Find AB.

Enter: [MATRX] 1 [MATRX] 2 [ENTER]

[[-8 -21 90]
[-3 -6 20]]

Find A^2.

Enter: [MATRX] 1 [x^2] [ENTER]

[[43 56]
[8 10]]

Find $B + AB$.

Enter: [MATRX] 2 [+] [MATRX] 1 [MATRX] 2 [ENTER]

[[-7 -21 98]
[-5 -9 26]]

Find the determinant of A. *The determinant of A is denoted det A.*

Enter: [MATRX] [▶] 1 [MATRX] 1 [ENTER]

5

Find the inverse of A. *The inverse of A is denoted A^{-1}.*

Enter: [MATRX] 1 [x^{-1}] [ENTER]

[[.4 -1.4]
[-.2 1.2]]

EXERCISES

Enter the matrices below into a graphing calculator. Then find each of the following.

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & -1 \\ 4 & 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 6 & -2 & 5 \\ 0 & 7 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 4 \\ -3 & 6 \\ 7 & -2 \end{bmatrix}$$

1. $-C$
2. $4B$
3. $\det A$
4. $-2A$
5. A^{-1}
6. CB
7. BC
8. $\det BC$
9. BA
10. $CB - A$
11. $\det CB$
12. $A + CB$
13. $(BC)^{-1}$
14. A^2
15. $(BC)^2$
16. $B + BA$
17. BAC
18. CBA

4–1 An Introduction to Matrices

What YOU'LL LEARN

- To perform scalar multiplication on a matrix,
- to solve matrices for variables, and
- to solve problems using matrix logic.

Why IT'S IMPORTANT

You can use matrices to make decisions and solve many types of problems.

The plural of matrix is matrices.

Real World APPLICATION

Decision Making

Emilio has been accepted at three colleges in Ohio: Denison University, Marietta College, and Muskingum College. He and his parents are trying to make a final decision based on cost, distance from home, campus life, and educational quality. Emilio rates each criteria on a scale from 1 (least favorable) to 10 (most favorable) and organizes the information in a **matrix** like the one shown below. A matrix is a rectangular array of variables or constants in horizontal rows and vertical columns, usually enclosed in brackets.

Matrices are often used as problem-solving tools.

$$\begin{array}{l} \\ \text{Denison} \\ \text{Marietta} \\ \text{Muskingum} \end{array} \begin{array}{c} \begin{array}{cccc} \text{cost} & \text{distance} & \text{campus} & \text{quality} \end{array} \\ \begin{bmatrix} 5 & 6 & 6 & 8 \\ 6 & 6 & 8 & 9 \\ 5 & 6 & 7 & 7 \end{bmatrix} \end{array}$$

When the information is shown in a matrix, it is easy to see that distances from home are not a useful criteria, because each college received the same score. You can also see that all of the entries in the second row are greater than the entries in either the first or third row. Based on these criteria, Emilio should attend Marietta College.

In a matrix, numbers or data are organized so that each position in the matrix has a purpose. Each value in the matrix is called an **element**.

$$C = \underbrace{\begin{bmatrix} 5 & 6 & 6 & 8 \\ 6 & 6 & 8 & 9 \\ 5 & 6 & 7 & 7 \end{bmatrix}}_{\text{4 columns}} \left.\vphantom{\begin{bmatrix} 5 \\ 6 \\ 5 \end{bmatrix}}\right\} \text{3 rows}$$

The element 9 is in row 2, column 4.

A matrix that has only one row is called a row matrix. A matrix that has only one column is called a column matrix.

A matrix is usually named using an uppercase letter, as in matrix C on the previous page. A matrix can also be named by using the matrix **dimensions** with the letter name. The dimensions tell how many rows and columns, in that order, are in the matrix. The matrix above would be named $C_{3\times4}$ since it has 3 rows and 4 columns.

Many problems can be solved using a method sometimes referred to as **matrix logic**. When you use matrix logic, you create a matrix that helps you organize all the information in the problem. By using the matrix, you can eliminate one possibility after another until you eventually arrive at a solution.

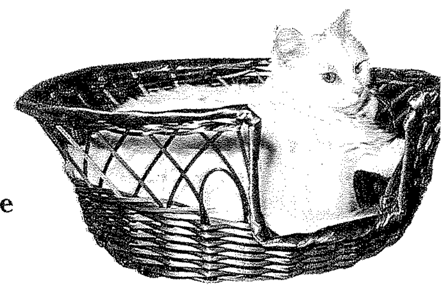
Example 1

PROBLEM SOLVING

Use Matrix Logic

Miko, Amanda, Latisha, and Tara are friends, and each has one of these pets: dog, cat, parrot, and gerbil. Use these clues to match each girl with her pet.

- **Latisha likes to visit the friend with the gerbil.**
- **Tara and Amanda frequently help their friend walk her dog.**
- **Miko cannot have a dog or a cat because she is allergic to them.**
- **Tara plans to teach her pet how to talk.**



Explore There are 4 girls and 4 pets. You must match each girl with her pet by using the information from the statements above.

Plan Make a 4×4 matrix to organize the information. Through the process of elimination, each girl can be matched with her pet.

A matrix that has the same number of rows and columns is called a square matrix.

Solve Put an $\times$ in the first row under gerbil to show that Latisha does not have the gerbil. Put two $\times$s to show that Tara and Amanda do not own the dog. Put two $\times$s to show that Miko cannot have a cat or dog. By the process of elimination, Latisha owns the dog. Put a circle in this box. Since only one girl owns the dog, put $\times$s in the rest of the boxes in that row. Continue to eliminate possibilities in this manner.

	dog	cat	parrot	gerbil
Latisha	O	×	×	×
Tara	×	×	O	×
Amanda	×	O	×	×
Miko	×	×	×	O

Miko has the gerbil, Amanda has the cat, Latisha has the dog, and Tara has the parrot.

Examine Check the result against the statements. The first statement says that Latisha likes to visit the girl with the gerbil, and the answer says that Miko has the gerbil. There is no conflict here. Using the same method for each sentence, you can see that there are no conflicts.

LOOK BACK

You can refer to Lesson 2-1 for information on the graphs of continuous and discrete functions.

Although matrices are sometimes used as a problem-solving tool, their importance extends to another branch of mathematics called **discrete mathematics**. Discrete mathematics deals with finite or discontinuous quantities. The distinction between continuous and discrete quantities is one that you have encountered before. Think of a staircase. You can slide your hand up the banister, but you have to climb the steps one by one. The banister represents a continuous quantity, like a linear function. However, each step represents a discrete quantity, like a point on a scatter plot or an element of a matrix.

Just as algebraic rules exist for functions, matrices have special algebraic rules. For example, you can multiply any matrix by a constant called a **scalar**. This is called **scalar multiplication**. When scalar multiplication is performed, each element is multiplied by that constant, and a new matrix is formed.

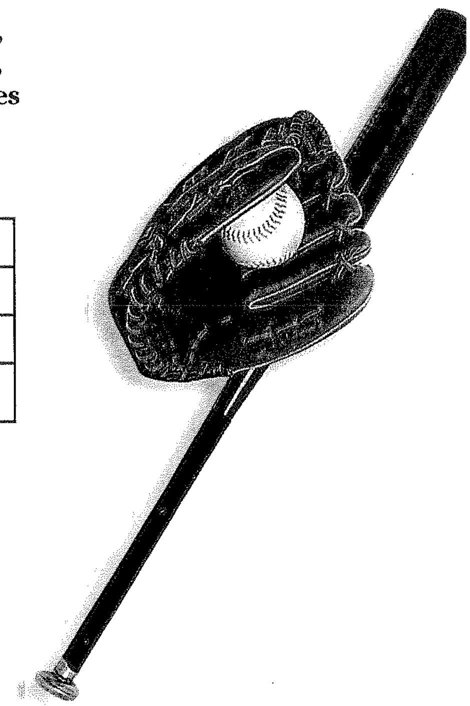
Scalar Multiplication of a Matrix	$k\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} ka & kb & kc \\ kd & ke & kf \end{bmatrix}$

Example 2

Real World APPLICATION

Business

The manager of Just Sports keeps track of monthly sales on a spreadsheet. The spreadsheet below shows the number of baseball and softball bats, balls, shoes, and gloves sold last May. This May, the store is going to have a promotion and hopes to increase sales by 8%. Write a matrix that shows the store's sales goals for this May.



	A	B	C	D	E
1		**bats**	**balls**	**shoes**	**gloves**
2	**baseball**	38	29	18	43
3	**softball**	42	25	16	51

First, write the matrix for last May.

$$\begin{bmatrix} 38 & 29 & 18 & 43 \\ 42 & 25 & 16 & 51 \end{bmatrix}$$

Multiply the matrix by 1.08 to show an increase of 8% for this May.

$$1.08\begin{bmatrix} 38 & 29 & 18 & 43 \\ 42 & 25 & 16 & 51 \end{bmatrix} = \begin{bmatrix} 41.04 & 31.32 & 19.44 & 46.44 \\ 45.36 & 27.00 & 17.28 & 55.08 \end{bmatrix}$$

TECHNOLOGY *Tips*

If you are using a graphing calculator, enter the sales matrix into matrix *A*. Then find 1.08*A*. You may need to use the arrow keys to see the entire matrix.

Two matrices are considered to be **equal** if they have the same dimensions and if each element of one matrix is equal to the corresponding element of the other matrix.

$$\begin{bmatrix} 2 & 5 & 4 \\ 8 & 6 & 1 \end{bmatrix} \neq \begin{bmatrix} 2 & 8 \\ 5 & 6 \\ 4 & 1 \end{bmatrix} \qquad \begin{bmatrix} 4 & 16 \\ 5 & 3 \end{bmatrix} \neq \begin{bmatrix} 4 & 16 \\ 5 & 3 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 3 \\ 9 & 1 \end{bmatrix} \neq \begin{bmatrix} 8 & 9 \\ 3 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 12 \\ 20 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 12 \\ 20 & 8 \end{bmatrix}$$

The definition of equal matrices can be used to find values when elements of the matrices are algebraic expressions.

Example 3 **Solve** $\begin{bmatrix} 2x \\ 2x + 3y \end{bmatrix} = \begin{bmatrix} y \\ 12 \end{bmatrix}$ **for x and y.**

Since the matrices are equal, the corresponding elements are equal. When you write the sentences that show this equality, two linear equations are formed.

$$2x = y$$
$$2x + 3y = 12$$

The first equation gives you a value for y that can be substituted into the second equation. Then you can find a value for x.

$$\begin{aligned} 2x + 3y &= 12 \\ 2x + 3(2x) &= 12 && \textit{Replace } y \textit{ with } 2x. \\ 2x + 6x &= 12 && \textit{Simplify.} \\ 8x &= 12 && \textit{Combine like terms.} \\ x &= 1.5 && \textit{Divide each side by 8.} \end{aligned}$$

To find a value for y, you can substitute 1.5 into either equation.

$$\begin{aligned} 2x &= y \\ 2(1.5) &= y && \textit{Replace } x \textit{ with 1.5.} \\ 3 &= y \end{aligned}$$

The solution is (1.5, 3).

Check your solution by substituting the values into the equation you did *not* use to find y.

$$\begin{aligned} 2x + 3y &= 12 \\ 2(1.5) + 3(3) &\stackrel{?}{=} 12 && \textit{Replace } x \textit{ with 1.5 and } y \textit{ with 3.} \\ 12 &= 12 \quad \checkmark \end{aligned}$$

LOOK BACK

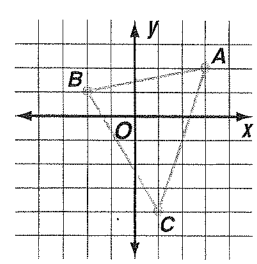
You can refer to Lesson 3-2 for information on using substitution to solve systems of equations.

A matrix containing coordinates of a geometric figure is often called a coordinate matrix.

Matrices are an important tool for integrating algebra and geometry because points and polygons can be represented by matrices. The ordered pair (x, y) is usually represented by the column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$, where the x-coordinate is in row 1, and the y-coordinate is in row 2. Similarly, polygons can be represented by grouping all of the column matrices of the coordinates of the vertices into one matrix.

coordinates of vertices

$$\triangle ABC = \begin{bmatrix} 3 & -2 & 1 \\ 2 & 1 & -4 \end{bmatrix} \begin{array}{l} \leftarrow x\textit{-coordinate} \\ \leftarrow y\textit{-coordinate} \end{array}$$



One of the ways that matrices help connect algebra and geometry is through **transformations.** Transformations are functions that map points of a shape onto its image. When a geometric figure is enlarged or reduced, this transformation is called a **dilation.** When the size of a figure changes, all linear measures of its image change in the same ratio. For example, if the perimeter of a figure triples, the length of each side of the figure also triples.

