49. **(a)** Since $f'(x) = 2^x \ln 2, f'(0) = 2^0 \ln 2 = \ln 2.$

(b) $f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{2^h - 2^0}{h} = \lim_{h\to 0} \frac{2^h - 1}{h}$

(c) Since quantities in parts (a) and (b) are equal,

$\lim_{h\to 0} \frac{2^h - 1}{h} = \ln 2.$

(d) By following the same procedure as above using $g(x) = 7^x$, we may see that $\lim_{h\to 0} \frac{7^h - 1}{h} = \ln 7.$

50. Recall that a point (a, b) is on the graph of $y = e^x$ if and only if the point (b, a) is on the graph of $y = \ln x$. Since there are points (x, e^x) on the graph of $y = e^x$ with arbitrarily large x-coordinates, there will be points $(x, \ln x)$ on the graph of $y = \ln x$ with arbitrarily large y-coordinates.

51. **(a)** The graph y_4 is a horizontal line at $y = a$.

(b) The graph of y_3 is always a horizontal line.

a	2	3	4	5
y_3	0.693147	1.098613	1.386295	1.609439
$\ln a$	0.693147	1.098612	1.386294	1.609438

We conclude that the graph of y_3 is a horizontal line at $y = \ln a$.

(c) $\frac{d}{dx} a^x = a^x$ if and only if $y_3 = \frac{y_2}{y_1} = 1.$

So if $y_3 = \ln a$, then $\frac{d}{dx} a^x$ will equal a^x if and only if $\ln a = 1$, or $a = e$.

(d) $y_2 = \frac{d}{dx} a^x = a^x \ln a$. This will equal $y_1 = a^x$ if and only if $\ln a = 1$, or $a = e$.

52. $\frac{d}{dx}\left(-\frac{1}{2}x^2 + k\right) = -x$ and $\frac{d}{dx}(\ln x + c) = \frac{1}{x}.$

Therefore, at any given value of x, these two curves will have perpendicular tangent lines.

53. **(a)** Since the line passes through the origin and has slope $\frac{1}{e}$, its equation is $y = \frac{x}{e}$.

(b) The graph of $y = \ln x$ lies below the graph of the line $y = \frac{x}{e}$ for all positive $x \neq e$. Therefore, $\ln x < \frac{x}{e}$ for all positive $x \neq e$.

(c) Multiplying by e, $e \ln x < x$ or $\ln x^e < x$.

(d) Exponentiating both sides of $\ln x^e < x$, we have $e^{\ln x^e} < e^x$, or $x^e < e^x$ for all positive $x \neq e$.

(e) Let $x = \pi$ to see that $\pi^e < e^\pi$. Therefore, e^π is bigger.

Chapter 3 Review Exercises

(pp. 172–175)

1. $\frac{dy}{dx} = \frac{d}{dx}\left(x^5 - \frac{1}{8}x^2 + \frac{1}{4}x\right) = 5x^4 - \frac{1}{4}x + \frac{1}{4}$

2. $\frac{dy}{dx} = \frac{d}{dx}(3 - 7x^3 + 3x^7) = -21x^2 + 21x^6$

3. $\frac{dy}{dx} = \frac{d}{dx}(2 \sin x \cos x)$

$= 2(\sin x)\frac{d}{dx}(\cos x) + 2(\cos x)\frac{d}{dx}(\sin x)$

$= -2 \sin^2 x + 2 \cos^2 x$

Alternate solution:

$\frac{dy}{dx} = \frac{d}{dx}(2 \sin x \cos x) = \frac{d}{dx} \sin 2x = (\cos 2x)(2)$

$= 2 \cos 2x$

4. $\frac{dy}{dx} = \frac{d}{dx} \frac{2x + 1}{2x - 1} = \frac{(2x - 1)(2) - (2x + 1)(2)}{(2x - 1)^2} = -\frac{4}{(2x - 1)^2}$

