment AD in ΔADC.

$\frac{\sin 42°}{13} = \frac{\sin 13.4°}{AD}$

$AD \approx \frac{13 \sin 13.4°}{\sin 42°}$

$AD \approx 4.5$

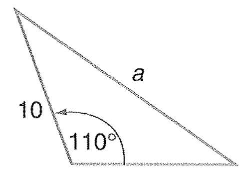
Therefore, $AD \approx 4.5$, $\angle ACD \approx 13.4°$, and $\angle ADC \approx 124.6°$.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

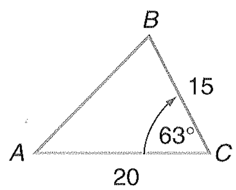
1. **State** the law of sines.
2. **Describe** a set of conditions for which the law of sines can be used.
3. **Explain** how you know when a triangle has no solution.
4. **Draw** ΔABC if $\angle ABC = 35°$, $BC = 8$, and $\angle BCA = 70°$.
5. **Choose** a value for a so that the triangle at the right has one solution.



Guided Practice

Write an equation that can be used to find the area of each triangle. Then solve the equation. Round to the nearest tenth.

6.



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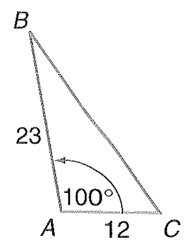
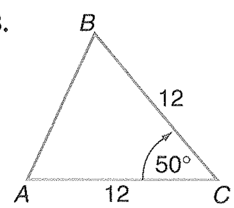
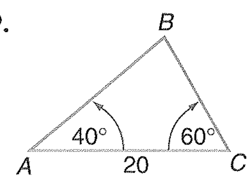


Solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

8.



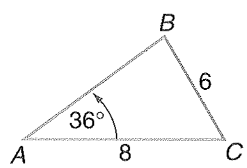
9.



10. $a = 8, A = 49°, B = 57°$

Determine whether each triangle has no solution, one solution, or two solutions. Then solve each triangle. Note that the triangles may not be drawn to scale.

11.



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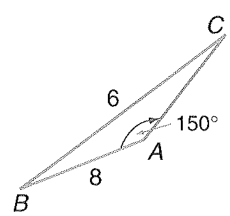


13. $a = 64, c = 90, C = 98°$

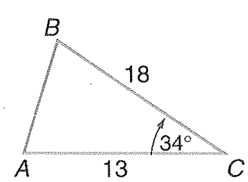
14. The longest side of a triangle is 67 inches. Two angles have measures of 47° and 55°. Solve the triangle.

EXERCISES

Practice

Write an equation that can be used to find the area of each triangle. Then solve the equation. Round to the nearest tenth.

15.



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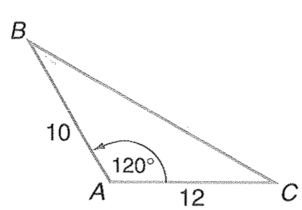


17. $b = 24, a = 20, C = 73°$

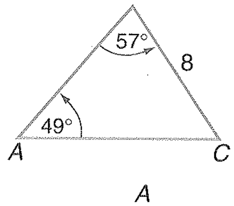
18. $b = 35, c = 47, A = 67°$

19. $a = 11.5, c = 19, B = 20°$

20. $a = 9.4, c = 13.5, B = 95°$

Solve each triangle. Round measures of sides and angles to the nearest tenth.

21.



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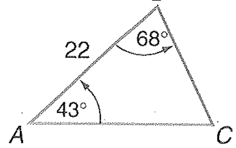

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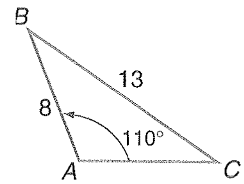

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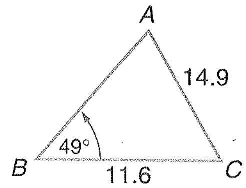

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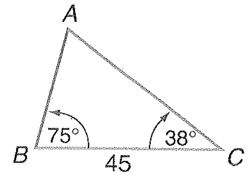

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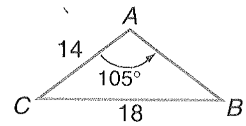


27. $A = 30°, C = 70°, c = 8$

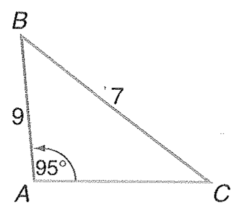
28. $c = 17, b = 15, C = 64°$

29. $a = 14, b = 7.5, A = 103°$

30. $a = 23, A = 73°, C = 24°$

Determine whether each triangle has no solution, one solution, or two solutions. Then solve each triangle. Note that the triangles may not be drawn to scale.

31.