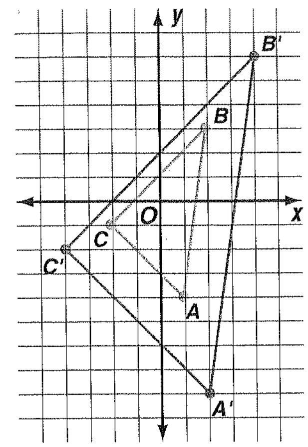
Example 4 INTEGRATION Geometry

$\triangle ABC$ has vertices $A(1, -4)$, $B(2, 3)$, and $C(-2, -1)$. Enlarge $\triangle ABC$ so that its perimeter is twice the original perimeter. What are the coordinates of the vertices of $\triangle A'B'C'$?

Graph $\triangle ABC$. Since perimeter is a linear measurement, multiply the coordinate matrix by the scalar 2.

$$2\begin{bmatrix} 1 & 2 & -2 \\ -4 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -4 \\ -8 & 6 & -2 \end{bmatrix}$$

The coordinates of the vertices of $\triangle A'B'C'$ are $A'(2, -8)$, $B'(4, 6)$, and $C'(-4, -2)$. Graph $\triangle A'B'C'$.



You can measure to verify that the perimeter of $\triangle A'B'C'$ is twice the original perimeter.

CHECK FOR UNDERSTANDING

Communicating Mathematics

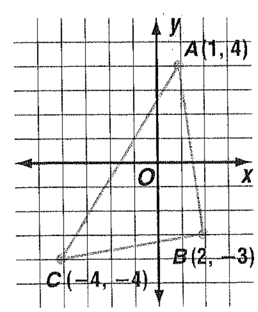
Study the lesson. Then complete the following.

1. **Define** a matrix in your own words.
2. **Find** an example of a matrix in a newspaper and name it using its dimensions.
3. **Choose** the matrix that represents the ordered pair $(-1, 3)$.

   a. $[-1, 3]$ b. $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

   c. $[3, -1]$ d. $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

4. **Write** a coordinate matrix for the triangle shown at the right.
5. **Explain** the meaning of *dilation.*



MATH JOURNAL

6. **Draw** a figure on a coordinate plane and write a coordinate matrix for its vertices. Explain what happens to the figure when the matrix is multiplied by a number greater than 1. Explain what happens when the matrix is multiplied by a number between 0 and 1. Use drawings to justify your answers.

Guided Practice

Perform the indicated operation.

7. $-2\begin{bmatrix} 7 & 3 & -1 \end{bmatrix}$

8. $4\begin{bmatrix} -1 & 0 \\ 3 & -2 \end{bmatrix}$

Solve for the variables.

9. $\begin{bmatrix} 2x & 3 & 3z \end{bmatrix} = \begin{bmatrix} 5 & 3y & 9 \end{bmatrix}$

10. $\begin{bmatrix} 6x \\ y \end{bmatrix} = \begin{bmatrix} 62 + 8y \\ 6 - 2x \end{bmatrix}$

11. **Business** On Monday, the Main Street Deli sold the following number of sandwiches: 15 turkey, 12 turkey and cheese, 8 ham, 10 ham and cheese, 8 roast beef, 11 roast beef and cheese. Organize the information into a 3×2 matrix.

12. **Geometry** Triangle ABC with $A(4, 5)$, $B(-3, -2)$, and $C(1, -4)$ is reduced so that its perimeter is one-half the original perimeter.
    a. Write the coordinate matrix for ΔABC.
    b. Write the coordinates of $\Delta A'B'C'$ in matrix form.
    c. Graph this situation.

EXERCISES

Practice

Perform the indicated operation.

13. $3\begin{bmatrix} 5 & -2 & 7 \\ -3 & 8 & 4 \end{bmatrix}$

14. $-2\begin{bmatrix} 6 & -4 \\ -2 & 4 \end{bmatrix}$

15. $\frac{1}{3}\begin{bmatrix} 6 & -5 \end{bmatrix}$

16. $0.2\begin{bmatrix} 10.50 \\ 8.75 \end{bmatrix}$

17. $-5\begin{bmatrix} 1.3 & 0 & 5.1 \\ 0.4 & 1.0 & 2.5 \end{bmatrix}$

18. $-0.3\begin{bmatrix} 8.95 & 7.50 \end{bmatrix}$

Solve for the variables.

19. $\begin{bmatrix} 4x & 3y \end{bmatrix} = \begin{bmatrix} 12 & -1 \end{bmatrix}$

20. $\begin{bmatrix} 2x + y \\ x - 3y \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \end{bmatrix}$

21. $x\begin{bmatrix} 4 & y \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 12 & -15 \\ 21 & z \end{bmatrix}$

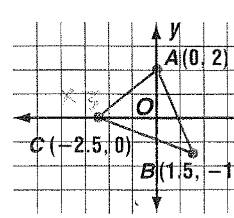
22. $4\begin{bmatrix} x & y - 1 \\ 3 & z \end{bmatrix} = \begin{bmatrix} 20 & 8 \\ 6z & x + y \end{bmatrix}$

23. $\begin{bmatrix} x^2 & 7 & 9 \\ 5 & 12 & 6 \end{bmatrix} = \begin{bmatrix} 25 & 7 & y \\ 5 & 2z & 6 \end{bmatrix}$

24. $\begin{bmatrix} x + 3y \\ 3x + y \end{bmatrix} = \begin{bmatrix} -13 \\ 1 \end{bmatrix}$

25. **Geometry** The vertex of the right angle of a right triangle is located at the origin with its other vertices at (0, 12) and (5, 0). Find the coordinates of the vertices of a similar triangle whose perimeter is one-fourth that of the original triangle.

26. **Geometry** Enlarge ΔABC shown at the right so that the resulting perimeter is three times the original perimeter.
    a. Graph ΔABC and $\Delta A'B'C'$.
    b. Write the coordinates of $\Delta A'B'C'$ in matrix form.



27. **Geometry** The coordinate matrix for ΔXYZ is $\begin{bmatrix} -2 & 4 & -1 \\ -1 & 2 & 3 \end{bmatrix}$. Explain what happens to the triangle when the matrix is multiplied by -0.5. Make a drawing to justify your answer.

Solve for the variables.

28. $\begin{bmatrix} r^2 - 24 & 17 \\ 7 & t^3 \end{bmatrix} = \begin{bmatrix} 1 & 2y + 3 \\ z^2 - 12 & 27 \end{bmatrix}$

29. $\begin{bmatrix} 5x - 7 & 11 \\ 5 & 23 \end{bmatrix} = \begin{bmatrix} 8 & 21 - m \\ r^3 - 3 & 4y + x \end{bmatrix}$

30. $\begin{bmatrix} 13 - 7y & a \\ 1 & 2b - 38 \end{bmatrix} = \begin{bmatrix} 5x & 2 - 6b \\ 2x + 3y & 5a \end{bmatrix}$

Critical Thinking

31. When the size of a figure changes, all linear measures of its image, such as the perimeter, change in the same ratio. Is it also true that the area of the figure changes in the same ratio? Justify your answer with matrices and a graph.

Applications and Problem Solving

32. **Use Matrix Logic** Fred, Ted, and Ed are taking Mary, Carrie, and Terri to the homecoming dance. Use these clues to find which couples will be attending the dance.

Mary is Ed's sister and lives on Fifth Avenue.

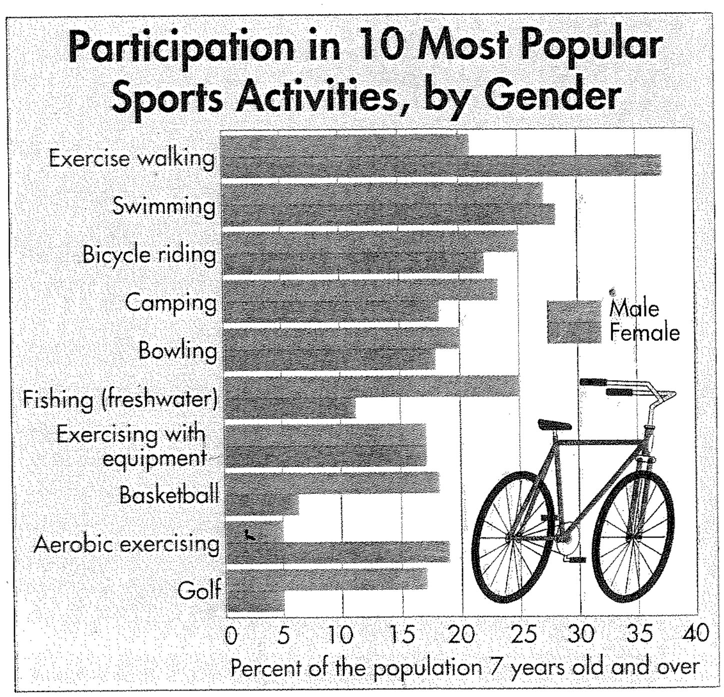
Ted drives a car to school each day.

Ed is taller than Terri's date.

Carrie and her date ride their bicycles to school every day.

Fred's date lives on State Street.

33. **Sports** The graph shows the percent of the U.S. population that participates in the ten most popular sporting activities for a recent year. Estimate the percents from the graph and organize the information in a matrix.



Source: National Sporting Goods Association

Mixed Review

34. Find the x-, y-, and z-intercepts for $3x + 6y - 8z = 24$. (Lesson 3–7)

35. Theater The Woodward Park High School auditorium seats 150 people. Admission to the spring play is \$2.00 for adults and \$1.00 for students. The Drama Club has already sold fifty student tickets and the rest are to be sold at the door. How many of each type of ticket should be sold for the Drama Club to earn the maximum amount of money? (Lesson 3–6)

36. ACT Practice A line represented by the equation $x = 4$ passes through points $A(4, 5)$ and $B(x, y)$. What is the slope of the line passing through A and B?

A 0 **B** 4 **C** s **D** undefined

E It cannot be determined from the information given.

37. Find the value of $f(-2)$, if $f(x) = x^2 - 4$. (Lesson 2–1)

38. Solve $|9 - 3t| > 5$. (Lesson 1–7)

39. Statistics The table lists the nine cities in the United States with the fewest average rainy days per year. (Lesson 1–3)

a. Find the median, mode, and mean of the average number of rainy days.

b. What do you notice about the locations of these cities?

City	Days
San Diego, CA	42
Bakersfield, CA	37
Los Angeles, CA	35
Long Beach, CA	32
Santa Barbara, CA	30
Bishop, CA	29
Las Vegas, NV	26
Phoenix, AZ	26
Yuma, AZ	17

For Extra Practice, see page 883.

WORKING ON THE In·ves·ti·ga·tion

Refer to the Investigation on pages 180–181.

3-2-1-Blast-Off!

Using the design of your launcher, conduct tests by shooting the table tennis ball various distances. During these tests, the launcher should be operated by one to four people in the group, the launcher should be shot from the floor, and the distance should be measured from the launcher's location to the spot where the table tennis ball hits the floor.

Also, use this testing time to calibrate your launcher to hit different distances in your firing range. In other words, determine the launch settings for several distances.

1. List your test results.
2. Describe your calibration system.
3. What is the process you used to hit a target at 50 cm, 100 cm, 150 cm, 200 cm, or 250 cm?
4. How accurate have your tests been? Explain.
5. List your launch settings for at least six target distances between 50 cm and 250 cm.
6. Create a table of launch settings and target distances. How can this data be put into a matrix? What would be the dimensions of the matrix? Create this matrix.
7. Draw a scatter plot of the relationship between the launch settings and the target distances. Describe the graph.
8. Determine a mathematical relationship between the two values.

Add the results of your work to your Investigation Folder.