5. $\frac{ds}{dt} = \frac{d}{dt} \cos(1 - 2t) = -\sin(1 - 2t)(-2) = 2 \sin(1 - 2t)$

6. $\frac{ds}{dt} = \frac{d}{dt} \cot\left(\frac{2}{t}\right) = -\csc^2\left(\frac{2}{t}\right)\frac{d}{dt}\left(\frac{2}{t}\right) = -\csc^2\left(\frac{2}{t}\right)\left(-\frac{2}{t^2}\right)$

$= \frac{2}{t^2} \csc^2\left(\frac{2}{t}\right)$

7. $\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x} + 1 + \frac{1}{\sqrt{x}}\right) = \frac{d}{dx}(x^{1/2} + 1 + x^{-1/2})$

$= \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} = \frac{1}{2\sqrt{x}} - \frac{1}{2x^{3/2}}$

8. $\frac{dy}{dx} = \frac{d}{dx}(x\sqrt{2x + 1}) = (x)\left(\frac{1}{2\sqrt{2x + 1}}\right)(2) + (\sqrt{2x + 1})(1)$

$= \frac{x + (2x + 1)}{\sqrt{2x + 1}} = \frac{3x + 1}{\sqrt{2x + 1}}$

9. $\frac{dr}{d\theta} = \frac{d}{d\theta} \sec(1 + 3\theta) = \sec(1 + 3\theta) \tan(1 + 3\theta)(3)$

$= 3 \sec(1 + 3\theta) \tan(1 + 3\theta)$

10. $\frac{dr}{d\theta} = \frac{d}{d\theta} \tan^2(3 - \theta^2)$

$= 2 \tan(3 - \theta^2)\frac{d}{d\theta} \tan(3 - \theta^2)$

$= 2 \tan(3 - \theta^2) \sec^2(3 - \theta^2)(-2\theta)$

$= -4\theta \tan(3 - \theta^2) \sec^2(3 - \theta^2)$

11. $\frac{dy}{dx} = \frac{d}{dx}(x^2 \csc 5x)$

$= (x^2)(-\csc 5x \cot 5x)(5) + (\csc 5x)(2x)$

$= -5x^2 \csc 5x \cot 5x + 2x \csc 5x$

12. $\frac{dy}{dx} = \frac{d}{dx} \ln \sqrt{x} = \frac{1}{\sqrt{x}} \frac{d}{dx} \sqrt{x} = \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x}, x > 0$

13. $\frac{dy}{dx} = \frac{d}{dx} \ln(1 + e^x) = \frac{1}{1 + e^x} \frac{d}{dx}(1 + e^x) = \frac{e^x}{1 + e^x}$

14. $\frac{dy}{dx} = \frac{d}{dx}(xe^{-x}) = (x)(e^{-x})(-1) + (e^{-x})(1) = -xe^{-x} +$

15. $\frac{dy}{dx} = \frac{d}{dx}(e^{1+\ln x}) = \frac{d}{dx}(e^1 e^{\ln x}) = \frac{d}{dx}(ex) = e$

16. $\frac{dy}{dx} = \frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{\cos x}{\sin x} = \cot x$, for values of x in the intervals $(k\pi, (k+1)\pi)$, where k is even.

17. $\frac{dr}{dx} = \frac{d}{dx}\ln(\cos^{-1} x) = \frac{1}{\cos^{-1} x}\frac{d}{dx}\cos^{-1} x$

$= \frac{1}{\cos^{-1} x}\left(-\frac{1}{\sqrt{1-x^2}}\right) = -\frac{1}{\cos^{-1} x\sqrt{1-x^2}}$

18. $\frac{dr}{d\theta} = \frac{d}{d\theta}\log_2(\theta^2) = \frac{1}{\theta^2 \ln 2}\frac{d}{d\theta}(\theta^2) = \frac{2\theta}{\theta^2\ln 2} = \frac{2}{\theta \ln 2}$