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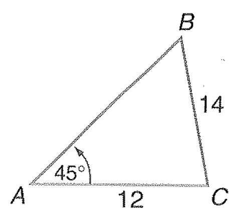

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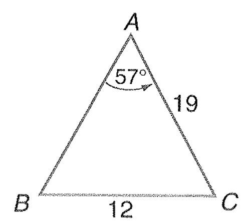


34. $a = 9, b = 20, A = 31°$

35. $a = 12, b = 14, A = 90°$

36. $a = 125, b = 150, A = 25°$

37. $A = 40°, b = 16, a = 10$

38. $a = 18, b = 20, A = 120°$

39. $A = 40°, b = 10, a = 8$

INTEGRATION
Geometry

40. An isosceles triangle has a base of 22 centimeters and exactly one angle measuring 36°. Find its perimeter.

41. The sides of a triangle measure 22, 13, and 8. Find the measure of the smallest angle.

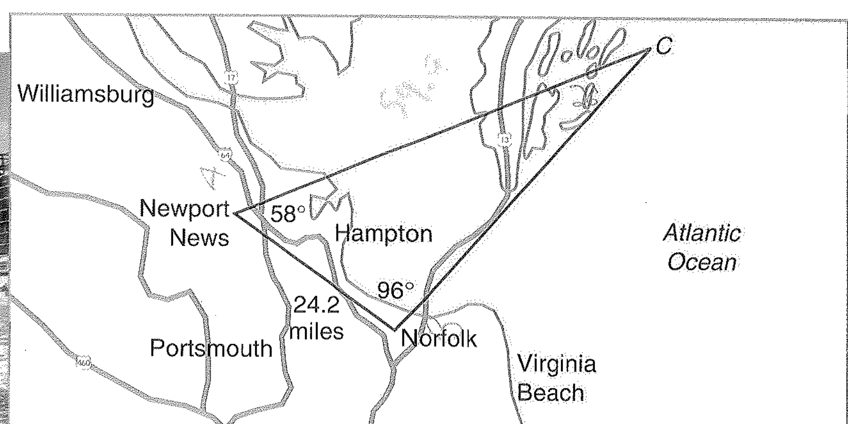
Critical Thinking

42. Prove that the law of sines holds for right triangles.

Applications and Problem Solving

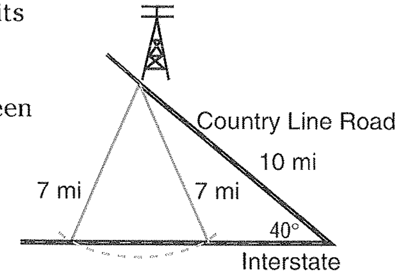
Real World

43. **Aviation** A pilot takes off from Newport News, Virginia, and flies toward the Atlantic Ocean. After reaching point C, the plane develops mechanical difficulties, and the pilot needs to return either to Newport News or Norfolk, Virginia. How far is it to the nearer airport?

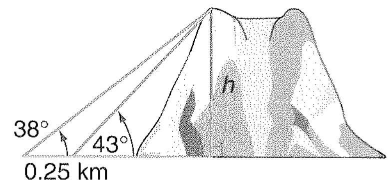


44. **Make a Drawing** Karen was given an assignment to draw and then construct a triangular model of three steel girders for her engineering class. Two of the girders measured 7 cm and 6 cm, and the angle opposite the 7 cm girder had to be 30°. Can she construct the triangle? If so, how long is the third girder?

45. **Communication** A low-watt radio station has its transmitter on County Line Road, 10 miles from where it intersects with the interstate highway. If the radio station has a range of 7 miles, between what two distances from the intersection can cars on the interstate hear the radio station?



46. **Geology** A geologist measured a 43° angle of elevation to the top of a volcano crater. After moving 0.25 kilometers farther away, the angle of elevation was 38°.
   a. Use the law of sines to find the height of the top of the volcano crater.
   b. This problem was solved in Example 6 on page 776. Compare and contrast the method used there with the method you used here.



Mixed Review

47. Find the exact values of the six trigonometric functions for an angle in standard position that measures 180°. (Lesson 13–3)

48. **Counting** How many ways can 6 different books be arranged on a shelf? (Lesson 12–2)

49. Find the sum of the first 50 terms of an arithmetic series where $a_1 = 5$ and $d = 25$. (Lesson 11–2)

50. **Finance** Use the formula $A = Pe^{rt}$ to determine whether Sean can buy a used car costing $2500 with the $1000 his grandparents invested for him 16 years ago at 7%. (Lesson 10–5)

51. Simplify $\frac{3x - 21}{x^2 - 49} \div \frac{3x}{x^2 + 7x}$.

52. **ACT Practice** If $x + y = 90°$ and x and y are positive, then $\frac{\cos x}{\sin y} =$
   A -1 B 0 C $\frac{1}{2}$ D 1
   E It cannot be determined from the information given.

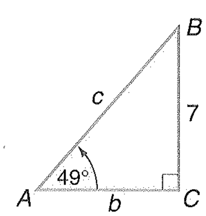
For Extra Practice, see page 907.

53. Factor $2x^2 - 11x - 21$. (Lesson 5–4)

54. Evaluate the determinant of $\begin{bmatrix} 6 & 4 \\ -3 & 2 \end{bmatrix}$. (Lesson 4–4)

SELF TEST

1. Solve the right triangle shown at the right. (Lesson 13–1)



Change each degree measure to radian measure. (Lesson 13–2)

2. 90° 3. 150° 4. −135°

Change each radian measure to degree measure. (Lesson 13–2)

5. $\frac{3\pi}{2}$ 6. $-\frac{7\pi}{4}$ 7. 2

8. Find sin θ, cos θ, and tan θ if the terminal side of θ in standard position contains $P(-2, 0)$. (Lesson 13–3)

9. **Electronics** The power P in watts absorbed by an AC circuit is given by the formula $P = IV \cos \theta$, where I is the current in amps, V is the voltage, and θ is the measure of the phase angle. Find the power absorbed by a circuit if its current is 2 amps, its voltage is 120 volts, and its phase angle is 70°. (Lesson 13–3)

10. Solve the triangle in which $A = 40°$, $b = 12$, and $a = 5$. (Lesson 13–4)