4-2 Adding and Subtracting Matrices

What YOU'LL LEARN

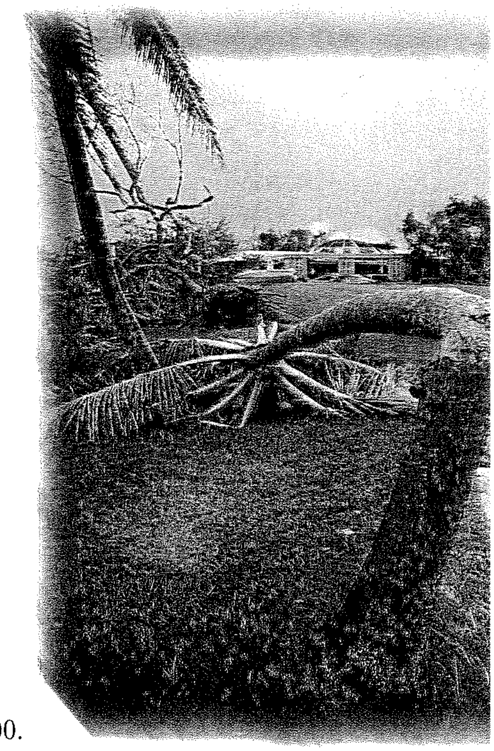
- To add and subtract matrices.

Why IT'S IMPORTANT

You can use matrices to solve problems involving meteorology, geography, and recreation.

Real World APPLICATION

Meteorology



At the turn of the century, weather forecasters were unable to give residents of coastal areas much warning of an approaching hurricane. On September 8, 1900, Isaac Cline, the head of the Galveston Weather Bureau in Texas, rode a horse along the beach front, urging people to evacuate as an unnamed hurricane approached. Even so, nearly 6000 people lost their lives later that day due to the flooding caused by the storm surge. In contrast, when hurricane Andrew hit the Florida and Louisiana coasts in 1992, only 23 deaths were attributed to it—most likely because of better forecasting and evacuation planning than in 1900.

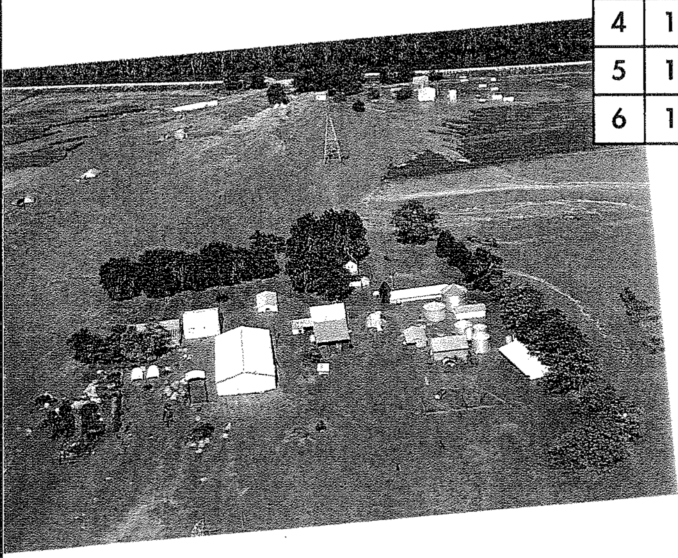
FYI

About 60% of flood victims die in their cars, trying to drive through water flowing across a road.

Other types of severe weather give little notice. Between 1940 and 1990, 21,447 people have died in the United States due to lightning, tornadoes, floods, and hurricanes. The data for each decade since the 1940s are shown in the spreadsheet below.

	A	B	C	D	E
1		lightning	tornadoes	floods	hurricanes
2	1940s	3293	1788	619	216
3	1950s	1841	1409	791	877
4	1960s	1332	935	1297	587
5	1970s	978	986	1819	217
6	1980s	726	521	1097	118

Source: National Weather Service



The information in the spreadsheet can also be represented by column matrices, with one for each type of severe weather. To find the total number of deaths due to severe weather, add the corresponding elements.

$$\begin{bmatrix} 3293 \\ 1841 \\ 1332 \\ 978 \\ 726 \end{bmatrix} + \begin{bmatrix} 1788 \\ 1409 \\ 935 \\ 986 \\ 521 \end{bmatrix} + \begin{bmatrix} 619 \\ 791 \\ 1297 \\ 1819 \\ 1097 \end{bmatrix} + \begin{bmatrix} 216 \\ 877 \\ 587 \\ 217 \\ 118 \end{bmatrix} = \begin{bmatrix} 5916 \\ 4918 \\ 4151 \\ 4000 \\ 2462 \end{bmatrix} \begin{array}{l} \leftarrow \textit{1940s} \\ \leftarrow \textit{1950s} \\ \leftarrow \textit{1960s} \\ \leftarrow \textit{1970s} \\ \leftarrow \textit{1980s} \end{array}$$

Since 1940, the following number of deaths occurred due to lightning, tornadoes, floods, or hurricanes: 1940s, 5916; 1950s, 4918; 1960s, 4151; 1970s, 4000; 1980s, 2462.

This example illustrates that in order to add matrices, they must have the same dimensions.

Addition of Matrices	**If A and B are two $m \times n$ matrices, then $A + B$ is an $m \times n$ matrix in which each element is the sum of the corresponding elements of A and B.** $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} a+j & b+k & c+l \\ d+m & e+n & f+o \\ g+p & h+q & i+r \end{bmatrix}$

Similarly, it is possible to subtract matrices.

Example 1

Real World APPLICATION

Recreation

The matrices below show sporting goods sales in the United States in millions of dollars for 1997 and 1998. By how many dollars did each category change from 1997 to 1998?

$$A = \begin{array}{l} \\ \text{athletic clothing} \\ \text{athletic shoes} \\ \text{athletic equipment} \\ \text{recreational vehicles} \end{array} \begin{array}{c} \textbf{1997} \\ \begin{bmatrix} 18{,}063 \\ 9485 \\ 17{,}244 \\ 14{,}233 \end{bmatrix} \end{array} \qquad B = \begin{array}{c} \textbf{1998} \\ \begin{bmatrix} 19{,}490 \\ 8735 \\ 17{,}470 \\ 15{,}495 \end{bmatrix} \end{array}$$

To determine the change in each category, find $B - A$.

$$B - A = \begin{bmatrix} 19{,}490 \\ 8735 \\ 17{,}470 \\ 15{,}495 \end{bmatrix} - \begin{bmatrix} 18{,}063 \\ 9485 \\ 17{,}244 \\ 14{,}233 \end{bmatrix} = \begin{bmatrix} 1427 \\ -750 \\ 226 \\ 1262 \end{bmatrix}$$

From 1997 to 1998, sales of athletic clothing increased \$1427 million, athletic shoes decreased \$750 million, athletic equipment increased \$226 million, and recreational vehicles increased \$1262 million.

interNET CONNECTION

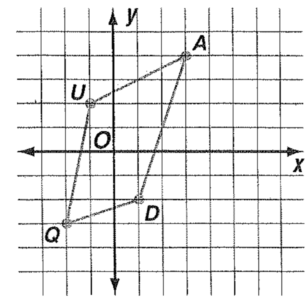
Data Update For more information on sporting goods sales, visit: **www.algebra2.glencoe.com**

In Lesson 4–1, you used matrices to represent polygons and their dilation images. Another type of transformation is a **translation.** A translation occurs when a figure is moved from one location to another on the coordinate plane without changing its size, shape, or orientation. You can use matrix addition to find the coordinates of translated figures.

Example 2 INTEGRATION **Geometry**

Find the coordinates of the vertices of quadrilateral *QUAD* with $Q(-2, -3)$, $U(-1, 2)$, $A(3, 4)$, and $D(1, -2)$ if it is moved 3 units to the right and 1 unit down.

Write the coordinates of quadrilateral *QUAD* as a coordinate matrix.



$$\begin{bmatrix} -2 & -1 & 3 & 1 \\ -3 & 2 & 4 & -2 \end{bmatrix}$$

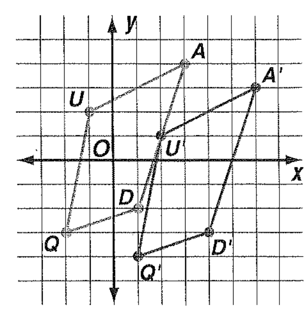
To translate the quadrilateral 3 units to the right means that each x-coordinate increases by 3. Translating 1 unit down means that each y-coordinate decreases by 1.

The matrix, called the **translation matrix,** that will increase each x-coordinate by 3 and decrease each y-coordinate by 1 is $\begin{bmatrix} 3 & 3 & 3 & 3 \\ -1 & -1 & -1 & -1 \end{bmatrix}$.

To find the coordinates of the vertices of the translated quadrilateral $Q'U'A'D'$, add the translation matrix to the coordinate matrix of *QUAD*.

$$\begin{bmatrix} -2 & -1 & 3 & 1 \\ -3 & 2 & 4 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 3 & 3 & 3 \\ -1 & -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 6 & 4 \\ -4 & 1 & 3 & -3 \end{bmatrix}$$

The coordinates of the vertices of $Q'U'A'D'$ are $Q'(1, -4)$, $U'(2,1)$, $A'(6, 3)$, and $D'(4, -3)$.



Graph the coordinates of $Q'U'A'D'$ to check the accuracy of your coordinates. The two quadrilaterals have the same size and shape. $Q'U'A'D'$ has moved to the right 3 units and 1 unit down from *QUAD*.

CHECK FOR UNDERSTANDING

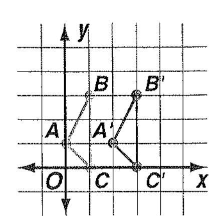
Communicating Mathematics

Study the lesson. Then complete the following.

1. **Explain** the conditions under which matrices can be added.

2. **Illustrate** the difference between a dilation and a translation.

3. **Write** the translation matrix for ΔABC and its image $\Delta A'B'C'$ shown at the right.



MATH JOURNAL

4. **Write** a convincing argument for the statement *Matrix addition is commutative and associative*. If the statement is not true, find a counterexample.

Guided Practice

Perform the indicated operations.

5. $\begin{bmatrix} 3 & 7 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 5 & -4 \end{bmatrix}$

6. $\begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 6 \\ -5 \\ 8 \end{bmatrix}$

7. $2[3 \;\; -1] + 3[5 \;\; 0]$

8. **Meteorology** Refer to the application at the beginning of the lesson. For each decade since 1940, how many more people died as a result of lightning than hurricanes?

9. **Geometry** Triangle ABC with vertices $A(-2, 2)$, $B(3, 5)$, and $C(5, -2)$ is translated so that A' is at $(1, -5)$.
   a. Draw a graph of this situation.
   b. Find the translation matrix.
   c. Write the coordinates of $A'B'C'$ in matrix form.

EXERCISES

Practice

Perform the indicated operations.

10. $\begin{bmatrix} 3 & -9 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} -8 & -4 \\ 3 & 10 \end{bmatrix}$

11. $[5 \;\; 8 \;\; -4] + [-1 \;\; 12 \;\; 5]$

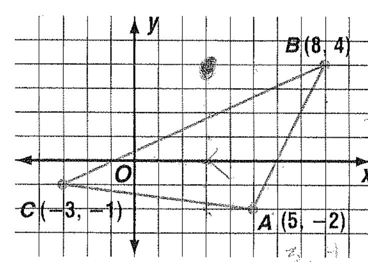
12. $4\begin{bmatrix} 2 & 7 \\ -3 & 6 \end{bmatrix} + 5\begin{bmatrix} -6 & -4 \\ 3 & 0 \end{bmatrix}$

13. $\frac{1}{2}\begin{bmatrix} 4 & 6 \\ 3 & 0 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 9 & 27 \\ 0 & 3 \end{bmatrix}$

14. $5\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} + 6\begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix} + 2\begin{bmatrix} -3 \\ 8 \\ -4 \end{bmatrix}$

15. $2\begin{bmatrix} -2 & 4 \\ 1 & -1 \\ 3 & 0 \end{bmatrix} - 3\begin{bmatrix} 5 & 3 \\ -3 & 2 \\ 8 & -9 \end{bmatrix} + \begin{bmatrix} 0 & -5 \\ 9 & -3 \\ -2 & 7 \end{bmatrix}$

16. Translate ΔABC shown at the right so that A' is at $(3, 4)$.
    a. Graph ΔABC and $\Delta A'B'C'$.
    b. Write the coordinates of $\Delta A'B'C'$ in matrix form.



17. Quadrilateral $BURT$ has vertices $B(6, 1)$, $U(3, 5)$, $R(-1, 4)$, and $T(-3, -5)$.
    a. What translation matrix would you need to use to translate $BURT$ so that R' has coordinates $(3, 2)$? 3, 2
    b. Use your translation matrix to find the coordinates of B', T', and U'.