19. $\frac{ds}{dt} = \frac{d}{dt}\log_5(t-7) = \frac{1}{(t-7)\ln 5}\frac{d}{dt}(t-7) = \frac{1}{(t-7)\ln 5}$, $t > 7$

20. $\frac{ds}{dt} = \frac{d}{dt}(8^{-t}) = 8^{-t}(\ln 8)\frac{d}{dt}(-t) = -8^{-t}\ln 8$

21. Use logarithmic differentiation.

$$y = x^{\ln x}$$
$$\ln y = \ln(x^{\ln x})$$
$$\ln y = (\ln x)(\ln x)$$
$$\frac{d}{dx}\ln y = \frac{d}{dx}(\ln x)^2$$
$$\frac{1}{y}\frac{dy}{dx} = 2\ln x \frac{d}{dx}\ln x$$
$$\frac{dy}{dx} = \frac{2y\ln x}{x}$$
$$\frac{dy}{dx} = \frac{2x^{\ln x}\ln x}{x}$$

22. $\frac{dy}{dx} = \frac{d}{dx}\frac{(2x)2^x}{\sqrt{x^2+1}}$

$$= \frac{\sqrt{x^2+1}\frac{d}{dx}[(2x)2^x] - (2x)(2^x)\frac{d}{dx}\sqrt{x^2+1}}{x^2+1}$$

$$= \frac{\sqrt{x^2+1}[(2x)(2^x)(\ln 2) + (2^x)(2)] - (2x)(2^x)\frac{1}{2\sqrt{x^2+1}}(2x)}{x^2+1}$$

$$= \frac{(x^2+1)(2^x)(2x\ln 2 + 2) - 2x^2(2^x)}{(x^2+1)^{3/2}}$$

$$= \frac{(2\cdot 2^x)[(x^2+1)(x\ln 2 + 1) - x^2]}{(x^2+1)^{3/2}}$$

$$= \frac{(2\cdot 2^x)(x^3\ln 2 + x^2 + x\ln 2 + 1 - x^2)}{(x^2+1)^{3/2}}$$

$$= \frac{(2\cdot 2^x)(x^3\ln 2 + x\ln 2 + 1)}{(x^2+1)^{3/2}}$$

Alternate solution, using logarithmic differentiation:

$$y = \frac{(2x)2^x}{\sqrt{x^2+1}}$$
$$\ln y = \ln(2x) + \ln(2^x) - \ln\sqrt{x^2+1}$$
$$\ln y = \ln 2 + \ln x + x\ln 2 - \frac{1}{2}\ln(x^2+1)$$
$$\frac{d}{dx}\ln y = \frac{d}{dx}[\ln 2 + \ln x + x\ln 2 - \frac{1}{2}\ln(x^2+1)]$$
$$\frac{1}{y}\frac{dy}{dx} = 0 + \frac{1}{x} + \ln 2 - \frac{1}{2}\frac{1}{x^2+1}(2x)$$
$$\frac{dy}{dx} = y\left(\frac{1}{x} + \ln 2 - \frac{x}{x^2+1}\right)$$
$$\frac{dy}{dx} = \frac{(2x)2^x}{\sqrt{x^2+1}}\left(\frac{1}{x} + \ln 2 - \frac{x}{x^2+1}\right)$$

23. $\frac{dy}{dx} = \frac{d}{dx}e^{\tan^{-1}x} = e^{\tan^{-1}x}\frac{d}{dx}\tan^{-1}x = \frac{e^{\tan^{-1}x}}{1+x^2}$

24. $\frac{dy}{du} = \frac{d}{du}\sin^{-1}\sqrt{1-u^2}$

$$= \frac{1}{\sqrt{1-(\sqrt{1-u^2})^2}}\frac{d}{du}\sqrt{1-u^2}$$

$$= \frac{1}{\sqrt{u^2}}\frac{1}{2\sqrt{1-u^2}}(-2u) = -\frac{u}{|u|\sqrt{1-u^2}}$$

25. $\frac{dy}{dt} = \frac{d}{dt}(t\sec^{-1}t - \frac{1}{2}\ln t)$

$$= (t)\left(\frac{1}{|t|\sqrt{t^2-1}}\right) + (\sec^{-1}t)(1) - \frac{1}{2t}$$

$$= \frac{t}{|t|\sqrt{t^2-1}} + \sec^{-1}t - \frac{1}{2t}$$

26. $\frac{dy}{dt} = \frac{d}{dt}[(1+t^2)\cot^{-1}2t]$

$$= (1+t^2)\left(-\frac{1}{1+(2t)^2}\right)(2) + (\cot^{-1}2t)(2t)$$

$$= -\frac{2+2t^2}{1+4t^2} + 2t\cot^{-1}2t$$

27. $\frac{dy}{dz} = \frac{d}{dz}(z\cos^{-1}z - \sqrt{1-z^2})$

$$= (z)\left(-\frac{1}{\sqrt{1-z^2}}\right) + (\cos^{-1}z)(1) - \frac{1}{2\sqrt{1-z^2}}(-2z)$$

$$= -\frac{z}{\sqrt{1-z^2}} + \cos^{-1}z + \frac{z}{\sqrt{1-z^2}}$$

$$= \cos^{-1}z$$

28. $\frac{dy}{dx} = \frac{d}{dx}(2\sqrt{x-1}\csc^{-1}\sqrt{x})$

$$= (2\sqrt{x-1})\left(-\frac{1}{|\sqrt{x}|\sqrt{(\sqrt{x})^2-1}}\right)\left(\frac{1}{2\sqrt{x}}\right)$$

$$+ (2\csc^{-1}\sqrt{x})\left(\frac{1}{2\sqrt{x-1}}\right)$$

$$= -\frac{\sqrt{x-1}}{(\sqrt{x})^2\sqrt{x-1}} + \frac{\csc^{-1}\sqrt{x}}{\sqrt{x-1}}$$

$$= -\frac{1}{x} + \frac{\csc^{-1}\sqrt{x}}{\sqrt{x-1}}$$

29. $\frac{dy}{dx} = \frac{d}{dx}\csc^{-1}(\sec x)$

$$= \left(-\frac{1}{|\sec x|\sqrt{\sec^2 x - 1}}\right)\frac{d}{dx}(\sec x)$$

$$= -\frac{1}{|\sec x|\sqrt{\tan^2 x}}\sec x \tan x$$

$$= -\frac{\sec x \tan x}{|\sec x \tan x|}$$

$$= -\frac{\frac{1}{\cos x}\frac{\sin x}{\cos x}}{\left|\frac{1}{\cos x}\frac{\sin x}{\cos x}\right|} = -\frac{\sin x}{|\sin x|}$$

$$= \begin{cases} -1, & 0 \le x < \pi, \quad x \ne \frac{\pi}{2} \\ 1, & \pi < x \le 2\pi, \quad x \ne \frac{3\pi}{2} \end{cases}$$

Alternate method:

On the domain $0 \le x \le 2\pi$, $x \ne \frac{\pi}{2}$, $x \ne \frac{3\pi}{2}$, we may rewrite the function as follows:

$$y = \csc^{-1}(\sec x)$$

$$= \frac{\pi}{2} - \sec^{-1}(\sec x)$$

$$= \frac{\pi}{2} - \cos^{-1}(\cos x)$$

$$= \begin{cases} \frac{\pi}{2} - x, & 0 \le x \le \pi, \quad x \ne \frac{\pi}{2} \\ \frac{\pi}{2} - (\pi - x), & \pi < x \le 2\pi, \quad x \ne \frac{3\pi}{2} \end{cases}$$

$$= \begin{cases} \frac{\pi}{2} - x, & 0 \le x \le \pi, \quad x \ne \frac{\pi}{2} \\ -\frac{\pi}{2} + x, & \pi < x \le 2\pi, \quad x \ne \frac{3\pi}{2} \end{cases}$$

Therefore, $\frac{dy}{dx} = \begin{cases} -1, & 0 \le x < \pi, \quad x \ne \frac{\pi}{2} \\ 1, & \pi < x \le 2\pi, \quad x \ne \frac{3\pi}{2} \end{cases}$

Note that the derivative exists at 0 and 2π only because these are the endpoints of the given domain; the two-sided derivative of $y = \csc^{-1}(\sec x)$ does not exist at these points.