18. Dilate and then translate ΔXYZ with vertices $X(-6, 2)$, $Y(-2, 8)$, and $Z(4, -5)$ so that X' has coordinates $(2, 2)$ and the perimeter of $\Delta X'Y'Z'$ is one-half the perimeter of ΔXYZ. State the coordinates of Y' and Z'.

19. Solve for the variables.

$$\begin{bmatrix} x \\ 7z \\ 2y \end{bmatrix} - \begin{bmatrix} 4z \\ -3y \\ 3x \end{bmatrix} + \begin{bmatrix} -2y \\ 2x \\ -5z \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \\ 18 \end{bmatrix}$$

Critical Thinking

20. **Geometry** Find the coordinates of the vertices of quadrilateral $MNPQ$ that is a translation of quadrilateral $XYZW$ whose vertices are $X(5, -3)$, $Y(2, 7)$, $Z(-3, 3)$, and $W(-5, 1)$, if M is located at the origin.

Applications and Problem Solving

Real World

21. **Geography** The matrices below represent the number of births and deaths in seven Atlantic seaboard states in a recent year.

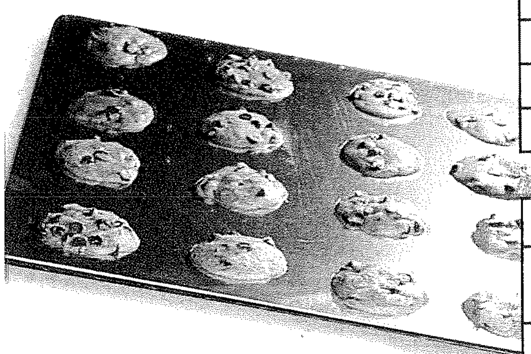
$$B = \begin{array}{l} \text{Delaware} \\ \text{Maryland} \\ \text{Virginia} \\ \text{North Carolina} \\ \text{South Carolina} \\ \text{Georgia} \\ \text{Florida} \end{array} \overset{\textbf{Births}}{\begin{bmatrix} 10{,}902 \\ 76{,}173 \\ 97{,}600 \\ 103{,}047 \\ 56{,}635 \\ 111{,}397 \\ 192{,}291 \end{bmatrix}} \qquad D = \begin{array}{l} \text{Delaware} \\ \text{Maryland} \\ \text{Virginia} \\ \text{North Carolina} \\ \text{South Carolina} \\ \text{Georgia} \\ \text{Florida} \end{array} \overset{\textbf{Deaths}}{\begin{bmatrix} 5937 \\ 37{,}806 \\ 49{,}541 \\ 59{,}478 \\ 30{,}609 \\ 53{,}288 \\ 140{,}401 \end{bmatrix}}$$

a. Does it make sense to find the sum of the matrices? Why or why not? If so, explain the meaning of the sum.

b. Does it make sense to find the difference of the matrices? Why or why not? If so, explain the meaning of the difference.

c. Suppose the Census Bureau predicts a 1% decrease in the number of deaths for this region. How would you show this in a matrix?

22. **Business** The Cookie Cutter Bakery keeps a log of each type of cookie sold in a spreadsheet at three of their branch stores so that they can monitor their purchases of supplies. Two days of sales are shown below.



A	B	C	D	E	
1	**Friday**	**chocolate chip**	**peanut butter**	**sugar**	**cut-out**
2	Store 1	120 30	97 24 1/4	64 16	75 18 3/4
3	Store 2	80 20	59 14 3/4	36 9	60 15
4	Store 3	72 18	84 21	29 7 1/4	48 12

A	B	C	D	E	
1	**Saturday**	**chocolate chip**	**peanut butter**	**sugar**	**cut-out**
2	Store 1	112 28	87 21 3/4	56 14	74 18 1/2
3	Store 2	84 21	65 16 1/4	39 9 3/4	70 17 1/2
4	Store 3	88 22	98 24 1/2	43 10 3/4	60 15

a. Write a matrix for each day's sales. Then find the sum of the two days' sales expressed as a matrix.

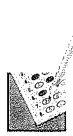
b. Each cookie takes approximately one-fourth cup of flour. If there are four cups of flour in one pound, how many pounds of flour were needed for these two days of baking?

Mixed Review

23. Find $4\begin{bmatrix} -7 & 5 & -11 \\ 2 & -4 & 9 \end{bmatrix}$. (Lesson 4–1)

24. **Geometry** In which octant does the point $(5, -1, 9)$ lie? (Lesson 3–7)

25. Graph $y > x + 4$. (Lesson 2–7)



26. **SAT Practice** If the average (arithmetic mean) of ten numbers is 18 and the average of six of these numbers is 12, what is the average of the other four numbers?

A 15   B 18   C 27   D 28   E 30

27. Solve $\frac{3}{4}t + 1 = 10$.

4-3 Multiplying Matrices

What YOU'LL LEARN

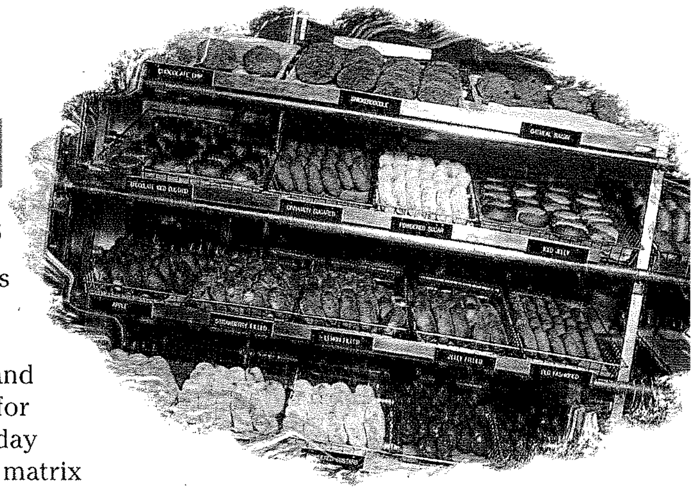
- To multiply matrices.

Why IT'S IMPORTANT

You can use matrices to solve problems involving probability and track and field.

Real World APPLICATION

Sales



The manager of DK's Donuts makes a daily report to the owner that summarizes the cost of each kind of donut and the number of donuts sold for that day. The sales for one day are summarized in the cost matrix C and sales matrix S shown below.

$$\begin{array}{c} \textbf{cost (\$)} \\ \begin{array}{ccccc} & \textit{plain} & \textit{jelly} & \textit{glazed} & \textit{specialty} \\ C = & [0.45 & 0.55 & 0.50 & 0.85] \end{array} \end{array} \qquad S = \begin{array}{l} \\ \textit{plain} \\ \textit{jelly} \\ \textit{glazed} \\ \textit{specialty} \end{array} \begin{array}{c} \textbf{number} \\ \begin{bmatrix} 191 \\ 122 \\ 98 \\ 69 \end{bmatrix} \end{array}$$

You can use matrix multiplication to find the income for the day. In this case, multiply each element in the cost matrix by its corresponding element in the sales matrix and find the total.

Notice that each element in the row matrix is multiplied by an element in the column matrix.

$$\begin{aligned} CS &= [0.45 \;\; 0.55 \;\; 0.50 \;\; 0.85] \cdot \begin{bmatrix} 191 \\ 122 \\ 98 \\ 69 \end{bmatrix} \\ &= [0.45(191) + 0.55(122) + 0.50(98) + 0.85(69)] \\ &= [85.95 + 67.10 + 49.00 + 58.65] \\ &= [260.70] \end{aligned}$$

The income for the day was \$260.70.

In general, the product of the two matrices is found by multiplying rows and columns.

Multiplying Matrices	**The product of $A_{m \times n}$ and $B_{n \times r}$ is $(AB)_{m \times r}$. The element in the *i*th row and the *j*th column of *AB* is the sum of the products of the corresponding elements in the *i*th row of *A* and the *j*th column of *B*.**

Notice that you can multiply two matrices only if the number of columns in the first matrix is equal to the number of rows in the second matrix.

$$\underset{2 \times 2}{\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}} \cdot \underset{2 \times 3}{\begin{bmatrix} 3 & -9 & 2 \\ 5 & 7 & -6 \end{bmatrix}} \qquad \underset{2 \times 3}{\begin{bmatrix} 3 & -9 & 2 \\ 5 & 7 & -6 \end{bmatrix}} \cdot \underset{2 \times 2}{\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}}$$

possible   *not possible*

The product on the left is defined, but the product on the right is not.

Example 1 If $A = \begin{bmatrix} 3 & -5 \\ 2 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 & -3 \\ 8 & -4 & 9 \end{bmatrix}$, find AB.

$$AB = \begin{bmatrix} 3(5) + (-5)(8) & 3(1) + (-5)(-4) & 3(-3) + (-5)(9) \\ 2(5) + 7(8) & 2(1) + 7(-4) & 2(-3) + 7(9) \end{bmatrix}$$

$$= \begin{bmatrix} 15 - 40 & 3 + 20 & -9 - 45 \\ 10 + 56 & 2 - 28 & -6 + 63 \end{bmatrix} \quad \textit{Note that } A_{2 \times 2} \cdot B_{2 \times 3} = (AB)_{2 \times 3}.$$

$$= \begin{bmatrix} -25 & 23 & -54 \\ 66 & -26 & 57 \end{bmatrix}$$

In many situations involving chance, matrices can be used to represent probabilities. Matrix multiplication can be used to predict future events.

Example 2 **A psychologist notes the behavior of mice at a certain point in a maze. For any particular trial, 70% of the mice that went right on the previous trial will go right on this trial, and 60% of those that went left on the previous trial will go right on this trial. This information can be represented by the following *transition matrix*.**

INTEGRATION

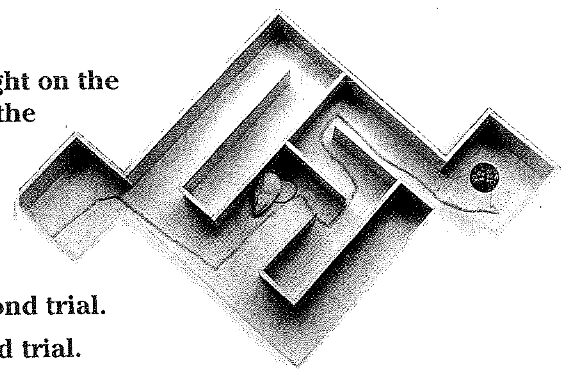
Probability

A transition matrix contains information about the transition from one event to another.

$$T = \begin{array}{c} \\ R \\ L \end{array} \begin{array}{c} \begin{array}{cc} R & L \end{array} \leftarrow \textit{second trial} \\ \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

first trial ↑

Suppose 50% of the mice went right on the first trial. This is represented by the following *probability matrix*.



$$P = \begin{array}{c} \begin{array}{cc} R & L \end{array} \\ [0.5 \;\; 0.5] \end{array}$$

a. Make a prediction for the second trial.

b. Make a prediction for the third trial.

a. To make a prediction about what will happen on the second trial, find PT.

$$PT = [0.5 \;\; 0.5] \cdot \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix} \text{ or } [0.65 \;\; 0.35]$$

On the second trial, 65% of the mice should go right, and 35% should go left.

b. For the third trial, the new probability matrix is $P = [0.65 \;\; 0.35]$. Find PT.

$$PT = [0.65 \;\; 0.35] \cdot \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix} \text{ or } [0.665 \;\; 0.335]$$

On the third trial, 66.5% of the mice should go right, and 33.5% should go left.

Another use of matrix multiplication is in transformational geometry. You have already learned how to translate a geometric figure and change its size by using matrices. Another type of transformation is a **rotation**. A rotation occurs when a figure is moved around a center point. To move a figure by rotation, you can use a rotation matrix.

MODELING MATHEMATICS

Rotations

Materials: grid paper, tracing paper, protractor

The matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ will rotate a figure on the coordinate plane about the origin. In this activity, you will determine the direction and degrees of rotation.

Your Turn

a. Draw a triangle on a coordinate plane and label it $\triangle ABC$. Write the coordinates of the vertices as a coordinate matrix.

b. Multiply the rotation matrix shown above by your coordinate matrix. Graph the resulting triangle on the same coordinate plane and label it $\triangle A'B'C'$. *Note that the rotation matrix should be on the left when multiplying.*

c. Place a piece of tracing paper over $\triangle ABC$ and trace it. With your pencil at the origin as a pivot point, slowly turn the tracing paper until the drawing of $\triangle ABC$ matches $\triangle A'B'C'$. Describe the motion of the triangle.

d. On the coordinate plane, draw $\overline{OA}$ and $\overline{OA'}$. Find the measure of $\angle A'OA$. Repeat for the remaining vertices.

e. Write a sentence that describes the effect of multiplying a coordinate matrix by the rotation matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

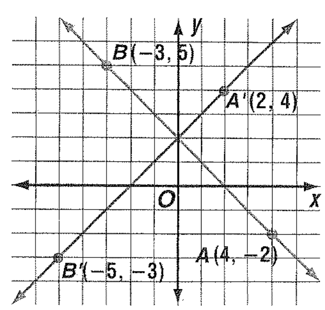
Example 3

INTEGRATION Geometry

Line AB passes through points $A(4, -2)$ and $B(-3, 5)$. Find the coordinates of two points on line $A'B'$ that has been rotated 90° counterclockwise about the origin. Draw its graph and describe the relationship between lines AB and $A'B'$.

Write the ordered pairs in a coordinate matrix. Then multiply the coordinate matrix by the rotation matrix.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & -3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 4 & -3 \end{bmatrix}$$



Coordinates of two points on the line are $A'(2, 4)$ and $B'(-5, -3)$. The two lines appear to be perpendicular.

You can check that the lines are perpendicular by finding the slope of each line.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **Name** the conditions under which two matrices can be multiplied.
2. **Find** the dimensions of matrix M if $M = A_{3\times 2} \cdot B_{2\times 4}$.
3. **Write** a convincing argument for the statement *Matrix multiplication is commutative*. If the statement is not true, find a counterexample.
4. **Give an example** of two matrices M and N for which the products MN and NM are both defined.
5. **You Decide** Brandon thinks two matrices can always be multiplied if they can be added. Dolores thinks that isn't necessarily true. Who is correct? Explain your reasoning.

MODELING MATHEMATICS

6. Apply a 90° counterclockwise rotation about the origin twice to a triangle on the coordinate plane. Compare the new coordinates to the original ones. Make a conjecture about what effect this rotation has on any figure.

Guided Practice

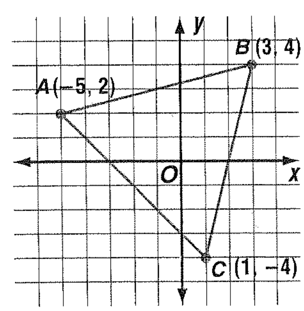
Find the dimensions of each matrix product.

7. $A_{3\times 5} \cdot B_{5\times 2}$
8. $P_{2\times 2} \cdot Q_{2\times 4}$

Perform the indicated operations, if possible.

9. $\begin{bmatrix} 4 & -2 & -7 \\ 6 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$
10. $\begin{bmatrix} 4 & -1 & 6 \\ 1 & 5 & -8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 9 & -6 \end{bmatrix}$

11. **Geometry** Find the new coordinates of the vertices of triangle ABC with vertices $A(-5, 2)$, $B(3, 4)$, and $C(1, -4)$, when the triangle is rotated 90° counterclockwise about the origin. Graph the original triangle and its rotation $A'B'C'$.



EXERCISES

Practice

Find the dimensions of each matrix product.

12. $A_{5\times 2} \cdot B_{2\times 5}$
13. $M_{4\times 2} \cdot N_{1\times 3}$ undef
14. $R_{2\times 3} \cdot S_{3\times 4}$
15. $X_{3\times 4} \cdot Y_{4\times 1}$ 3×1
16. $P_{1\times 5} \cdot Q_{5\times 1}$
17. $A_{3\times 2} \cdot B_{3\times 2}$

Perform the indicated operations, if possible.

18. $[2 \;\; -1] \cdot \begin{bmatrix} 5 \\ 3 \end{bmatrix}$
19. $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 & -2 \\ 5 & 7 & -6 \end{bmatrix}$
20. $\begin{bmatrix} 4 & -1 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \end{bmatrix}$
21. $\begin{bmatrix} 5 & -2 & -1 \\ 8 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} -4 & 2 \\ 1 & 0 \end{bmatrix}$

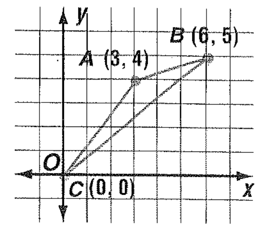
Perform the indicated operations, if possible.

22. $3\begin{bmatrix} 5 & 7 \\ 1 & -2 \end{bmatrix} + 2\begin{bmatrix} -3 & 0 \\ -4 & 2 \end{bmatrix}$

2x2

23. $\begin{bmatrix} 0 & 8 \\ 3 & 1 \\ -1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 & -2 \\ 0 & 8 & -5 \end{bmatrix}$

24. Geometry Find the new coordinates of the vertices of ΔABC shown at the right after it has been rotated 90° counterclockwise about the origin.



25. Geometry Given that A is any 2×2 matrix, and R is the rotation matrix, is $RA = AR$? Explain.

Use the matrices *A*, *B*, *C*, and *D* to evaluate each expression.

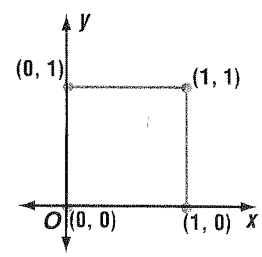
$A = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 0 & -3 \\ 7 & -5 & 9 \end{bmatrix}$ $C = \begin{bmatrix} -6 & 4 \\ -2 & 8 \\ 3 & 0 \end{bmatrix}$ $D = \begin{bmatrix} -1 & 0 \\ 3 & 7 \end{bmatrix}$

26. $AB + B$ **27.** $CB + B$ **28.** $AD + CB$ **29.** $AD + BC$

30. Geometry After a triangle was rotated 90° counterclockwise about the origin, the coordinates of the vertices are $(-3, -5)$, $(-2, 7)$, and $(1, 4)$. What were the coordinates of the vertices of the triangle in its original position?

Graphing Calculator

Use a graphing calculator to determine the effect of multiplying the unit square matrix, $S = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, illustrated in the graph at the right, by each matrix below.



31. $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ **32.** $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

33. $C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **34.** $D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Critical Thinking

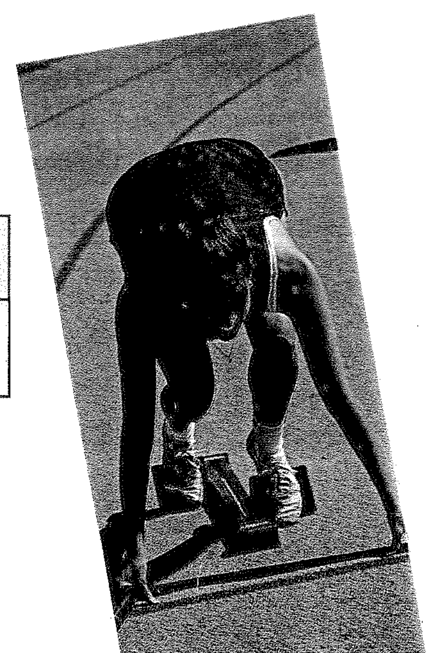
35. Find the values of w, x, y, and z to make the statement $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ true. If the matrix containing w, x, y, and z were multiplied by any other matrix containing two columns, what do you think the result would be?

Applications and Problem Solving

Real World

36. Track and Field In a three-team track meet, the following numbers of first-, second-, and third-place finishes were recorded.

School	First Place	Second Place	Third Place
Blendon	4	10	6
Walnut Springs	7	6	9
Heritage	8	3	4



If 5 points are awarded for first, 3 for second, and 1 for third, use matrix multiplication to find the final scores for each school.

37. **Probability** A weather station in a certain area gathers data about the chance of precipitation. It predicts that, if it rains on a given day, 50% of the time it will rain on the next day. If it is not raining, it will rain on the next day only 30% of the time. The weather forecast for Monday predicts the chance of rain is 80%. Find the chance of rain on Wednesday using this pattern.

38. **Health** Due to a flu epidemic, the school nurse estimates that 30% of the students who are well today will be sick tomorrow and 50% of the students who are sick today will be well tomorrow.
   a. Write a transition matrix to show this situation.
   b. If 80% of the student population is well today, predict what percent will be sick tomorrow.

39. **Finance** Isabel "bought" shares of stock in three U.S. companies for a project in her economics class. She bought 150 shares of a utility company, 100 shares of a food company, and 200 shares of a car manufacturer. At the time she purchased the stocks, the utility company was \$54 per share, the food company was \$60 per share, and the car manufacturer was \$43.50 per share.
   a. Organize the data into two matrices and use matrix multiplication to find the total amount she spent for the shares of stock.
   b. At the end of the project, she "sold" all of her stock. The utility company was \$55.25 per share, the food company was \$61 per share, and the car manufacturer was \$41.75 per share. Use matrix operations to determine how much money Isabel "made" or "lost" in her project.

Mixed Review

40. Find $[4 \ \ 1 \ \ -3] + [6 \ \ -5 \ \ 8]$. (Lesson 4–2)

41. Find $-6\begin{bmatrix} 2 & -1 \\ -5 & 7 \end{bmatrix}$. (Lesson 4–1)

42. **School** Your semester test in English consists of short answers and essay questions. Each short answer question is worth 5 points, and each essay question is worth 15 points. You may choose up to 20 questions of any type to answer. It takes 2 minutes to answer each short answer question and 12 minutes to answer each essay question. If you have one hour to complete the test, and assuming you answer all of the questions that you attempt correctly, how many of each type of question should you answer to earn the highest score? (Lesson 3–6)

43. **SAT Practice** If the circumference of a circle is $\frac{4\pi}{3}$, then what is half of its area?
   A $\frac{2\pi}{9}$ B $\frac{4\pi}{9}$ C $\frac{8\pi}{9}$ D $\frac{2\pi^2}{9}$ E $\frac{4\pi^2}{9}$

44. Write the slope-intercept form of the equation of the line with slope of -2 that passes through the point (3, 1). (Lesson 2–4)

45. State the x- and y-intercepts of the graph of the line with equation $3x - 12y = 24$. (Lesson 2–2)

46. **Telecommunications** A call to a sports hotline costs \$3.38 for the first three minutes and \$0.96 for each minute thereafter. What is the cost of a 12-minute phone call? (Lesson 1–4)

For Extra Practice, see page 883.

4-4 Matrices and Determinants

What **YOU'LL LEARN**

- To evaluate the determinant of a 3×3 matrix, and
- to find the area of a triangle, given the coordinates of its vertices.

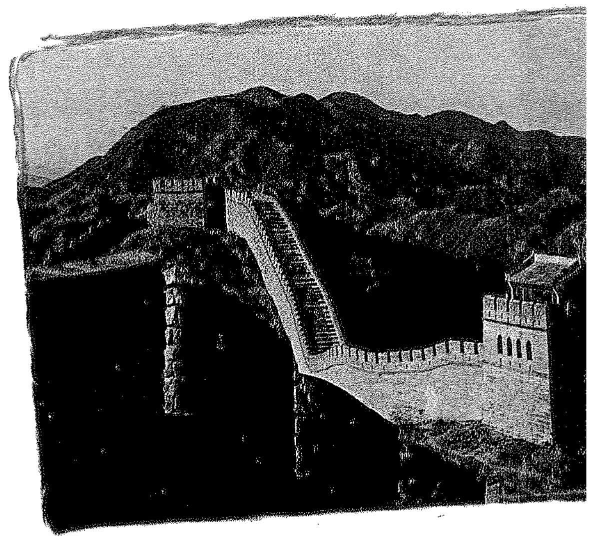
Why **IT'S IMPORTANT**

You can use matrices and determinants to solve problems involving geometry and geography.

CONNECTION

Math History

Although matrices are a relatively new notation in mathematics, the *idea* of matrices can be traced back to the Chinese book *Nine Chapters on the Mathematical Art*, which was published about 250 B.C. This book contained many problems that are solved by using matrices. The Chinese matrix was a large counting board resembling a checkerboard, and bamboo rods were placed on the squares to represent equations.



When the Japanese mathematician Seki Kowa (1683) investigated the Chinese system of solving systems of equations, his calculations were similar to those used today to simplify a **determinant**.

Every square matrix has a number associated with it, called its determinant. The notation for the determinant of $\begin{bmatrix} -5 & -7 \\ 11 & 8 \end{bmatrix}$ is $\begin{vmatrix} -5 & -7 \\ 11 & 8 \end{vmatrix}$. To evaluate the determinant, use the rule for second-order determinants.

$$\begin{vmatrix} -5 & -7 \\ 11 & 8 \end{vmatrix} = -5(8) - (-7)(11) \quad \textit{Recall that } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$$= -40 + 77$$

$$= 37$$

LOOK BACK

You can refer to Lesson 3-3 for information about second-order determinants.

Determinants of 3×3 matrices are called **third-order determinants**. One method of evaluating third-order determinants is called **expansion by minors**. The **minor** of an element is the determinant formed when the row and column containing that element are deleted. For the determinant $\begin{vmatrix} -2 & 3 & 8 \\ 6 & 7 & -1 \\ -4 & 5 & 9 \end{vmatrix}$, the minor of 5 is $\begin{vmatrix} -2 & 3 & 8 \\ 6 & 7 & -1 \\ -4 & 5 & 9 \end{vmatrix}$ or $\begin{vmatrix} -2 & 8 \\ 6 & -1 \end{vmatrix}$.

To use expansion by minors with third-order determinants, each member of one row is multiplied by its minor. The signs of the products alternate, beginning with a positive sign in the first and third row and a negative sign in the second row. The following definition shows an expansion using the elements in the first row of the determinant. However, any row can be used.

Expansion of a Third-Order Determinant	$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

Example 1 **Evaluate $\begin{vmatrix} 2 & 3 & 4 \\ 6 & 5 & 7 \\ -1 & 9 & 8 \end{vmatrix}$ using expansion by minors.**

Decide which row of elements you will use for the expansion. Let's use the first row.

$$\begin{vmatrix} 2 & 3 & 4 \\ 6 & 5 & 7 \\ -1 & 9 & 8 \end{vmatrix} = 2\begin{vmatrix} 5 & 7 \\ 9 & 8 \end{vmatrix} - 3\begin{vmatrix} 6 & 7 \\ -1 & 8 \end{vmatrix} + 4\begin{vmatrix} 6 & 5 \\ -1 & 9 \end{vmatrix}$$

$$= 2(40 - 63) - 3(48 + 7) + 4(54 + 5)$$
$$= -46 - 165 + 236$$
$$= 25$$

You can check your work by evaluating the determinant again using a different row of elements.

Another method for evaluating a third-order determinant is using diagonals. In this method, you begin by writing the first two columns on the right side of the determinant.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \rightarrow \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

Next, draw diagonals from each element of the top row of the determinant downward to the right. Find the product of the elements on each diagonal.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \rightarrow \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

$aei \quad bfg \quad cdh$

Then, draw diagonals from the elements in the third row of the determinant upward to the right. Find the product of the elements on each diagonal.

$gec \quad hfa \quad idb$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \rightarrow \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

To find the value of the determinant, add the products of the first set of diagonals and then subtract the products of the second set of diagonals. The value is $aei + bfg + cdh - gec - hfa - idb$.

Example 2 **Evaluate $\begin{vmatrix} -1 & 0 & 8 \\ 7 & 3 & 4 \\ 2 & 2 & 5 \end{vmatrix}$ using diagonals.**

TECHNOLOGY Tips

If you are using a graphing calculator, press MATRX and use the arrow keys to highlight the MATH menu. Item 1:det is used to calculate the determinant.

First, rewrite the first two columns to the right of the determinant.

$$\begin{vmatrix} -1 & 0 & 8 \\ 7 & 3 & 4 \\ 2 & 2 & 5 \end{vmatrix}\begin{matrix} -1 & 0 \\ 7 & 3 \\ 2 & 2 \end{matrix}$$

Next, find the products of the elements of the diagonals.

$$\begin{matrix} & & & 48 & -8 & 0 \\ -1 & 0 & 8 & -1 & 0 & \\ 7 & 3 & 4 & 7 & 3 & \\ 2 & 2 & 5 & 2 & 2 & \\ & & -15 & 0 & 112 & \end{matrix}$$

Then, add the bottom products and subtract the top products.

$$-15 + 0 + 112 - 48 - (-8) - 0 = 57$$

The value of the determinant is 57.

One very powerful application of determinants is finding the areas of polygons. The formula below shows how determinants serve as a mathematical tool to find the area of a triangle when the coordinates of the three vertices are given.

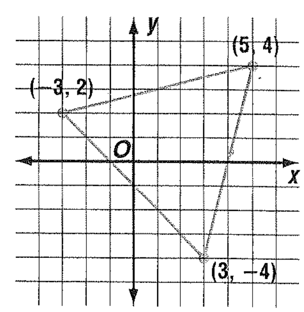
Area of Triangles	**The area of a triangle having vertices at (a, b), (c, d), and (e, f) is $\lvert A \rvert$, where** $A = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$.

Notice that it is necessary to use the absolute value of A to guarantee a nonnegative value for area.

Example 3 **Find the area of the triangle whose vertices are located at (3, −4), (5, 4), and (−3, 2).**

INTEGRATION
Geometry

Assign values to a, b, c, d, e, and f and substitute them into the area formula and evaluate.



$$A = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix} \quad \begin{matrix} (a, b) = (3, -4) \\ (c, d) = (5, 4) \\ (e, f) = (-3, 2) \end{matrix}$$

$$= \frac{1}{2}\begin{vmatrix} 3 & -4 & 1 \\ 5 & 4 & 1 \\ -3 & 2 & 1 \end{vmatrix} \quad \textit{Substitute.}$$

$$= \frac{1}{2}[(3)4 + (-4)(-3) + (5)(2) - (-3)(4) - (2)(3) - (5)(-4)]$$

$$= \frac{1}{2}[12 + 12 + 10 - (-12) - 6 - (-20)]$$

$$= \frac{1}{2}(60) \text{ or } 30$$

The area of the triangle is 30 square units.

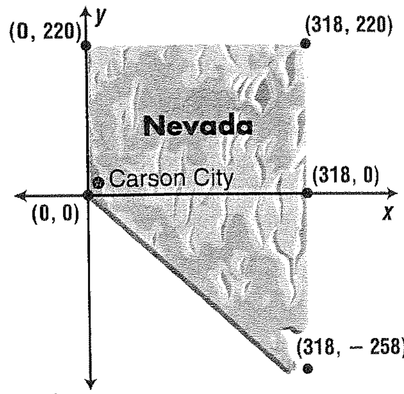
Maps usually have grids, similar to a coordinate system, that make it easier for you to locate cities, states, or landmarks. A coordinate system can also be used to find the area of large regions.

Example 4

Real World APPLICATION

Cartography

The figure at the right shows a map of the state of Nevada that has been placed on a coordinate plane in which 1 unit = 1 mile. Estimate the area of Nevada from the coordinates of the vertices.



The x-axis separates the map into a triangular region and a rectangular region. Find the area of the rectangular region.

$A = \ell w$

$= 318 \cdot 220$ or 69,960

Now find the area of the triangular region using the coordinates (0, 0), (318, 0), and (318, −258). Use expansion by minors.

$$A = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix} \quad \begin{matrix} (a, b) = (0, 0) \\ (c, d) = (318, 0) \\ (e, f) = (318, -258) \end{matrix}$$

$$= \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 318 & 0 & 1 \\ 318 & -258 & 1 \end{vmatrix} = \frac{1}{2} \cdot 1\begin{vmatrix} 318 & 0 \\ 318 & -258 \end{vmatrix} \text{ or } -41{,}022$$

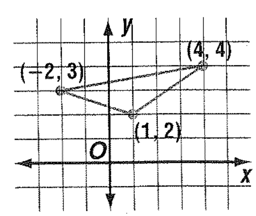
Finally, add the areas of the two regions. The area of Nevada is 69,960 + 41,022 or about 111,000 square miles. *Compare this to the actual area.*

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **Explain** how $\begin{bmatrix} 7 & 8 \\ 3 & -2 \end{bmatrix}$ and $\begin{vmatrix} 7 & 8 \\ 3 & -2 \end{vmatrix}$ are different.
2. **Describe** how to find the minor of 8 in $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.
3. **State** the condition(s) under which a matrix has a determinant.
4. **Write** a matrix that will help you use a determinant to find the area of the triangle shown at the right.



Guided Practice

Determine whether each matrix has a determinant. Write *yes* or *no*. If *yes*, find the value of the determinant.

5. $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$
6. $\begin{bmatrix} -8 & 0 \\ 5 & -4 \end{bmatrix}$
7. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. Evaluate $\begin{vmatrix} 2 & 3 & 4 \\ 6 & 5 & 7 \\ 1 & 2 & 8 \end{vmatrix}$ using expansion by minors.

9. Evaluate $\begin{vmatrix} -1 & 4 & 0 \\ 3 & -2 & -5 \\ -3 & -1 & 2 \end{vmatrix}$ using diagonals.

10. **Geometry** Use a determinant to find the area of the triangle whose vertices have coordinates $(-2, 3)$, $(5, 8)$, and $(1, 2)$.

EXERCISES

Practice

Determine whether each matrix has a determinant. Write *yes* or *no*. If *yes*, find the value of the determinant.

11. $\begin{bmatrix} -3 & 5 \\ 6 & -10 \end{bmatrix}$

12. $\begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix}$

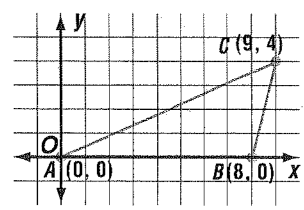
13. $\begin{bmatrix} 4 & 3 \\ 8 & -1 \\ 7 & 2 \end{bmatrix}$

14. $\begin{bmatrix} -5 & 8 \\ 3 & 0 \end{bmatrix}$

15. $\begin{bmatrix} -2 & 0 & 1 \\ 1 & 2 & 0 \\ 4 & -1 & 1 \end{bmatrix}$

16. $\begin{bmatrix} 5 & 7 & -2 \\ 3 & -2 & 6 \\ 1 & -4 & 3 \end{bmatrix}$

17. **Geometry** Use a determinant to find the area of $\triangle ABC$ shown at the right. Check your answer by using the formula $A = \frac{1}{2}bh$.



Evaluate each determinant using expansion by minors.

18. $\begin{vmatrix} -3 & 0 & 6 \\ 6 & 5 & -2 \\ 1 & 4 & 2 \end{vmatrix}$

19. $\begin{vmatrix} 0 & -4 & 0 \\ 3 & -2 & 5 \\ 2 & -1 & 1 \end{vmatrix}$

20. $\begin{vmatrix} -2 & 7 & -2 \\ 4 & 6 & 2 \\ 1 & 0 & -1 \end{vmatrix}$

Evaluate each determinant using diagonals.

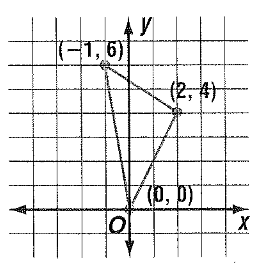
21. $\begin{vmatrix} 1 & 6 & 4 \\ -2 & 3 & 1 \\ 1 & 6 & 4 \end{vmatrix}$

22. $\begin{vmatrix} 1 & -1 & 1 \\ 3 & 3 & 1 \\ 0 & 5 & 2 \end{vmatrix}$

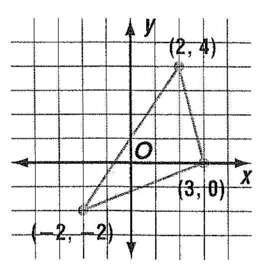
23. $\begin{vmatrix} 2 & -3 & 4 \\ -2 & 1 & 5 \\ 5 & 3 & -2 \end{vmatrix}$

Use a determinant to find the area of each triangle below.

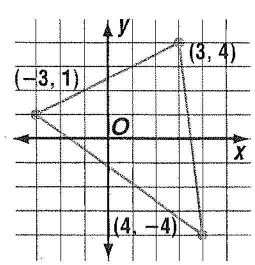
24.



25.



26.



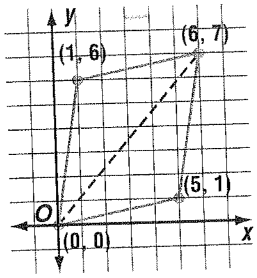
Solve for the variable.

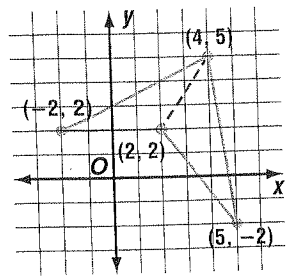
27. $\begin{vmatrix} 2 & x \\ 5 & -3 \end{vmatrix} = 24$

28. $\begin{vmatrix} 4 & x & -2 \\ -x & -3 & 1 \\ -6 & 2 & 3 \end{vmatrix} = -3$

INTEGRATION
Geometry

Use determinants to find the area of each polygon below.

29. 

30. 

31. Find the value of x such that the area of a triangle whose vertices have coordinates (6, 5), (8, 2), and (x, 11) is 30.

Graphing Calculator

32. Use a graphing calculator to find two third-order matrices that are not equal, but have equal determinants.

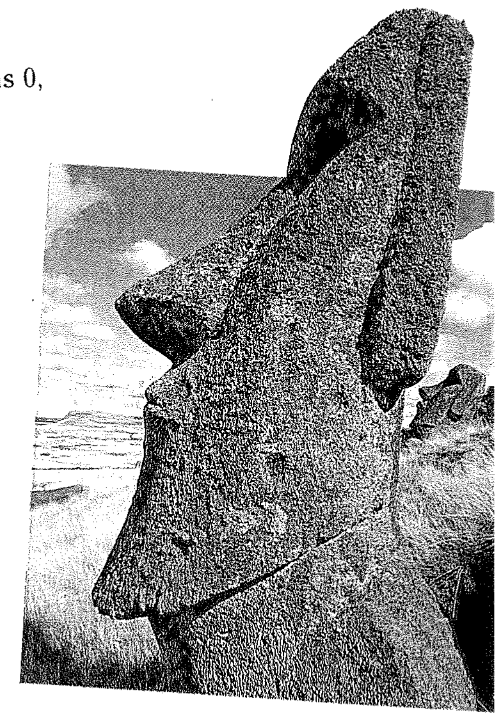
Critical Thinking

33. Find a third-order matrix in which no element is 0, but for which the determinant is 0.

Applications and Problem Solving

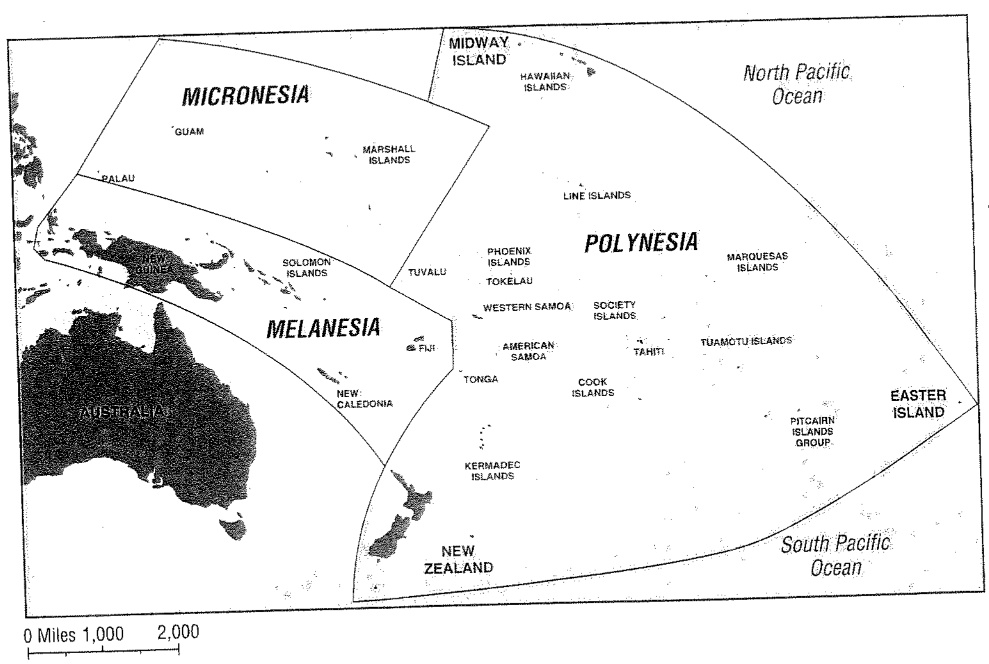
Real World

34. **World Cultures** Easter Island is the easternmost island of Polynesia, which is a triangular area containing thousands of islands in the South Pacific. Its northern vertex is Midway Island and the southern boundary runs from New Zealand to Easter Island. A map of Polynesia is shown below. Explain how to use a coordinate grid to estimate the area of Polynesia.

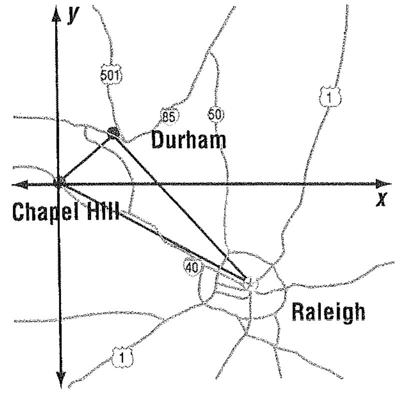


GLOBAL CONNECTIONS

Easter Island is famous for its enormous statues, called *moai*, which were carved hundreds of years ago. Today, more than 600 statues, some as tall as 40 feet, are scattered over the island.



35. **Geography** The region in North Carolina bounded by Chapel Hill, Durham, and Raleigh is known as the Research Triangle Park. If a coordinate grid in which 1 unit = 1 mile is placed over the map of North Carolina with Chapel Hill at the origin, the coordinates of these three cities are (0, 0), (6, 3.5) and (21, −10.5). Estimate the area of Research Triangle Park.



Mixed Review

36. Find $\begin{bmatrix} 4 & 0 & -8 \\ 7 & -2 & 10 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 \\ 6 & 0 \end{bmatrix}$. (Lesson 4–3)

37. Solve $\begin{bmatrix} 2 & x \\ y & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & z \end{bmatrix}$ for the variables. (Lesson 4–1)

X Y

38. Given $f(x, y) = 12x - 8y$, find the value of $f(-2, -4)$. (Lesson 3–5)

39. Use Cramer's rule to solve the system of equations. (Lesson 3–3)

$2x - y = 7$

$x + 3y = 7$

40. Graph $f(x) = |x - 3|$. (Lesson 2–6)

41. **SAT Practice** If $3 < x < 5 < y < 10$, then which of the following best defines $\frac{x}{y}$?

A $\frac{3}{10} < \frac{x}{y} < 1$ B $\frac{3}{10} < \frac{x}{y} < \frac{1}{2}$ C $\frac{3}{5} < \frac{x}{y} < \frac{1}{2}$

D $\frac{3}{5} < \frac{x}{y} < 1$ E $\frac{3}{5} < \frac{x}{y} < 2$

For Extra Practice, see page 884.

42. Name the property illustrated by $x(a + b) = xa + xb$. (Lesson 1–2)

SELF TEST

1. **Investments** Three women and their husbands invested a total of \$5400 in a sandwich shop. The women invested \$2400 in all; Sue invested \$200 more than Tamara, and Elisa invested \$200 more than Sue. Lou invested half as much as his wife, Bob invested the same as his wife, and Mateo invested twice as much as his wife. Who is married to whom? (Lesson 4–1)

2. Find the dimensions of the matrix product $A_{3\times4} \cdot B_{4\times3}$. (Lesson 4–3)

Perform the indicated operations, if possible. (Lessons 4–1, 4–2, and 4–3)

3. $\begin{bmatrix} -2 & 1.5 \\ 3 & -0.25 \end{bmatrix} - \begin{bmatrix} -6 & 2 \\ 3 & 1.25 \end{bmatrix}$

4. $\begin{bmatrix} 5.3 & -1.2 \\ 1.6 & 2 \end{bmatrix} + \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix}$

5. $2\begin{bmatrix} 5 & 4 \\ -1 & 6 \end{bmatrix} - 3\begin{bmatrix} -3 & 0 \\ -2 & 0 \end{bmatrix}$

6. $-4\begin{bmatrix} -1 & -0.25 \\ 0 & 2 \\ 0.5 & 4 \end{bmatrix}$

7. $\begin{bmatrix} -2 & 3 \\ 1 & 10 \\ 0 & -6 \end{bmatrix} \cdot \begin{bmatrix} 9 & 3 \\ 1 & 4 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & 2 \\ 3 & 5 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

9. Evaluate the determinant of $\begin{bmatrix} -1 & 3 & 4 \\ 0 & 5 & 1 \\ 6 & -2 & 3 \end{bmatrix}$. (Lesson 4–4)

10. **Geometry** Use determinants to find the area of a triangle with vertices having coordinates (−1, −2), (5, 3), and (2, 6). (Lesson 4–4)

4-5 Identity and Inverse Matrices

What YOU'LL LEARN

- To write the identity matrix for any square matrix, and
- to find the inverse of a 2×2 matrix.

Why IT'S IMPORTANT

You can use matrices to solve problems involving cryptology and geometry.

FYI

One of the few unbroken codes in military history was constructed by a group of American Indians called the Navaho Code Talkers. During World War II, the entire military operation at Iwo Jima was directed through orders communicated by the Code Talkers. The Japanese were never able to break the code.

Real World APPLICATION

Cryptology

Cryptology deals with coding messages so that only people with the key can decipher them. In ancient Greece, Spartans wound a belt in a spiral around a stick and wrote messages along the length of the stick. When they unwound the belt, only those people who had a stick exactly the same size as the first could read the message. Since then, cryptology has been important in military communications, particularly in time of war. Today, programmers use cryptology to protect secret data stored in computers.

Two U.S. Marines use their Navajo code during battle, 1943

An important advancement in cryptology occurred in the 1930s when American mathematician Lester Hill used matrices to encode messages. Here's a simplified version of how it works.

Step 1

Suppose the first word in a message is MEET. Assign each letter a number based on its position in the alphabet ($A = 1, B = 2, \ldots, Y = 25, Z = 26$). Thus, $M = 13$, $E = 5$, and $T = 20$. Write the numbers in a matrix.

$$\begin{bmatrix} M & E \\ E & T \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 5 & 20 \end{bmatrix}$$

Step 2

Multiply the matrix by a *coding matrix*. Let's use $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 13 & 5 \\ 5 & 20 \end{bmatrix} = \begin{bmatrix} 5 & 20 \\ 18 & 25 \end{bmatrix}$$

Step 3

Assign a letter to each number of the matrix based on Step 1.

$$\begin{bmatrix} 5 & 20 \\ 18 & 25 \end{bmatrix} = \begin{bmatrix} E & T \\ R & Y \end{bmatrix}$$

Therefore, MEET would encode as ETRY. *Notice that each E in the uncoded message is assigned to a different letter in the coded message.*

When the person receives the message, he or she needs to decipher it by *undoing* the multiplication to get back to the original matrix. Mathematically, this means finding the **inverse** of the coding matrix. *You will use this code in Example 4.*

Recall from your work with real numbers that the inverse and identity of a number are related. In real numbers, 1 is the identity for multiplication because $a \cdot 1 = 1 \cdot a = a$. Similarly, the inverse of a matrix is related to the **identity matrix**. The identity matrix is a square matrix that, when multiplied by another matrix, equals that same matrix.

With 2×2 matrices, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix because $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The identity matrix is symbolized by I. Since one of the properties of the identity matrix is that it is commutative, only a square matrix can have an identity. In an identity matrix, the principal diagonal goes from upper left to lower right and consists only of ones.

Identity Matrix for Multiplication

The identity matrix for multiplication, *I*, is a square matrix with 1 for every element of the principal diagonal and 0 in all other positions. For any square matrix *A* of the same order as *I*,

$$A \cdot I = I \cdot A = A.$$

Example 1 **If $N = \begin{bmatrix} -2 & 4 & 7 \\ 5 & -3 & 6 \\ -8 & 2 & -1 \end{bmatrix}$, find I so that $N \cdot I = N$.**

The dimensions of N are 3×3. So, I must also be 3×3. The principal diagonal contains only 1s. Complete the matrix with 0s.

$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ The 3×3 identity matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} -2 & 4 & 7 \\ 5 & -3 & 6 \\ -8 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 7 \\ 5 & -3 & 6 \\ -8 & 2 & -1 \end{bmatrix}$$

Therefore, $N \cdot I = N$.

Another property of real numbers is that every real number, except 0, has a multiplicative inverse. That is, $\frac{1}{a}$ is the multiplicative inverse of a because $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$. Likewise, if matrix A has an inverse named A^{-1}, then $A \cdot A^{-1} = A^{-1} \cdot A = I$. The following example shows how the inverse of a 2×2 matrix can be found.

When we refer to the inverse of a matrix, it implies the multiplicative inverse unless otherwise stated.

Example 2 **If $M = \begin{bmatrix} 7 & 4 \\ 2 & 3 \end{bmatrix}$, find M^{-1}. Check your result.**

TECHNOLOGY Tips

The x^{-1} key on a graphing calculator is used to find the inverse of a matrix.

Let M^{-1} be $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$. By the definition of an inverse, $M \cdot M^{-1} = I$.

$$\begin{bmatrix} 7 & 4 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 7w + 4y & 7x + 4z \\ 2w + 3y & 2x + 3z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \textit{Multiply.}$$

When two matrices are equal, their corresponding elements are equal. So the following equations can be generated from the two equal matrices.

(1) $7w + 4y = 1$ (2) $7x + 4z = 0$ (3) $2w + 3y = 0$ (4) $2x + 3z = 1$

Use equations (1) and (3) to find values for w and y.

First solve for w. Then substitute the w value into one of the equations to find y.

$$\begin{array}{l} 7w + 4y = 1 \\ 2w + 3y = 0 \end{array} \rightarrow \begin{array}{r} 21w + 12y = 3 \\ \underline{(-)\ 8w + 12y = 0} \\ 13w \qquad = 3 \\ w = \frac{3}{13} \end{array} \rightarrow \begin{array}{r} 7w + 4y = 1 \\ 7\left(\frac{3}{13}\right) + 4y = 1 \\ 4y = -\frac{8}{13} \\ y = -\frac{2}{13} \end{array}$$

Use equations (2) and (4) to find values for x and z.

First solve for z. Then substitute the z value into one of the equations to find x.

$$\begin{array}{l} 7x + 4z = 0 \\ 2x + 3z = 1 \end{array} \rightarrow \begin{array}{r} 14x + 8z = 0 \\ \underline{(-)\ 14x + 21z = 7} \\ -13z = -7 \\ z = \frac{7}{13} \end{array} \rightarrow \begin{array}{r} 7x + 4z = 0 \\ 7x + 4\left(\frac{7}{13}\right) = 0 \\ 7x = -\frac{28}{13} \\ x = -\frac{4}{13} \end{array}$$

Therefore, $M^{-1} = \begin{bmatrix} \frac{3}{13} & -\frac{4}{13} \\ -\frac{2}{13} & \frac{7}{13} \end{bmatrix}$.

Check: $\begin{bmatrix} 7 & 4 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{13} & -\frac{4}{13} \\ -\frac{2}{13} & \frac{7}{13} \end{bmatrix} = \begin{bmatrix} \frac{21}{13} - \frac{8}{13} & -\frac{28}{13} + \frac{28}{13} \\ \frac{6}{13} - \frac{6}{13} & -\frac{8}{13} + \frac{21}{13} \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

You should also check to be sure that $M^{-1} \cdot M = I$.

The same method used in Example 2 can be used to develop the general form of the inverse of a 2×2 matrix.

$$\text{The inverse of } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is } \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} \text{ or } \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Notice that $ad - bc$ is the value of the determinant of the matrix. Remember that $\frac{1}{ad - bc}$ is not defined when $ad - bc = 0$. Therefore, if the value of the determinant of a matrix is 0, the matrix cannot have an inverse.

Inverse of a 2×2 Matrix	**Any matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ will have an inverse M^{-1} if and only if $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$. Then $M^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.**

Example 3 **If $Q = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}$, find Q^{-1}. Check your result.**

Find the value of the determinant.

$$\begin{vmatrix} 2 & -1 \\ 1 & -3 \end{vmatrix} = -6 - (-1) \text{ or } -5$$

Since the determinant does not equal 0, Q^{-1} exists.

$$Q^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix} -3 & 1 \\ -1 & 2 \end{bmatrix}$$

Check: $-\frac{1}{5}\begin{bmatrix} -3 & 1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -6+1 & 3-3 \\ -2+2 & 1-6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

TECHNOLOGY Tips

If you get a SINGULAR MATRIX error on a graphing calculator when trying to find an inverse, it means that the matrix has no inverse.

In the application at the beginning of the lesson, the coding matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ was used to encode the word MEET as ETRY. In the following Example, you will *decode* part of the message by finding the inverse of the coding matrix.

Example 4

Real World Application

Cryptology

Suppose a person receives the message ETRYATNYEMYULAMMIXCQ that has been encoded using the matrix $C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. You already know that the first four letters, ETRY, correspond to MEET. Decode the next four letters in the message.

First, write the letters in a 2×2 matrix and assign each letter a number based on its position in the alphabet.

$$\begin{bmatrix} A & T \\ N & Y \end{bmatrix} = \begin{bmatrix} 1 & 20 \\ 14 & 25 \end{bmatrix}$$

(continued on the next page)

fabulous FIRSTS

Rebecca Marier (1974–)

For the first time in the U.S. Military Academy's 193-year history, a woman—Rebecca Marier—graduated first in her class among 987 other senior cadets on June 3, 1995, at West Point, New York.

Now, find the inverse of the coding matrix $C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. The determinant is $0 - 1$ or -1.

$$C^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

Finally, multiply the inverse matrix by the first matrix and assign letters to the elements in the product.

$$\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 20 \\ 14 & 25 \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 1 & 20 \end{bmatrix} \text{ or } \begin{bmatrix} M & E \\ A & T \end{bmatrix}$$

Therefore, the next four letters of the message are MEAT.

The first eight letters of the message are MEETMEAT. You will decode the rest of the message in Exercises 10 and 33.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **Explain** how the multiplicative inverse and identity for real numbers are similar to the matrix inverse and identity.
2. **Write** the 4×4 identity matrix.
3. **Choose** the inverse of $\begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$.

   a. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **c.** $\begin{bmatrix} 2 & -1 \\ -\frac{3}{2} & 1 \end{bmatrix}$ **d.** $\begin{bmatrix} 4 & -2 \\ -3 & 2 \end{bmatrix}$

4. **Create** a square matrix that does not have an inverse.
5. **You Decide** Miyoki says that the matrix $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ does not have a multiplicative identity. Hector says the identity is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ because $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Who is correct? Explain your reasoning.

Guided Practice

Find the inverse of each matrix, if it exists. If it does not exist, explain why not.

6. $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$
7. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
8. $\begin{bmatrix} 6 & 3 \\ 8 & 4 \end{bmatrix}$
9. $\begin{bmatrix} -5 & 1 \\ 7 & 4 \end{bmatrix}$
10. **Cryptology** Decode the next four letters, EMYU, of the message in Example 4.

EXERCISES

Practice

Find the inverse of each matrix, if it exists. If it does not exist, explain why not.

11. $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$

12. $\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$

13. $\begin{bmatrix} 4 & -8 \\ -1 & 2 \end{bmatrix}$

14. $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$

15. $\begin{bmatrix} 4 & -3 \\ 2 & 7 \end{bmatrix}$

16. $\begin{bmatrix} -1 & 5 & -2 \\ 4 & 2 & -3 \end{bmatrix}$

17. $\begin{bmatrix} 2 & -5 \\ 6 & 1 \end{bmatrix}$

18. $\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$

19. $\begin{bmatrix} -2 & 0 \\ 5 & 6 \end{bmatrix}$

Determine whether each statement is *true* or *false*.

20. $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = I$

21. $\begin{bmatrix} 2 & 1 & -4 \\ -3 & 6 & 5 \\ 1 & 2 & -2 \end{bmatrix} \cdot I = \begin{bmatrix} 2 & 1 & -4 \\ -3 & 6 & 5 \\ 1 & 2 & -2 \end{bmatrix}$

22. $\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = I$

23. $\begin{bmatrix} 1 & 5 \\ 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{7} & \frac{5}{7} \\ \frac{1}{7} & -\frac{1}{7} \end{bmatrix} = I$

24. The inverse of $\begin{bmatrix} 3 & 1 & 2 \\ -2 & 0 & 4 \\ 3 & 5 & 2 \end{bmatrix}$ is $-\frac{1}{64}\begin{bmatrix} -20 & 8 & 4 \\ 16 & 0 & -16 \\ -10 & -12 & 2 \end{bmatrix}$.

25. All square matrices have multiplicative identities.
26. Only square matrices have multiplicative inverses.
27. Some square matrices do not have multiplicative inverses.
28. Some square matrices do not have multiplicative identities.
29. All multiplicative identities are square matrices.

Graphing Calculator

30. When Mike used a graphing calculator to find the inverse of $\begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix}$, there was an ERROR statement. Explain why.

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Critical Thinking

31. Prove that $A \cdot I = I \cdot A = A$ for all second-order matrices.

Applications and Problem Solving

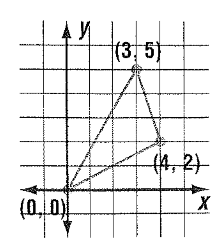
Real World

32. **Geometry** Recall that the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ will rotate a figure on a coordinate plane 90° counterclockwise about the origin.

a. Find the inverse of this rotation matrix.

b. Make a conjecture about what movement the inverse describes on a coordinate plane.

c. Test your conjecture on the triangle shown at the right. Make a drawing to verify your conjecture.



33. **Cryptology** Refer to Example 4 and Exercise 10.
   a. Decode the last eight letters, LAMMIXCQ, of the coded message. (*Hint:* Negative integers and zero are assigned letters as follows: $0 = Z$, $-1 = Y$, $-2 = X$, $-3 = W$, and so on.)
   b. Write the entire decoded message from Example 4.
   c. Write a message and code it using your own coding matrix. (*Hint:* Use a coding matrix whose determinant is 1 or -1.) Trade messages with a partner and decode the messages.

Mixed Review

34. Solve $\begin{vmatrix}4 & -a \\ 7 & 3a\end{vmatrix} = 57$. (Lesson 4–4)

35. **Geometry** The perimeter of a right triangle is 24 centimeters. Three times the length of the longer leg minus two times the length of the shorter leg exceeds the hypotenuse by 2 centimeters. The length of the shorter leg is one centimeter more than half the hypotenuse. What are the lengths of all three sides? (Lesson 3–7)

36. **ACT Practice** If $x + 3y = 12$ and $\frac{2}{3}x - y = 5$, then $x =$?
   A 1 **B** 8 **C** 9 **D** 13.5 **E** 27

37. Find the value of $g(12)$ when $g(x) = \frac{26 - x}{2}$. (Lesson 2–1)

For Extra Practice, see page 884.

38. Solve $5 < 2x - 9 < 11$. (Lesson 1–6)

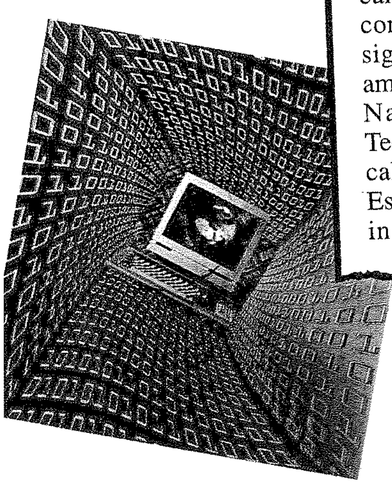
39. Solve $|a + 5| + 5 = 3$. (Lesson 1–5)

Mathematics and Society

DSS Code

The excerpt below appeared in an article in *Popular Science* in December, 1994.

> THE GROWING AMOUNT OF BUSINESS being conducted electronically these days raises thorny questions: How does your broker, lawyer, or accountant know that your e-mail message really came from you? And just how valid is a computerized contract when there are no signatures on the bottom line? These are among the questions being addressed by the National Institute of Standards and Technology, which has proposed a solution called the Digital Signal Standard (DSS). Essentially, the DSS is a method for creating a mathematical "signature" on your documents . . . . The DSS relies on a well-known cryptology concept called public and private keys. You use a private key—a long number you keep to yourself—to generate an encoded, "signed" version of your message. A recipient verifies this signature using your public key, another long number. The public key is tied to the private key by a mathematical equation that makes it easy to compute the public key from the private key, but nearly impossible to perform the reverse calculation. If the math indicates a match, your signature has been verified. ■



1. You could call the DSS code a "one-way" function because it is relatively easy to do in one direction, but much more difficult to do in the reverse direction. Can you think of another example of a one-way function?
2. If the federal government used the DSS, who might want to keep a "master key" so they could decode all messages flowing through the government's systems?
3. List some of the advantages and disadvantages of having a "master key."