30. $\frac{dr}{d\theta} = \frac{d}{d\theta}\left(\frac{1 + \sin\theta}{1 - \cos\theta}\right)^2$

$$= 2\left(\frac{1 + \sin\theta}{1 - \cos\theta}\right)\left(\frac{(1 - \cos\theta)(\cos\theta) - (1 + \sin\theta)(\sin\theta)}{(1 - \cos\theta)^2}\right)$$

$$= 2\left(\frac{1 + \sin\theta}{1 - \cos\theta}\right)\left(\frac{\cos\theta - \cos^2\theta - \sin\theta - \sin^2\theta}{(1 - \cos\theta)^2}\right)$$

$$= 2\left(\frac{1 + \sin\theta}{1 - \cos\theta}\right)\left(\frac{\cos\theta - \sin\theta - 1}{(1 - \cos\theta)^2}\right)$$

31. Since $y = \ln x^2$ is defined for all $x \ne 0$ and $\frac{dy}{dx} = \frac{1}{x^2}\frac{d}{dx}(x^2) = \frac{2x}{x^2} = \frac{2}{x}$, the function is differentiable for all $x \ne 0$.

32. Since $y = \sin x - x\cos x$ is defined for all real x and $\frac{dy}{dx} = \cos x - (x)(-\sin x) - (\cos x)(1) = x\sin x$, the function is differentiable for all real x.

33. Since $y = \sqrt{\frac{1 - x}{1 + x^2}}$ is defined for all $x < 1$ and

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\frac{1 - x}{1 + x^2}}}\frac{(1 + x^2)(-1) - (1 - x)(2x)}{(1 + x^2)^2}$$

$$= \frac{x^2 - 2x - 1}{2\sqrt{1 - x}\,(1 + x^2)^{3/2}},$$ which is defined only for $x < 1$, the function is differentiable for all $x < 1$.

34. Since $y = (2x - 7)^{-1}(x + 5) = \frac{x + 5}{2x - 7}$ is defined for all $x \ne \frac{7}{2}$ and $\frac{dy}{dx} = \frac{(2x - 7)(1) - (x + 5)(2)}{(2x - 7)^2} = -\frac{17}{(2x - 7)^2}$, the function is differentiable for all $x \ne \frac{7}{2}$.

35. Use implicit differentiation.

$$xy + 2x + 3y = 1$$

$$\frac{d}{dx}(xy) + \frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}(1)$$

$$x\frac{dy}{dx} + (y)(1) + 2 + 3\frac{dy}{dx} = 0$$

$$(x + 3)\frac{dy}{dx} = -(y + 2)$$

$$\frac{dy}{dx} = -\frac{y + 2}{x + 3}$$

36. Use implicit differentiation.

$$5x^{4/5} + 10y^{6/5} = 15$$

$$\frac{d}{dx}(5x^{4/5}) + \frac{d}{dx}(10y^{6/5}) = \frac{d}{dx}(15)$$

$$4x^{-1/5} + 12y^{1/5}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{4x^{-1/5}}{12y^{1/5}} = -\frac{1}{3(xy)^{1/5}}$$

37. Use implicit differentiation.

$$\sqrt{xy} = 1$$

$$\frac{d}{dx}\sqrt{xy} = \frac{d}{dx}(1)$$

$$\frac{1}{2\sqrt{xy}}\left[x\frac{dy}{dx} + (y)(1)\right] = 0$$

$$x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

Alternate method: