Chapter 5 Overview

We have seen how the need to calculate instantaneous rates of change led the discoverers of calculus to an investigation of the slopes of tangent lines and, ultimately, to the derivative—to what we call *differential* calculus. But they knew that derivatives revealed only half the story. In addition to a calculation method (a "calculus") to describe how functions were changing at a given instant, they also needed a method to describe how those instantaneous changes could accumulate over an interval to produce the function. That is why they were also investigating *areas under curves,* an investigation that ultimately led to the second main branch of calculus, called *integral* calculus.

Once they had the calculus for finding slopes of tangent lines and the calculus for finding areas under curves—two geometric operations that would seem to have nothing at all to do with each other—the challenge for Newton and Leibniz was to prove the connection that they knew intuitively had to be there. The discovery of this connection (called the Fundamental Theorem of Calculus) brought differential and integral calculus together to become the single most powerful insight mathematicians had ever acquired for understanding how the universe worked.

5.1 Estimating with Finite Sums

Distance Traveled • Rectangular Approximation Method (RAM) • Volume of a Sphere • Cardiac Output

Distance Traveled

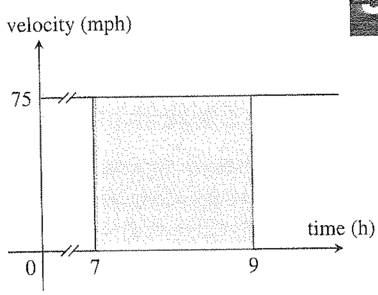


Figure 5.1 The distance traveled by a 75 mph train in 2 hours is 150 miles, which corresponds to the area of the shaded rectangle.

We know why a mathematician pondering motion problems might have been led to consider slopes of curves, but what do those same motion problems have to do with areas under curves? Consider the following problem from a typical elementary school textbook:

> A train moves along a track at a steady rate of 75 miles per hour from 7:00 A.M. to 9:00 A.M. What is the total distance traveled by the train?

Applying the well-known formula *distance* = *rate* × *time,* we find that the answer is 150 miles. Simple. Now suppose that you are Isaac Newton trying to make a connection between this formula and the graph of the velocity function.

You might notice that the distance traveled by the train (150 miles) is exactly the *area* of the rectangle whose base is the time interval $[7, 9]$ and whose height at each point is the value of the constant velocity function $v = 75$ (Figure 5.1). This is no accident, either, since *the distance traveled* and *the area* in this case are both found by multiplying the rate (75) by the change in time (2).

This same connection between distance traveled and rectangle area could be made no matter how fast the train was going or how long or short the time interval was. But what if the train had a velocity v that *varied* as a function of time? The graph (Figure 5.2) would no longer be a horizontal line, so the region under the graph would no longer be rectangular.

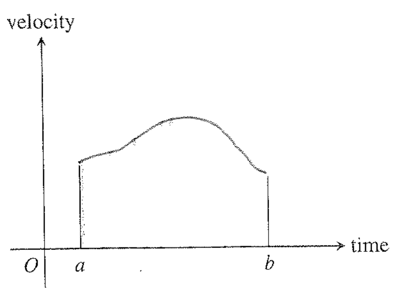


Figure 5.2 If the velocity varies over the time interval $[a, b]$, does the shaded region give the distance traveled?

Would the area of this irregular region still give the total distance traveled over the time interval? Newton and Leibniz (and, actually, many others who had considered this question) thought that it obviously would, and that is why they were interested in a calculus for finding areas under curves. They imagined the time interval being partitioned into many tiny subintervals, each one so small that the velocity over it would essentially be constant. Geometrically, this was equivalent to slicing the irregular region into narrow strips, each of which would be nearly indistinguishable from a narrow rectangle (Figure 5.3).

They argued that, just as the total area could be found by summing the areas of the (essentially rectangular) strips, the total distance traveled could be found by summing the small distances traveled over the tiny time intervals.

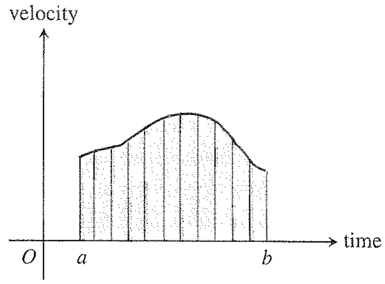


Figure 5.3 The region is partitioned into vertical strips. If the strips are narrow enough, they are almost indistinguishable from rectangles. The sum of the areas of these "rectangles" will give the total area and can be interpreted as distance traveled.

Example 1 FINDING DISTANCE TRAVELED WHEN VELOCITY VARIES

A particle starts at $x = 0$ and moves along the x-axis with velocity $v(t) = t^2$ for time $t \geq 0$. Where is the particle at $t = 3$?

Solution We graph v and partition the time interval $[0, 3]$ into subintervals of length Δt. (Figure 5.4 shows twelve subintervals of length 3/12 each.)

Notice that the region under the curve is partitioned into thin strips with bases of length 1/4 and *curved* tops that slope upward from left to right. You might not know how to find the area of such a strip, but you can get a good approximation of it by finding the area of a suitable rectangle. In Figure 5.5, we use the rectangle whose height is the y-coordinate of the function at the midpoint of its base.

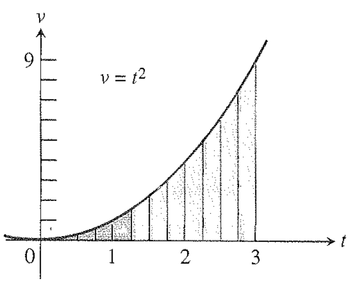


Figure 5.4 The region under the parabola $v = t^2$ from $t = 0$ to $t = 3$ is partitioned into 12 thin strips, each with base $\Delta t = 1/4$. The strips have curved tops. (Example 1)

The area of this narrow rectangle approximates the distance traveled over the time subinterval. Adding all the areas (distances) gives an approximation of the total area under the curve (total distance traveled) from $t = 0$ to $t = 3$ (Figure 5.6).

Computing this sum of areas is straightforward. Each rectangle has a base of length $\Delta t = 1/4$, while the height of each rectangle can be found by evaluating the function at the midpoint of the subinterval. Table 5.1 shows the computations for the first four rectangles.

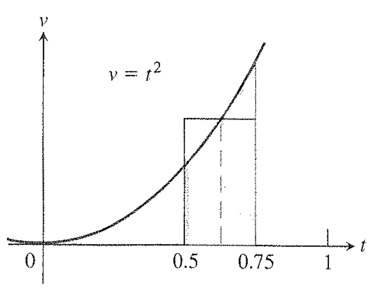


Figure 5.5 The area of the shaded region is approximated by the area of the rectangle whose height is the function value at the midpoint of the interval. (Example 1)

Table 5.1

Subinterval	$\left[0, \frac{1}{4}\right]$	$\left[\frac{1}{4}, \frac{1}{2}\right]$	$\left[\frac{1}{2}, \frac{3}{4}\right]$	$\left[\frac{3}{4}, 1\right]$
Midpoint m_i	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{5}{8}$	$\frac{7}{8}$
Height $= (m_i)^2$	$\frac{1}{64}$	$\frac{9}{64}$	$\frac{25}{64}$	$\frac{49}{64}$
Area $= (1/4)(m_i)^2$	$\frac{1}{256}$	$\frac{9}{256}$	$\frac{25}{256}$	$\frac{49}{256}$

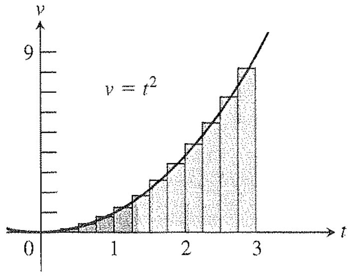


Figure 5.6 These rectangles have approximately the same areas as the strips in Figure 5.4. Each rectangle has height m_i^2, where m_i is the midpoint of its base. (Example 1)

Continuing in this manner, we derive the area $(1/4)(m_i)^2$ for each of the twelve subintervals and add them:

$$\frac{1}{256}+\frac{9}{256}+\frac{25}{256}+\frac{49}{256}+\frac{81}{256}+\frac{121}{256}+\frac{169}{256}+\frac{225}{256}+$$

$$\frac{289}{256}+\frac{361}{256}+\frac{441}{256}+\frac{529}{256}=\frac{2300}{256}\approx 8.98.$$

Since this number approximates the area and hence the total distance traveled by the particle, we conclude that the particle has moved approximately 9 units in 3 seconds. If it starts at $x = 0$, then it is very close to $x = 9$ when $t = 3$.

To make it easier to talk about approximations with rectangles, we now introduce some new terminology.

Approximation by Rectangles

Approximating irregularly-shaped regions by regularly-shaped regions for the purpose of computing areas is not new. Archimedes used the idea more than 2200 years ago to find the area of a circle, demonstrating in the process that π was located between 3.140845 and 3.142857. He also used approximations to find the area under a parabolic arch, anticipating the answer to an important seventeenth-century question nearly 2000 years before anyone thought to ask it. The fact that we still measure the area of anything—even a circle—in "square units" is obvious testimony to the historical effectiveness of using rectangles for approximating areas.

Rectangular Approximation Method (RAM)

In Example 1 we used the *Midpoint Rectangular Approximation Method (MRAM)* to approximate the area under the curve. The name suggests the choice we made when determining the heights of the approximating rectangles: We evaluated the function at the midpoint of each subinterval. If instead we had evaluated the function at the left-hand endpoint we would have obtained the *LRAM* approximation, and if we had used the right-hand endpoints we would have obtained the *RRAM* approximation. Figure 5.7 shows what the three approximations look like graphically when we approximate the area under the curve $y = x^2$ from $x = 0$ to $x = 3$ with six subintervals.

No matter which RAM approximation we compute, we are adding products of the form $f(x_i) \cdot \Delta x$, or, in this case, $(x_i)^2 \cdot (3/6)$.

LRAM:

$$\left(0\right)^2\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)+\left(1\right)^2\left(\frac{1}{2}\right)+\left(\frac{3}{2}\right)^2\left(\frac{1}{2}\right)+\left(2\right)^2\left(\frac{1}{2}\right)+\left(\frac{5}{2}\right)^2\left(\frac{1}{2}\right)=6.875$$

MRAM:

$$\left(\frac{1}{4}\right)^2\left(\frac{1}{2}\right)+\left(\frac{3}{4}\right)^2\left(\frac{1}{2}\right)+\left(\frac{5}{4}\right)^2\left(\frac{1}{2}\right)+\left(\frac{7}{4}\right)^2\left(\frac{1}{2}\right)+\left(\frac{9}{4}\right)^2\left(\frac{1}{2}\right)+\left(\frac{11}{4}\right)^2\left(\frac{1}{2}\right)=8.9375$$

RRAM:

$$\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)+\left(1\right)^2\left(\frac{1}{2}\right)+\left(\frac{3}{2}\right)^2\left(\frac{1}{2}\right)+\left(2\right)^2\left(\frac{1}{2}\right)+\left(\frac{5}{2}\right)^2\left(\frac{1}{2}\right)+\left(3\right)^2\left(\frac{1}{2}\right)=11.375$$

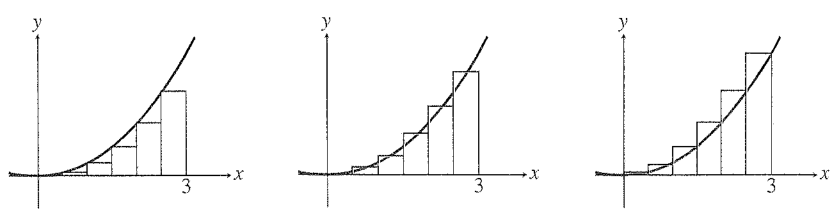


Figure 5.7 LRAM, MRAM, and RRAM approximations to the area under the graph of $y = x^2$ from $x = 0$ to $x = 3$.

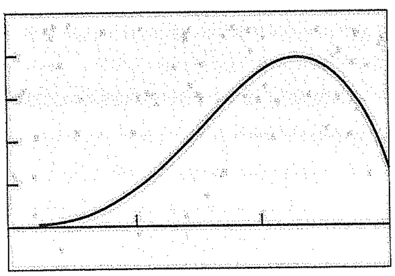
As we can see from Figure 5.7, LRAM is smaller than the true area and RRAM is larger. MRAM appears to be the closest of the three approximations. However, observe what happens as the number n of subintervals increases:

n	LRAM_n	MRAM_n	RRAM_n
6	6.875	8.9375	11.375
12	7.90625	8.984375	10.15625
24	8.4453125	8.99609375	9.5703125
48	8.720703125	8.999023438	9.283203125
100	8.86545	8.999775	9.13545
1000	8.9865045	8.99999775	9.0135045

We computed the numbers in this table using a graphing calculator and a summing program called RAM. A version of this program for most graphing calculators can be found in the *Technology Resource Manual* that accompanies this textbook. *All three sums* approach the same number (in this case, 9).

Example 2 ESTIMATING AREA UNDER THE GRAPH OF A NONNEGATIVE FUNCTION

Figure 5.8 shows the graph of $f(x) = x^2 \sin x$ on the interval $[0, 3]$. Estimate the area under the curve from $x = 0$ to $x = 3$.



[0, 3] by [–1, 5]

Figure 5.8 The graph of $y = x^2 \sin x$ over the interval $[0, 3]$. (Example 2)

Solution We apply our RAM program to get the numbers in this table.

n	LRAM_n	MRAM_n	RRAM_n
5	5.15480	5.89668	5.91685
10	5.52574	5.80685	5.90677
25	5.69079	5.78150	5.84320
50	5.73615	5.77788	5.81235
100	5.75701	5.77697	5.79511
1000	5.77476	5.77667	5.77857

It is not necessary to compute all three sums each time just to approximate the area, but we wanted to show again how all three sums approach the same number. With 1000 subintervals, all three agree in the first three digits. (The *exact* area is $-7 \cos 3 + 6 \sin 3 - 2$, which is 5.77666752456 to twelve digits.)

Exploration 1 **Which RAM is the Biggest?**

You might think from the previous two RAM tables that LRAM is always a little low and RRAM a little high, with MRAM somewhere in between. That, however, depends on n and on the shape of the curve.

1. Graph $y = 5 - 4\sin(x/2)$ in the window $[0, 3]$ by $[0, 5]$. Copy the graph on paper and sketch the rectangles for the LRAM, MRAM, and RRAM sums with $n = 3$. Order the three approximations from greatest to smallest.
2. Graph $y = 2\sin(5x) + 3$ in the same window. Copy the graph on paper and sketch the rectangles for the LRAM, MRAM, and RRAM sums with $n = 3$. Order the three approximations from greatest to smallest.
3. If a positive, continuous function is increasing on an interval, what can we say about the relative sizes of LRAM, MRAM, and RRAM? Explain.
4. If a positive, continuous function is decreasing on an interval, what can we say about the relative sizes of LRAM, MRAM, and RRAM? Explain.

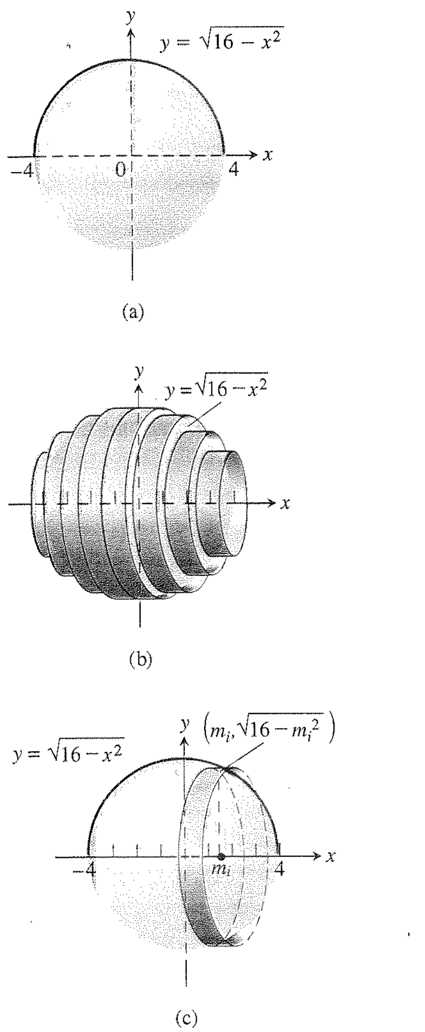


Figure 5.9 (a) The semicircle $y = \sqrt{16 - x^2}$ revolved about the x-axis to generate a sphere. (b) Slices of the solid sphere approximated with cylinders (drawn for $n = 8$). (c) The typical approximating cylinder has radius $f(m_i) = \sqrt{16 - m_i^2}$. (Example 3)

Volume of a Sphere

Although the visual representation of RAM approximation focuses on area, remember that our original motivation for looking at sums of this type was to find distance traveled by an object moving with a nonconstant velocity. The connection between Examples 1 and 2 is that in each case, we have a function f defined on a closed interval and estimate what we want to know with a sum of function values multiplied by interval lengths. Many other physical quantities can be estimated this way.

Example 3 ESTIMATING THE VOLUME OF A SPHERE

Estimate the volume of a solid sphere of radius 4.

Solution We picture the sphere as if its surface were generated by revolving the graph of the function $f(x) = \sqrt{16 - x^2}$ about the x-axis (Figure 5.9a). We partition the interval $-4 \le x \le 4$ into n subintervals of equal length $\Delta x = 8/n$. We then slice the sphere with planes perpendicular to the x-axis at the partition points, cutting it like a round loaf of bread into n parallel slices of width Δx. When n is large, each slice can be approximated by a cylinder, a familiar geometric shape of known volume, $\pi r^2 h$. In our case, the cylinders lie on their sides and h is Δx while r varies according to where we are on the x-axis (Figure 5.9b). A logical radius to choose for each cylinder is $f(m_i) = \sqrt{16 - m_i^2}$, where m_i is the midpoint of the interval where the i^{th} slice intersects the x-axis (Figure 5.9c).

We can now approximate the volume of the sphere by using MRAM to sum the cylinder volumes,

$$\pi r^2 h = \pi(\sqrt{16 - m_i^2})^2 \Delta x.$$

Keeping Track of Units

Notice in Example 3 that we are summing products of the form $\pi(16 - x^2)\times$ (a cross section area, measured in square units) times Δx (a length, measured in units). The products are therefore measured in cubic units, which are the correct units for volume.

The function we use in the RAM program is $\pi(\sqrt{16 - x^2})^2 = \pi(16 - x^2)$. The interval is $[-4, 4]$.

Number of Slices (n)	MRAM_n
10	269.42299
25	268.29704
50	268.13619
100	268.09598
1000	268.08271

The value for $n = 1000$ compares *very* favorably with the true volume,

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(4)^3 = \frac{256\pi}{3} \approx 268.0825731.$$

Even for $n = 10$ the difference between the MRAM approximation and the true volume is a small percentage of V:

$$\frac{|\text{MRAM}_{10} - V|}{V} = \frac{\text{MRAM}_{10} - 256\pi/3}{256\pi/3} \le 0.005.$$

That is, the error percentage is about one half of one percent!

Cardiac Output

So far we have seen applications of the RAM process to finding distance traveled and volume. These applications hint at the usefulness of this technique. To suggest its versatility we will present an application from human physiology.

The number of liters of blood your heart pumps in a fixed time interval is called your *cardiac output*. For a person at rest, the rate might be 5 or 6 liters per minute. During strenuous exercise the rate might be as high as 30 liters per minute. It might also be altered significantly by disease. How can a physician measure a patient's cardiac output without interrupting the flow of blood?

One technique is to inject a dye into a main vein near the heart. The dye is drawn into the right side of the heart and pumped through the lungs and out the left side of the heart into the aorta, where its concentration can be measured every few seconds as the blood flows past. The data in Table 5.2 and the plot in Figure 5.10 (obtained from the data) show the response of a healthy, resting patient to an injection of 5.6 mg of dye.

Table 5.2 Dye Concentration Data

Seconds after Injection t	Dye Concentration (adjusted for recirculation) c
5	0
7	3.8
9	8.0
11	6.1
13	3.6
15	2.3
17	1.45
19	0.91
21	0.57
23	0.36
25	0.23
27	0.14
29	0.09
31	0

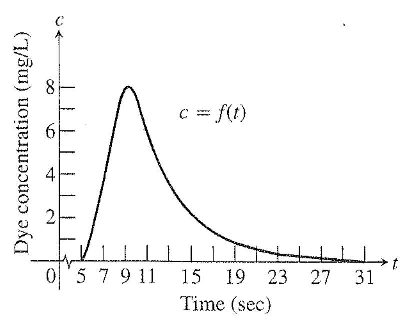


Figure 5.10 The dye concentration data from Table 5.2, plotted and fitted with a smooth curve. Time is measured with $t = 0$ at the time of injection. The dye concentration is zero at the beginning while the dye passes through the lungs. It then rises to a maximum at about $t = 9$ sec and tapers to zero by $t = 31$ sec.

Charles Richard Drew

1904–1950

Millions of people are alive today because of Charles Drew's pioneering work on blood plasma and the preservation of human blood for transfusion. After directing the Red Cross program that collected plasma for the Armed Forces in World War II, Dr. Drew went on to become Head of Surgery at Howard University and Chief of Staff at Freedmen's Hospital in Washington, D. C.

The graph shows dye concentration (measured in milligrams of liter of blood) as a function of time (in seconds). How can we use this obtain the cardiac output (measured in liters of blood per second)? T is to divide the *number of mg of dye* by the *area under the dye conce curve*. You can see why this works if you consider what happens to th

$$\frac{\text{mg of dye}}{\text{units of area under curve}} = \frac{\text{mg of dye}}{\dfrac{\text{mg of dye}}{\text{L of blood}} \cdot \text{sec}}$$

$$= \frac{\text{mg of dye}}{\text{sec}} \cdot \frac{\text{L of blood}}{\text{mg of dye}}$$

$$= \frac{\text{L of blood}}{\text{sec}}.$$

So you are now ready to compute like a cardiologist.

Example 4 COMPUTING CARDIAC OUTPUT FROM DYE CONCENTRATION

Estimate the cardiac output of the patient whose data appear in Table and Figure 5.10. Give the estimate in liters per minute.

Solution We have seen that we can obtain the cardiac output by divi the amount of dye (5.6 mg for our patient) by the area under the curv in Figure 5.10. Now we need to find the area. Our geometry formulas not apply to this irregularly shaped region, and the RAM program is u less without a formula for the function. Nonetheless, we can draw the MRAM rectangles ourselves and estimate their heights from the grapl In Figure 5.11 each rectangle has a base 2 units long and a height $f(m$ equal to the height of the curve above the midpoint of the base.

The area of each rectangle, then, is $f(m_i)$ times 2, and the sum of the : tangular areas is the MRAM estimate for the area under the curve:

$$\text{Area} \approx f(6) \cdot 2 + f(8) \cdot 2 + f(10) \cdot 2 + \cdots + f(28) \cdot 2$$
$$\approx 2 \cdot (1.4 + 6.3 + 7.5 + 4.8 + 2.8 + 1.9 + 1.1 + 0.7 + 0.5 + 0.3 + 0.2 + 0.1)$$
$$= 2 \cdot (27.6) = 55.2 \text{ (mg/L)} \cdot \text{sec}.$$

Dividing 5.6 mg by this figure gives an estimate for cardiac output in lit per second. Multiplying by 60 converts the estimate to liters per minute

$$\frac{5.6 \text{ mg}}{55.2 \text{ mg} \cdot \text{sec/L}} \cdot \frac{60 \text{ sec}}{1 \text{ min}} \approx 6.09 \text{ L/min}.$$

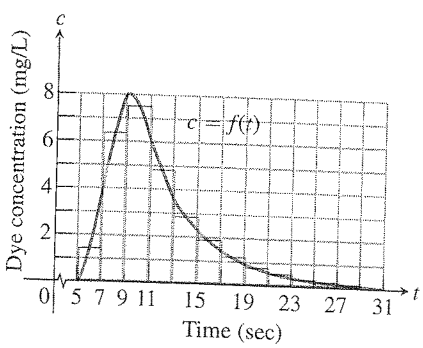


Figure 5.11 The region under the concentration curve of Figure 5.10 is approximated with rectangles. We ignore the portion from $t = 29$ to $t = 31$; its concentration is negligible. (Example 4)

Quick Review 5.1

As you answer the questions in Exercises 1–10, try to associate the answers with area, as in Figure 5.1.

1. A train travels at 80 mph for 5 hours. How far does it travel?
2. A truck travels at an average speed of 48 mph for 3 hours. How far does it travel?
3. Beginning at a standstill, a car maintains a constant acceleration of 10 ft/sec^2 for 10 seconds. What is its velocity after 10 seconds? Give your answer in ft/sec and then convert it to mi/h.
4. In a vacuum, light travels at a speed of 300,000 kilometers per second. How many kilometers does it travel in a year? (This distance equals one *light-year.*)
5. A long distance runner ran a race in 5 hours, averaging 6 mph for the first 3 hours and 5 mph for the last 2 hours. How far did she run?
6. A pump working at 20 gallons/minute pumps for an hc How many gallons are pumped?
7. At 8:00 P.M. the temperature began dropping at a rate of 1 degree Celsius per hour. Twelve hours later it began ri at a rate of 1.5 degrees per hour for six hours. What wa net change in temperature over the 18-hour period?
8. Water flows over a spillway at a steady rate of 300 cul feet per second. How many cubic feet of water pass ov the spillway in one day?
9. A city has a population density of 350 people per squai mile in an area of 50 square miles. What is the populat the city?
10. A hummingbird in flight beats its wings at a rate of 70 per second. How many times does it beat its wings in a hour if it is in flight 70% of the time?

Section 5.1 Exercises

Exercises 1–4 refer to the region R enclosed between the graph of the function $y = 2x - x^2$ and the x-axis for $0 \leq x \leq 2$.

1. **(a)** Sketch the region R.

   (b) Partition $[0, 2]$ into 4 subintervals and show the four rectangles that LRAM uses to approximate the area of R. Compute the LRAM sum without a calculator.
2. Repeat Exercise 1(b) for RRAM and MRAM.
3. Using a calculator program, find the RAM sums that complete the following table.

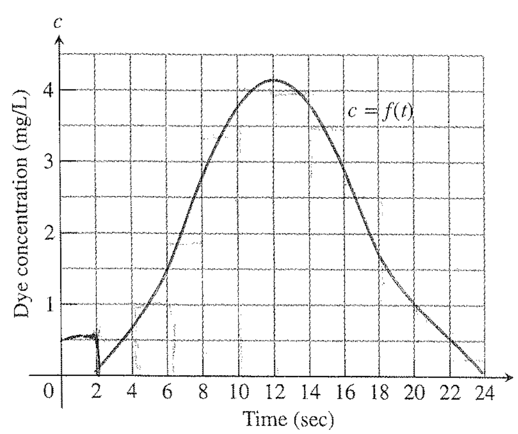
n	LRAM_n	MRAM_n	RRAM_n
10			
50			
100			
500			

4. Make a conjecture about the area of the region R.

In Exercises 5–8, use RAM to estimate the area of the region enclosed between the graph of f and the x-axis for $a \leq x \leq b$.

5. $f(x) = x^2 - x + 3, \quad a = 0, \quad b = 3$
6. $f(x) = \dfrac{1}{x}, \quad a = 1, \quad b = 3$
7. $f(x) = e^{-x^2}, \quad a = 0, \quad b = 2$
8. $f(x) = \sin x, \quad a = 0, \quad b = \pi$
9. *Cardiac Output* The following table gives dye concentrations for a dye-concentration cardiac-output determination like the one in Example 4. The amount of dye injected in this patient was 5 mg instead of 5.6 mg. Use rectangles to estimate the area under the dye concentra curve and then go on to estimate the patient's cardiac o

Seconds after Injection t	Dye Concentration (adjusted for recirculation) c
2	0
4	0.6
6	1.4
8	2.7
10	3.7
12	4.1
14	3.8
16	2.9
18	1.7
20	1.0
22	0.5
24	0



10. *Distance Traveled* The table below shows the velocity of a model train engine moving along a track for 10 sec. Estimate the distance traveled by the engine, using 10 subintervals of length 1 with **(a)** left-endpoint values (LRAM) and **(b)** right-endpoint values (RRAM).

Time (sec)	Velocity (in./sec)	Time (sec)	Velocity (in./sec)
0	0	6	11
1	12	7	6
2	22	8	2
3	10	9	6
4	5	10	0
5	13		

11. *Distance Traveled Upstream* You are walking along the bank of a tidal river watching the incoming tide carry a bottle upstream. You record the velocity of the flow every 5 minutes for an hour, with the results shown in the table below. About how far upstream does the bottle travel during that hour? Find the **(a)** LRAM and **(b)** RRAM estimates using 12 subintervals of length 5.

Time (min)	Velocity (m/sec)	Time (min)	Velocity (m/sec)
0	1	35	1.2
5	1.2	40	1.0
10	1.7	45	1.8
15	2.0	50	1.5
20	1.8	55	1.2
25	1.6	60	0
30	1.4		

12. *Length of a Road* You and a companion are driving along a twisty stretch of dirt road in a car whose speedometer works but whose odometer (mileage counter) is broken. To find out how long this particular stretch of road is, you record the car's velocity at 10-sec intervals, with the results shown in the table below. (The velocity was converted from mi/h to ft/sec using 30 mi/h = 44 ft/sec.) Estimate the length of the road by averaging the LRAM and RRAM sums.

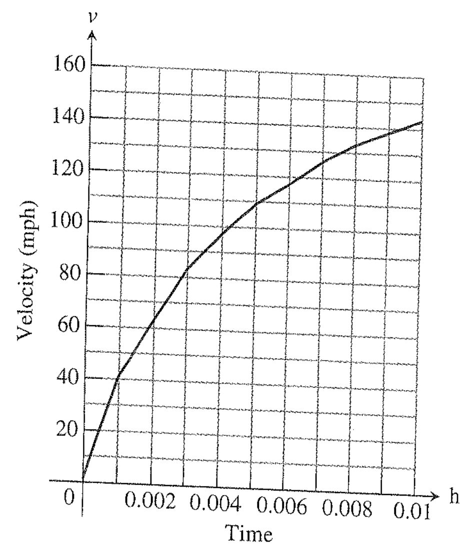
Time (sec)	Velocity (ft/sec)	Time (sec)	Velocity (ft/sec)
0	0	70	15
10	44	80	22
20	15	90	35
30	35	100	44
40	30	110	30
50	44	120	35
60	35		

13. *Distance from Velocity Data* The table below gi for the velocity of a vintage sports car accelerating 0 to 142 mi/h in 36 sec (10 thousandths of an hou

Time (h)	Velocity (mi/h)	Time (h)	Velocit (mi/h
0.0	0	0.006	116
0.001	40	0.007	125
0.002	62	0.008	132
0.003	82	0.009	137
0.004	96	0.010	142
0.005	108		

(a) Use rectangles to estimate how far the car travel during the 36 sec it took to reach 142 mi/h.

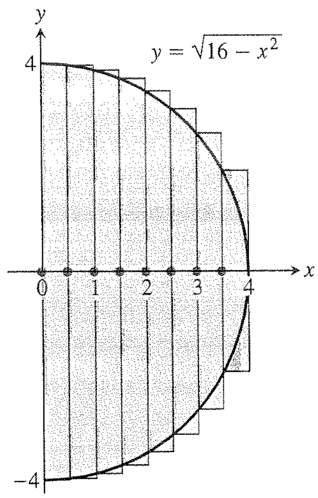
(b) Roughly how many seconds did it take the car t the halfway point? About how fast was the car goin



14. *(Continuation of Example 3)* Use the slicing techniqu Example 3 to find the MRAM sums that approximate volume of a sphere of radius 5. Use $n = 10, 20, 40,$ and 160.

15. *(Continuation of Exercise 14)* Use a geometry formul find the volume V of the sphere in Exercise 14 and fin the error and **(b)** the percentage error in the MRAM approximation for each value of n given.

16. *Volume of a Solid Hemisphere* To estimate the volume of a solid hemisphere of radius 4, imagine its axis of symmetry to be the interval [0, 4] on the x-axis. Partition [0, 4] into eight subintervals of equal length and approximate the solid with cylinders based on the circular cross sections of the hemisphere perpendicular to the x-axis at the subintervals' left endpoints. (See the accompanying profile view.)



(a) Writing to Learn Find the sum S_8 of the volumes of the cylinders. Do you expect S_8 to overestimate V? Give reasons for your answer.

(b) Express $|V - S_8|$ as a percentage of V to the nearest percent.

17. Repeat Exercise 16 using cylinders based on cross sections at the right endpoints of the subintervals.

18. *Volume of Water in a Reservoir* A reservoir shaped like a hemispherical bowl of radius 8 m is filled with water to a depth of 4 m.

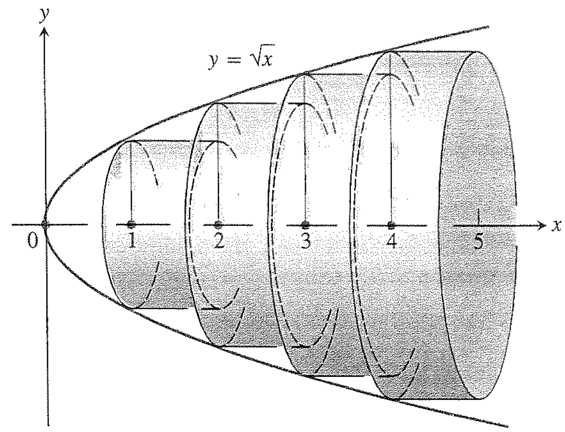
(a) Find an estimate S of the water's volume by approximating the water with eight circumscribed solid cylinders.

(b) It can be shown that the water's volume is $V = (320\pi)/3$ m^3. Find the error $|V - S|$ as a percentage of V to the nearest percent.

19. *Volume of Water in a Swimming Pool* A rectangular swimming pool is 30 ft wide and 50 ft long. The table below shows the depth $h(x)$ of the water at 5-ft intervals from one end of the pool to the other. Estimate the volume of water in the pool using **(a)** left-endpoint values and **(b)** right-endpoint values.

Position (ft) x	Depth (ft) $h(x)$	Position (ft) x	Depth (ft) $h(x)$
0	6.0	30	11.5
5	8.2	35	11.9
10	9.1	40	12.3
15	9.9	45	12.7
20	10.5	50	13.0
25	11.0		

20. *Volume of a Nose Cone* The nose "cone" of a rocket is a *paraboloid* obtained by revolving the curve $y = \sqrt{.}$ $0 \le x \le 5$ about the x-axis, where x is measured in fee Estimate the volume V of the nose cone by partitioning [0, 5] into five subintervals of equal length, slicing the c with planes perpendicular to the x-axis at the subinterva left endpoints, constructing cylinders of height 1 based cross sections at these points, and finding the volumes c these cylinders. (See the accompanying figure.)



21. *Volume of a Nose Cone* Repeat Exercise 20 using cylinders based on cross sections at the *midpoints* of the subintervals.

22. *Free Fall with Air Resistance* An object is dropped straight down from a helicopter. The object falls faster faster but its acceleration (rate of change of its velocity decreases over time because of air resistance. The acceleration is measured in ft/sec^2 and recorded every second after the drop for 5 sec, as shown in the table b

t	0	1	2	3	4	5
a	32.00	19.41	11.77	7.14	4.33	2.63

(a) Use LRAM$_5$ to find an upper estimate for the speed when $t = 5$.

(b) Use RRAM$_5$ to find a lower estimate for the speed when $t = 5$.

(c) Use upper estimates for the speed during the first se second second, and third second to find an upper estima for the distance fallen when $t = 3$.

23. *Distance Traveled by a Projectile* An object is shot straight upward from sea level with an initial velocity c 400 ft/sec.

(a) Assuming gravity is the only force acting on the obj give an upper estimate for its velocity after 5 sec have elapsed. Use $g = 32$ ft/sec^2 for the gravitational const

(b) Find a lower estimate for the height attained after 5

24. *Water Pollution* Oil is leaking out of a tanker damaged at sea. The damage to the tanker is worsening as evidenced by the increased leakage each hour, recorded in the table below.

Time (h)	0	1	2	3	4
Leakage (gal/h)	50	70	97	136	190

Time (h)	5	6	7	8
Leakage (gal/h)	265	369	516	720

(a) Give an upper and lower estimate of the total quantity of oil that has escaped after 5 hours.

(b) Repeat (a) for the quantity of oil that has escaped after 8 hours.

(c) The tanker continues to leak 720 gal/h after the first 8 hours. If the tanker originally contained 25,000 gal of oil, approximately how many more hours will elapse in the worst case before all of the oil has leaked? in the best case?

25. *Air Pollution* A power plant generates electricity by burning oil. Pollutants produced by the burning process are removed by scrubbers in the smokestacks. Over time the scrubbers become less efficient and eventually must be replaced when the amount of pollutants released exceeds government standards. Measurements taken at the end of each month determine the rate at which pollutants are released into the atmosphere as recorded in the table below.

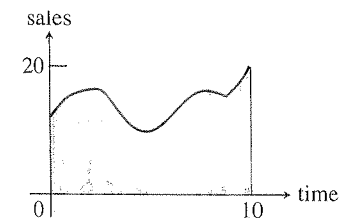
Month	Jan	Feb	Mar	Apr	May	Jun
Pollutant Release Rate (tons/day)	0.20	0.25	0.27	0.34	0.45	0.52

Month	Jul	Aug	Sep	Oct	Nov	Dec
Pollutant Release Rate (tons/day)	0.63	0.70	0.81	0.85	0.89	0.95

(a) Assuming a 30-day month and that new scrubbers allow only 0.05 ton/day released, give an upper estimate of the total tonnage of pollutants released by the end of June. What is a lower estimate?

(b) In the best case, approximately when will a total of 125 tons of pollutants have been released into the atmosphere?

26. Writing to Learn The graph shows the sales record f company over a 10-year period. If sales are measured i millions of units per year, explain what information can obtained from the area of the region, and why.



Exploration

27. *Area of a Circle* *Work in groups of two or three.* Inscril a regular n-sided polygon inside a circle of radius 1 and compute the area of the polygon for the following values

(a) 4 (square) **(b)** 8 (octagon) **(c)** 16

(d) Compare the areas in parts (a), (b), and (c) with the of the circle.

Extending the Ideas

28. *Rectangular Approximation Methods* Prove or disp the following statement: MRAM_n is always the average LRAM_n and RRAM_n.

29. *Rectangular Approximation Methods* Show that if is a nonnegative function on the interval $[a, b]$ and the line $x = (a + b)/2$ is a line of symmetry of the graph $y = f(x)$, then $\text{LRAM}_n f = \text{RRAM}_n f$ for every posit integer n.

30. *(Continuation of Exercise 27)*

(a) Inscribe a regular n-sided polygon inside a circle of radius 1 and compute the area of one of the n congruen triangles formed by drawing radii to the vertices of the polygon.

(b) Compute the limit of the area of the inscribed polyg as $n \to \infty$.

(c) Repeat the computations in (a) and (b) for a circle o radius r.

5.2 Definite Integrals

Riemann Sums • Terminology and Notation of Integration • Definite Integral and Area • Constant Functions • Integrals on Calculator • Discontinuous Integrable Functions

Riemann Sums

In the preceding section, we estimated distances, areas, and volumes with f sums. The terms in the sums were obtained by multiplying selected fun values by the lengths of intervals. In this section we move beyond finite s to see what happens in the limit, as the terms become infinitely small and number infinitely large.

Sigma notation enables us to express a large sum in compact form:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n.$$

The Greek capital letter Σ (sigma) stands for "sum." The index k tells us w to begin the sum (at the number below the Σ) and where to end (at the nu above). If the symbol ∞ appears above the Σ, it indicates that the terms g indefinitely.

The sums in which we will be interested are called *Riemann* ("*ree*-ma *sums,* after Georg Friedrich Bernhard Riemann (1826–1866). LRAM, MR and RRAM in the previous section are all examples of Riemann sums- because they estimated area, but because they were constructed in a parti way. We now describe that construction formally, in a more general contex does not confine us to nonnegative functions.

We begin with an arbitrary continuous function $f(x)$ defined on a cl interval $[a, b]$. Like the function graphed in Figure 5.12, it may have neg values as well as positive values.

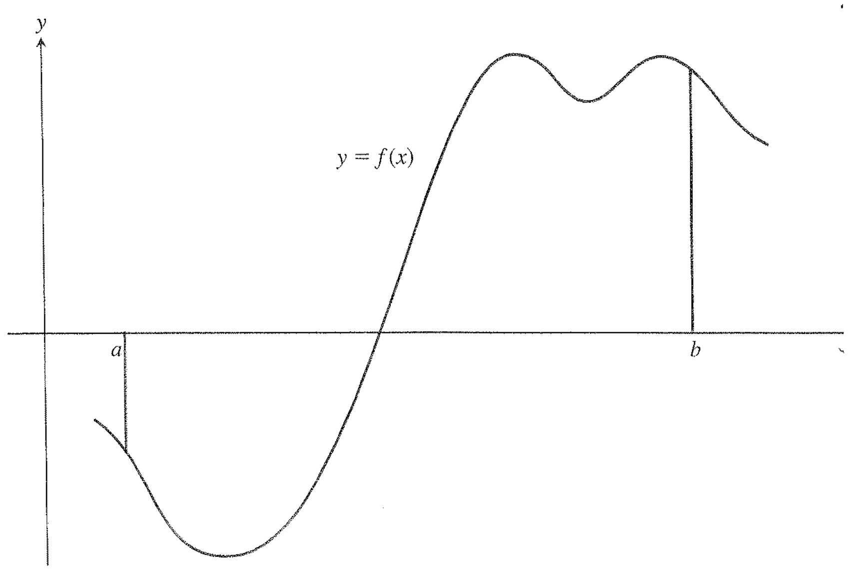


Figure 5.12 The graph of a typical function $y = f(x)$ over a closed interval $[a,$

We then partition the interval $[a, b]$ into n subintervals by choosing n points, say $x_1, x_2, \cdots, x_{n-1}$, between a and b subject only to the condition t

$$a < x_1 < x_2 < \cdots < x_{n-1} < b.$$

To make the notation consistent, we denote a by x_0 and b by x_n. The set

$$P = \{x_0, x_1, x_2, \cdots, x_n\}$$

is called a **partition** of $[a, b]$.

The partition P determines n closed **subintervals,** as shown in Fi 5.13. The k^{th} subinterval is $[x_{k-1}, x_k]$, which has length $\Delta x_k = x_k - x_{k-1}$.

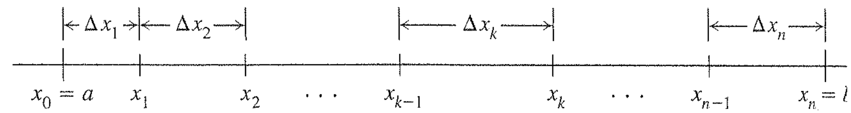


Figure 5.13 The partition $P = \{a = x_0, x_1, x_2, \cdots, x_n = b\}$ divides $[a, b]$ into n subintervals of lengths $\Delta x_1, \Delta x_2, \cdots, \Delta x_n$. The k^{th} subinterval has length Δx_k.

In each subinterval we select some number. Denote the number ch from the k^{th} subinterval by c_k.

Then, on each subinterval we stand a vertical rectangle that reaches 1 the x-axis to touch the curve at $(c_k, f(c_k))$. These rectangles could lie e above or below the x-axis (Figure 5.14).

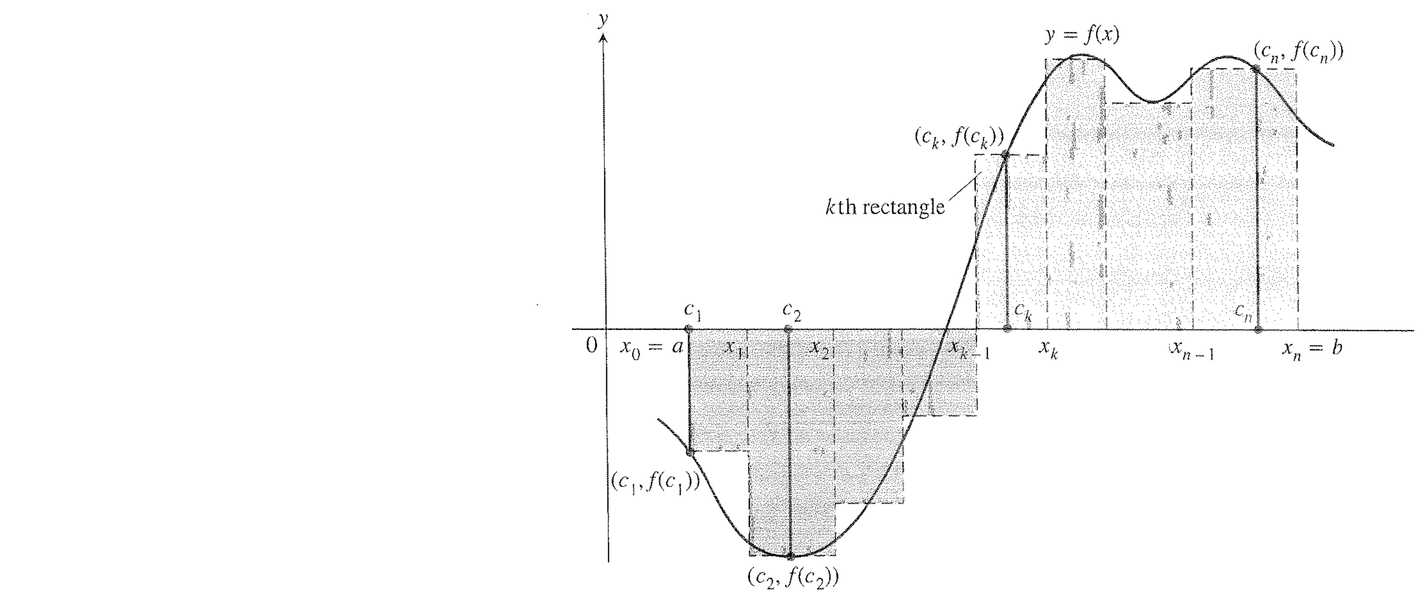


Figure 5.14 Rectangles extending from the x-axis to intersect the curve at the points $(c_k, f(c_k))$. The rectangles approximate the region between the x-axis and graph of the function.

On each subinterval, we form the product $f(c_k) \cdot \Delta x_k$. This product c positive, negative, or zero, depending on $f(c_k)$.

Finally, we take the sum of these products:

$$S_n = \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k.$$

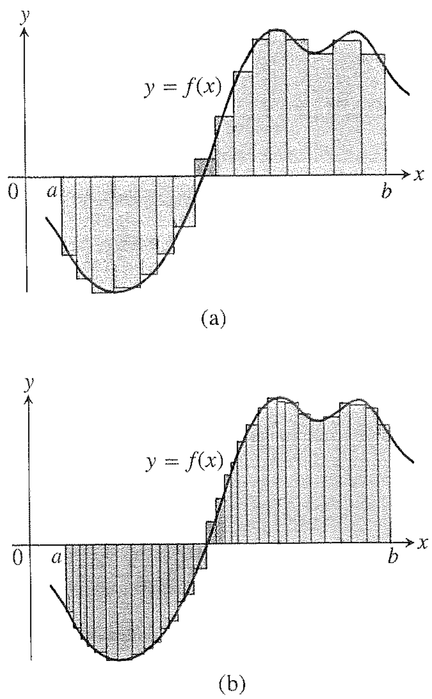


Figure 5.15 The curve of Figure 5.12 with rectangles from finer partitions of $[a, b]$. Finer partitions create more rectangles, with shorter bases.

This sum, which depends on the partition P and the choice of the numbe is a **Riemann sum for f on the interval $[a, b]$.**

As the partitions of $[a, b]$ become finer and finer, we would expe rectangles defined by the partitions to approximate the region betwee x-axis and the graph of f with increasing accuracy (Figure 5.15).

Just as LRAM, MRAM, and RRAM in our earlier examples converg a common value in the limit, *all* Riemann sums for a given function on | converge to a common value, as long as the lengths of the subintervals al to zero. This latter condition is assured by requiring the longest subint length (called the **norm** of the partition and denoted by $||P||$) to tend to

Definition The Definite Integral as a Limit of Riemann Sum

Let f be a function defined on a closed interval $[a, b]$. For any partition P of $[a, b]$, let the numbers c_k be chosen arbitrarily in the subintervals $[x_{k-1}, x_k]$.

If there exists a number I such that

$$\lim_{||P||\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = I$$

no matter how P and the c_k's are chosen, then f is **integrable** on $[a,$ and I is the **definite integral** of f over $[a, b]$.

Despite the potential for variety in the sums $\sum f(c_k)\Delta x_k$ as the part change and as the c_k's are chosen arbitrarily in the intervals of each part the sums always have the same limit as $||P||\to 0$ as long as f is *conti* on $[a, b]$.

Theorem 1 The Existence of Definite Integrals

All continuous functions are integrable. That is, if a function f is continuous on an interval $[a, b]$, then its definite integral over $[a, b]$ exist

Georg Riemann (1826–1866)

The mathematicians of the 17th and 18th centuries blithely assumed the existence of limits of Riemann sums (as we admittedly did in our RAM explorations of the last section), but the existence was not established mathematically until Georg Riemann proved Theorem 1 in 1854. You can find a current version of Riemann's proof in most advanced calculus books.

Because of Theorem 1, we can get by with a simpler construction fo inite integrals of continuous functions. Since we know for these function the Riemann sums tend to the same limit for *all* partitions in which $||P$ we need only to consider the limit of the so-called **regular partitio** which all the subintervals have the same length.

The Definite Integral of a Continuous Function on $[a, b]$

Let f be continuous on $[a, b]$, and let $[a, b]$ be partitioned into n subintervals of equal length $\Delta x = (b - a)/n$. Then the definit integral of f over $[a, b]$ is given by

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x,$$

where each c_k is chosen arbitrarily in the k^{th} subinterval.

Terminology and Notation of Integration

Leibniz's clever choice of notation for the derivative, dy/dx, had the adv of retaining an identity as a "fraction" even though both numerator and d inator had tended to zero. Although not really fractions, derivatives can l like fractions, so the notation makes profound results like the Chain Rul

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

seem almost simple.

The notation that Leibniz introduced for the definite integral was e inspired. In his derivative notation, the Greek letters ("Δ" for "differe switch to Roman letters ("d" for "differential") in the limit,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

In his definite integral notation, the Greek letters again become R letters in the limit,

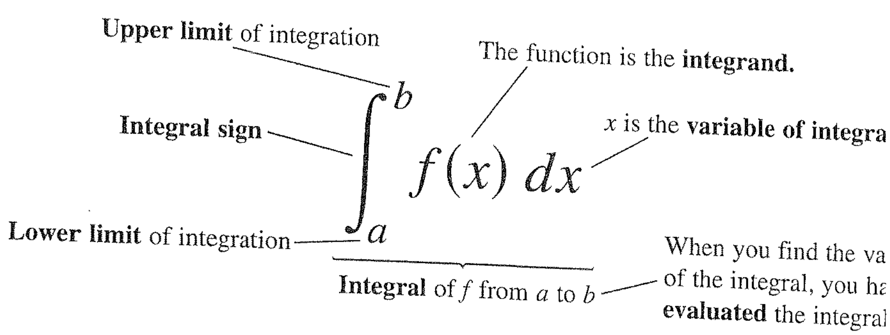
$$\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k)\Delta x = \int_a^b f(x)\, dx.$$

Notice that the difference Δx has again tended to zero, becoming a d ential dx. The Greek "Σ" has become an elongated Roman "S," so thɛ integral can retain its identity as a "sum." The c_k's have become so cro together in the limit that we no longer think of a choppy selection of x v between a and b, but rather of a continuous, unbroken sampling of x v from a to b. It is as if we were summing *all* products of the form $f(x)\, dx$ goes from a to b, so we can abandon the k and the n used in the finite expression.

The symbol

$$\int_a^b f(x)\, dx$$

is read as "the integral from a to b of f of x dee x," or sometimes as "the i gral from a to b of f of x with respect to x." The component parts also l names:



The value of the definite integral of a function over any particular inter depends on the function and not on the letter we choose to represent its in pendent variable. If we decide to use t or u instead of x, we simply write integral as

$$\int_a^b f(t)\, dt \quad \text{or} \quad \int_a^b f(u)\, du \quad \text{instead of} \quad \int_a^b f(x)\, dx.$$

No matter how we represent the integral, it is the same *number,* defined as a of Riemann sums. Since it does not matter what letter we use to run from a the variable of integration is called a **dummy variable.**

Example 1 USING THE NOTATION

The interval $[-1, 3]$ is partitioned into n subintervals of equal length $\Delta x = 4/n$. Let m_k denote the midpoint of the k^{th} subinterval. Express the limit

$$\lim_{n \to \infty} \sum_{k=1}^{n} (3(m_k)^2 - 2m_k + 5)\, \Delta x$$

as an integral.

Solution Since the midpoints m_k have been chosen from the subinterv of the partition, this expression is indeed a limit of Riemann sums. (Th points chosen did not have to be midpoints; they could have been chose from the subintervals in any arbitrary fashion.) The function being integrated is $f(x) = 3x^2 - 2x + 5$ over the interval $[-1, 3]$. Therefore,

$$\lim_{n \to \infty} \sum_{k=1}^{n} (3(m_k)^2 - 2m_k + 5)\, \Delta x = \int_{-1}^{3} (3x^2 - 2x + 5)\, dx.$$

Definite Integral and Area

If an integrable function $y = f(x)$ is nonnegative throughout an interval [each nonzero term $f(c_k)\Delta x_k$ is the area of a rectangle reaching from the x up to the curve $y = f(x)$. (See Figure 5.16.)

The Riemann sum

$$\sum f(c_k)\, \Delta x_k,$$

which is the sum of the areas of these rectangles, gives an estimate of the of the region between the curve and the x-axis from a to b. Since the re gles give an increasingly good approximation of the region as we use parti with smaller and smaller norms, we call the limiting value the area unde curve.

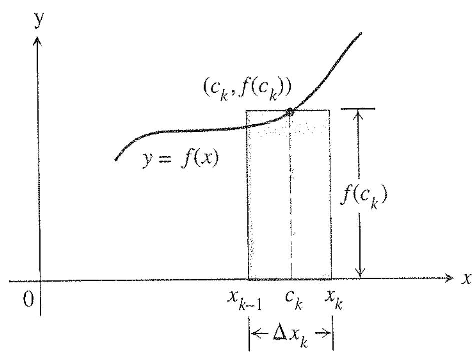


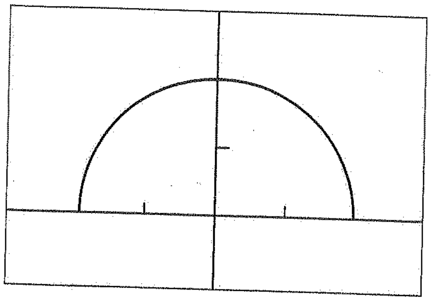
Figure 5.16 A term of a Riemann sum $\sum f(c_k)\Delta x_k$ for a nonnegative function f is either zero or the area of a rectangle such as the one shown.

Definition Area Under a Curve (as a Definite Integral)

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b$ then the **area under the curve $y = f(x)$ from a to b** is the integral c f from a to b,

$$A = \int_a^b f(x)\, dx.$$

This definition works both ways: We can use integrals to calculate *and* we can use areas to calculate integrals.



[−3, 3] by [−1, 3]

Figure 5.17 A square viewing window on $y = \sqrt{4 - x^2}$. The graph is a semicircle because $y = \sqrt{4 - x^2}$ is the same as $y^2 = 4 - x^2$, or $x^2 + y^2 = 4$, with $y \geq 0$. (Example 2)

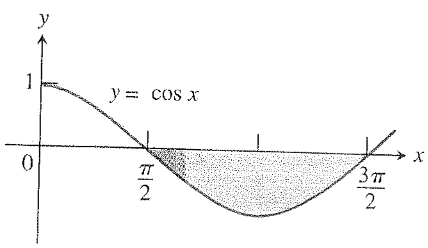


Figure 5.18 Because $f(x) = \cos x$ is nonpositive on $[\pi/2, 3\pi/2]$, the integral of f is a negative number. The area of the shaded region is the opposite of this integral,

$$\text{Area} = -\int_{\pi/2}^{3\pi/2} \cos x\, dx.$$

Example 2 REVISITING AREA UNDER A CURVE

Evaluate the integral $\int_{-2}^{2} \sqrt{4 - x^2}\, dx$.

Solution We recognize $f(x) = \sqrt{4 - x^2}$ as a function whose graph semicircle of radius 2 centered at the origin (Figure 5.17).

The area between the semicircle and the x-axis from -2 to 2 can be co puted using the geometry formula

$$\text{Area} = \frac{1}{2} \cdot \pi r^2 = \frac{1}{2} \cdot \pi (2)^2 = 2\pi.$$

Because the area is also the value of the integral of f from -2 to 2,

$$\int_{-2}^{2} \sqrt{4 - x^2}\, dx = 2\pi.$$

If an integrable function $y = f(x)$ is nonpositive, the nonzero t $f(c_k)\Delta x_k$ in the Riemann sums for f over an interval $[a, b]$ are negativ rectangle areas. The limit of the sums, the integral of f from a to b, is there the *negative* of the area of the region between the graph of f and the x- (Figure 5.18).

$$\int_a^b f(x)\, dx = -(\text{the area}) \quad \text{if} \quad f(x) \leq 0.$$

Or, turning this around,

$$\text{Area} = -\int_a^b f(x)\, dx \quad \text{when} \quad f(x) \leq 0.$$

If an integrable function $y = f(x)$ has both positive and negative value an interval $[a, b]$, then the Riemann sums for f on $[a, b]$ add areas of rectan that lie above the x-axis to the negatives of areas of rectangles that lie below x-axis, as in Figure 5.19. The resulting cancellations mean that the limit value is a number whose magnitude is less than the total area between the cu and the x-axis. The value of the integral is the area above the x-axis minus area below.

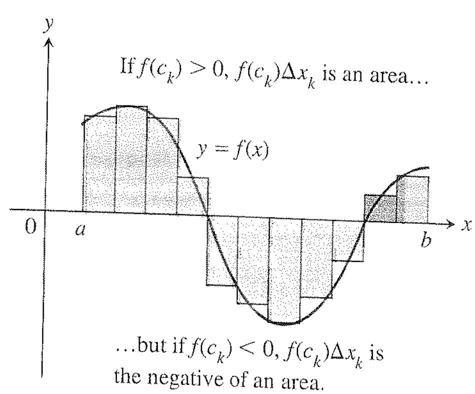


Figure 5.19 An integrable function f with negative as well as positive values.

Net Area

Sometimes $\int_a^b f(x)\,dx$ is called the *net area* of the region determined by the curve $y = f(x)$ and the x-axis between $x = a$ and $x = b$.

For any integrable function,

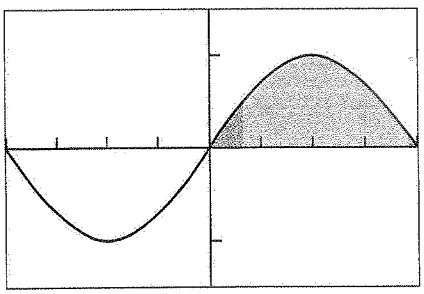
$$\int_a^b f(x)\,dx = (\text{area above the } x\text{-axis}) - (\text{area below the } x\text{-axis}).$$

Exploration 1 **Finding Integrals by Signed Areas**

It is a fact (which we will revisit) that $\int_0^{\pi} \sin x\,dx = 2$ (Figure 5.20
With that information, what you know about integrals and areas, whi
you know about graphing curves, and sometimes a bit of intuition, determine the values of the following integrals. Give as convincing an argument as you can for each value, based on the graph of the function.

1. $\int_{\pi}^{2\pi} \sin x\,dx$
2. $\int_0^{2\pi} \sin x\,dx$
3. $\int_0^{\pi/2} \sin x\,dx$
4. $\int_0^{\pi} (2 + \sin x)\,dx$
5. $\int_0^{\pi} 2 \sin x\,dx$
6. $\int_2^{\pi+2} \sin (x - 2)$
7. $\int_{-\pi}^{\pi} \sin u\,du$
8. $\int_0^{2\pi} \sin (x/2)\,dx$
9. $\int_0^{\pi} \cos x\,dx$
10. Suppose k is *any* positive number. Make a conjecture about $\int_{-k}^{k} \sin x\,dx$ and support your conjecture.

$y = \sin x$



$[-\pi, \pi]$ by $[-1.5, 1.5]$

Figure 5.20 $\int_0^{\pi} \sin x\,dx = 2$. (Exploration 1)

Constant Functions

Integrals of constant functions are easy to evaluate. Over a closed interva
are simply the constant times the length of the interval (Figure 5.21).

Theorem 2 The Integral of a Constant

If $f(x) = c$, where c is a constant, on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = \int_a^b c\,dx = c(b - a).$$

Proof A constant function is continuous, so the integral exists, a
can evaluate it as a limit of Riemann sums with subintervals of equal $(b - a)/n$. Any such sum looks like

$$\sum_{k=1}^{n} f(c_k) \cdot \Delta x, \quad \text{which is} \quad \sum_{k=1}^{n} c \cdot \frac{b - a}{n}.$$

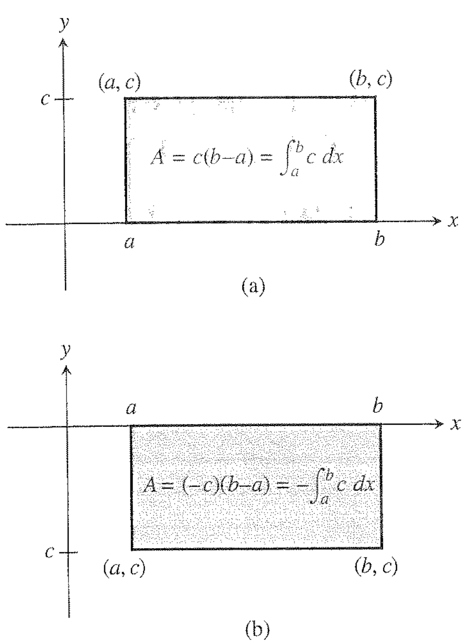


Figure 5.21 (a) If c is a positive constant, then $\int_a^b c\,dx$ is the area of the rectangle shown. (b) If c is negative, then $\int_a^b c\,dx$ is the opposite of the area of the rectangle.

Then

$$\sum_{k=1}^{n} c \cdot \frac{b-a}{n} = c \cdot (b-a) \sum_{k=1}^{n} \frac{1}{n}$$
$$= c(b-a) \cdot n\left(\frac{1}{n}\right)$$
$$= c(b-a).$$

Since the sum is *always* $c(b-a)$ for any value of n, it follows that the lim the sums, the integral to which they converge, is also $c(b-a)$.

Example 3 REVISITING THE TRAIN PROBLEM

A train moves along a track at a steady 75 miles per hour from 7:00 A.M 9:00 A.M. Express its total distance traveled as an integral. Evaluate the integral using Theorem 2.

Solution (See Figure 5.22.)

$$\text{Distance traveled} = \int_7^9 75\, dt = 75 \cdot (9-7) = 150$$

Since the 75 is measured in miles/hour and the $(9-7)$ is measured in hours, the 150 is measured in miles. The train traveled 150 miles.

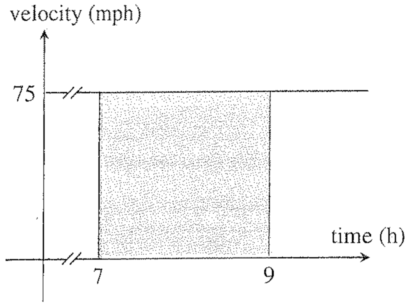


Figure 5.22 The area of the rectangle is a special case of Theorem 2. (Example 3)

Integrals on a Calculator

You do not have to know much about your calculator to realize that finding limit of a Riemann sum is exactly the kind of thing that it does best. We h seen how effectively it can approximate areas using MRAM, but most moc calculators have sophisticated built-in programs that converge to integrals v much greater speed and precision than that. We will assume that your calcul has such a numerical integration capability, which we will denote as **NINT** particular, we will use NINT $(f(x), x, a, b)$ to denote a calculator (or compu approximation of $\int_a^b f(x)\,dx$. When we write

$$\int_a^b f(x)\,dx = \text{NINT}\,(f(x), x, a, b),$$

we do so with the understanding that the right-hand side of the equation is approximation of the left-hand side.

Example 4 USING NINT

Evaluate the following integrals numerically.

(a) $\int_{-1}^{2} x \sin x\, dx$ (b) $\int_0^1 \frac{4}{1+x^2}\,dx$ (c) $\int_0^5 e^{-x^2}\,dx$

Solution

(a) NINT $(x \sin x, x, -1, 2) \approx 2.04$

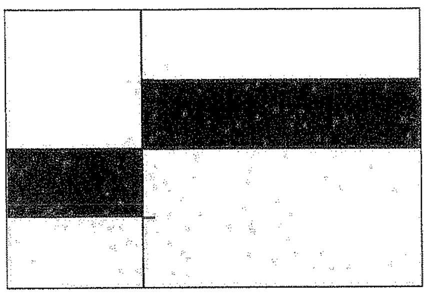
(b) NINT $(4/(1+x^2), x, 0, 1) \approx 3.14$

(c) NINT $(e^{-x^2}, x, 0, 5) \approx 0.89$

Bounded Functions

We say a function is *bounded* on a given domain if its range is confined between some minimum value m and some maximum value M. That is, given any x in the domain, $m \le f(x) \le M$. Equivalently, the graph of $y = f(x)$ lies between the horizontal lines $y = m$ and $y = M$.

$y = |x|/x$



[−1, 2] by [−2, 2]

Figure 5.23 A discontinuous integrable function:
$$\int_{-1}^{2} \frac{|x|}{x}\,dx =$$
−(area below x-axis) + (area above x-axis). (Example 5)

We will eventually be able to confirm that the exact value for Examp is $-2\cos 2 + \sin 2 - \cos 1 + \sin 1$. You might want to conjecture for self what the exact answer to Example 4b might be. As for Example 4 explicit *exact* value has ever been found for this integral! The best we ca in this case (and in many like it) is to approximate the integral numeri Here, technology is not only useful, it is essential.

Discontinuous Integrable Functions

Theorem 1 guarantees that all continuous functions are integrable. But functions with discontinuities are also integrable. For example, a bou function (see margin note) that has a finite number of points of disconti on an interval $[a, b]$ will still be integrable on the interval if it is contir everywhere else.

Example 5 INTEGRATING A DISCONTINUOUS FUNCTION

Find $\displaystyle\int_{-1}^{2} \frac{|x|}{x}\,dx.$

Solution This function has a discontinuity at $x = 0$, where the grapl jumps from $y = -1$ to $y = 1$. The graph, however, determines two rectangles, one below the x-axis and one above (Figure 5.23).

Using the idea of net area, we have

$$\int_{-1}^{2} \frac{|x|}{x}\,dx = -1 + 2 = 1.$$

Exploration 2 **More Discontinuous Integrands**

1. Explain why the function
$$f(x) = \frac{x^2 - 4}{x - 2}$$
is not continuous on $[0, 3]$. What kind of discontinuity occurs?
2. Use areas to show that
$$\int_{0}^{3} \frac{x^2 - 4}{x - 2}\,dx = 10.5.$$
3. Use areas to show that
$$\int_{0}^{5} \operatorname{int}(x)\,dx = 10.$$

A Nonintegrable Function

How "bad" does a function have to be before it is *not* integrable? One way to defeat integrability is to be unbounded (like $y = 1/x$ near $x = 0$), which can prevent the Riemann sums from tending to a finite limit. Another, more subtle, way is to be bounded but badly discontinuous, like the *characteristic function of the rationals:*

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

No matter what partition we take of the closed interval $[0, 1]$, every subinterval contains both rational and irrational numbers. That means that we can always form a Riemann sum with all rational c_k's (a Riemann sum of 1) or all irrational c_k's (a Riemann sum of 0). The sums can therefore never tend toward a unique limit.

Quick Review 5.2

In Exercises 1–3, evaluate the sum.

1. $\sum_{n=1}^{5} n^2$

2. $\sum_{k=0}^{4} (3k - 2)$

3. $\sum_{j=0}^{4} 100\,(j + 1)^2$

In Exercises 4–6, write the sum in sigma notation.

4. $1 + 2 + 3 + \cdots + 98 + 99$

5. $0 + 2 + 4 + \cdots + 48 + 50$

6. $3(1)^2 + 3(2)^2 + \cdots + 3(500)^2$

In Exercises 7 and 8, write the expression as a single sum in sigma notation.

7. $2\sum_{x=1}^{50} x^2 + 3\sum_{x=1}^{50} x$

8. $\sum_{k=0}^{8} x^k + \sum_{k=9}^{20} x^k$

9. Find $\sum_{k=0}^{n} (-1)^k$ if n is odd.

10. Find $\sum_{k=0}^{n} (-1)^k$ if n is even.

Section 5.2 Exercises

In Exercises 1–6, express the limit as a definite integral.

1. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} c_k{}^2 \Delta x_k$, where P is any partition of $[0, 2]$

2. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (c_k{}^2 - 3c_k)\,\Delta x_k$, where P is any partition of $[-7, 5]$

3. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \frac{1}{c_k}\,\Delta x_k$, where P is any partition of $[1, 4]$

4. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \frac{1}{1 - c_k}\,\Delta x_k$, where P is any partition of $[2, 3]$

5. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \sqrt{4 - c_k{}^2}\,\Delta x_k$, where P is any partition of $[0, 1]$

6. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (\sin^3 c_k)\,\Delta x_k$, where P is any partition of $[-\pi, \pi]$

In Exercises 7–12, evaluate the integral.

7. $\int_{-2}^{1} 5\,dx$

8. $\int_{3}^{7} (-20)\,dx$

9. $\int_{0}^{3} (-160)\,dt$

10. $\int_{-4}^{-1} \frac{\pi}{2}\,d\theta$

11. $\int_{-2.1}^{3.4} 0.5\,ds$

12. $\int_{\sqrt{2}}^{\sqrt{18}} \sqrt{2}\,dr$

In Exercises 13–22, use the graph of the integrand and areas to evaluate the integral.

13. $\int_{-2}^{4} \left(\frac{x}{2} + 3\right) dx$

14. $\int_{1/2}^{3/2} (-2x + 4)\,dx$

15. $\int_{-3}^{3} \sqrt{9 - x^2}\,dx$

16. $\int_{-4}^{0} \sqrt{16 - x^2}\,dx$

17. $\int_{-2}^{1} |x|\,dx$

18. $\int_{-1}^{1} (1 - |x|)\,dx$

19. $\int_{-1}^{1} (2 - |x|)\,dx$

20. $\int_{-1}^{1} (1 + \sqrt{1 - x^2})\,d$:

21. $\int_{\pi}^{2\pi} \theta\,d\theta$

22. $\int_{\sqrt{2}}^{5\sqrt{2}} r\,dr$

In Exercises 23–28, use areas to evaluate the integral.

23. $\int_{0}^{b} x\,dx, \quad b > 0$

24. $\int_{0}^{b} 4x\,dx, \quad b > 0$

25. $\int_{a}^{b} 2s\,ds, \quad 0 < a < b$

26. $\int_{a}^{b} 3t\,dt, \quad 0 < a < b$

27. $\int_{a}^{2a} x\,dx, \quad a > 0$

28. $\int_{a}^{\sqrt{3}a} x\,dx, \quad a > 0$

In Exercises 29–38, *work in groups of two or three.* Use grap your knowledge of area, and the fact that

$$\int_{0}^{1} x^3\,dx = \frac{1}{4}$$

to evaluate the integral.

29. $\int_{-1}^{1} x^3\,dx$

30. $\int_{0}^{1} (x^3 + 3)\,dx$

31. $\int_{2}^{3} (x - 2)^3\,dx$

32. $\int_{-1}^{1} |x|^3\,dx$

33. $\int_{0}^{1} (1 - x^3)\,dx$

34. $\int_{-1}^{2} (|x| - 1)^3\,dx$

35. $\int_{0}^{2} \left(\frac{x}{2}\right)^3 dx$

36. $\int_{-8}^{8} x^3\,dx$

37. $\int_0^1 (x^3 - 1)\, dx$

38. $\int_0^1 \sqrt[3]{x}\, dx$

In Exercises 39–42, use NINT to evaluate the expression.

39. $\int_0^5 \frac{x}{x^2 + 4}\, dx$

40. $3 + 2\int_0^{\pi/3} \tan x\, dx$

41. Find the area enclosed between the x-axis and the graph of $y = 4 - x^2$ from $x = -2$ to $x = 2$.

42. Find the area enclosed between the x-axis and the graph of $y = x^2 e^{-x}$ from $x = -1$ to $x = 3$.

In Exercises 43–46, **(a)** find the points of discontinuity of the integrand on the interval of integration, and **(b)** use area to evaluate the integral.

43. $\int_{-2}^3 \frac{x}{|x|}\, dx$

44. $\int_{-6}^5 2 \text{ int } (x - 3)\, dx$

45. $\int_{-3}^4 \frac{x^2 - 1}{x + 1}\, dx$

46. $\int_{-5}^6 \frac{9 - x^2}{x - 3}\, dx$

Extending the Ideas

47. **Writing to Learn** The function

$$f(x) = \begin{cases} \frac{1}{x^2}, & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$$

is defined on $[0, 1]$ and has a single point of discontinuity at $x = 0$.

(a) What happens to the graph of f as x approaches 0 from the right?

(b) The function f is not integrable on $[0, 1]$. Give a convincing argument based on Riemann sums to explain why it is not.

48. It can be shown by mathematical induction (see Appendix 2) that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Use this fact to give a formal proof that

$$\int_0^1 x^2\, dx = \frac{1}{3}$$

by following the steps given below.

(a) Partition $[0, 1]$ into n subintervals of length $1/n$. Sh that the RRAM Riemann sum for the integral is

$$\sum_{k=1}^{n} \left(\left(\frac{k}{n} \right)^2 \cdot \frac{1}{n} \right).$$

(b) Show that this sum can be written as

$$\frac{1}{n^3} \cdot \sum_{k=1}^{n} k^2.$$

(c) Show that the sum can therefore be written as

$$\frac{(n+1)(2n+1)}{6n^2}.$$

(d) Show that

$$\lim_{n \to \infty} \sum_{k=1}^{n} \left(\left(\frac{k}{n} \right)^2 \cdot \frac{1}{n} \right) = \frac{1}{3}.$$

(e) Explain why the equation in (d) proves that

$$\int_0^1 x^2\, dx = \frac{1}{3}.$$

5.3 Definite Integrals and Antiderivatives

Properties of Definite Integrals • Average Value of a Function • Mean Value Theorem for Definite Integrals • Connecting Differe and Integral Calculus

Properties of Definite Integrals

In defining $\int_a^b f(x)$ as a limit of sums $\sum c_k\, \Delta x_k$, we moved from left to across the interval $[a, b]$. What would happen if we integrated in the *op direction*? The integral would become $\int_b^a f(x)\, dx$—again a limit of sums form $\sum f(c_k)\Delta x_k$—but this time each of the Δx_k's would be negative x-values *decreased* from b to a. This would change the signs of all the in each Riemann sum, and ultimately the sign of the definite integral. This suggests the rule

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Since the original definition did not apply to integrating backwards over an interval, we can treat this rule as a logical extension of the definition.

Although $[a, a]$ is technically not an interval, another logical extension of the definition is that $\int_a^a f(x)\,dx = 0$.

These are the first two rules in Table 5.3. The others are inherited from rules that hold for Riemann sums. However, the limit step required to *prove* that these rules hold in the limit (as the norms of the partitions tend to zero) places their mathematical verification beyond the scope of this course. You can see intuitively why each rule makes sense, however.

Table 5.3 Rules for Definite Integrals

1. *Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A definition
2. *Zero:*	$\int_a^a f(x)\,dx = 0$	Also a definition
3. *Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any number k
	$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$	$k = -1$
4. *Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
5. *Additivity:*	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	

6. *Max-Min Inequality:* If max f and min f are the maximum and minimum values of f on $[a, b]$, then

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a).$$

7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

$$f(x) \ge 0 \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0 \quad g = 0$$

Example 1 USING THE RULES FOR DEFINITE INTEGRALS

Suppose

$$\int_{-1}^{1} f(x)\,dx = 5, \qquad \int_{1}^{4} f(x)\,dx = -2, \qquad \text{and} \qquad \int_{-1}^{1} h(x)\,dx = 7.$$

Find each of the following integrals, if possible.

(a) $\int_{4}^{1} f(x)\,dx$ **(b)** $\int_{-1}^{4} f(x)\,dx$ **(c)** $\int_{-1}^{1} [2f(x) + 3h(x)]$

(d) $\int_{0}^{1} f(x)\,dx$ **(e)** $\int_{-2}^{2} h(x)\,dx$ **(f)** $\int_{-1}^{4} [f(x) + h(x)]\,d$

Solution

(a) $\int_{4}^{1} f(x)\,dx = -\int_{1}^{4} f(x)\,dx = -(-2) = 2$

(b) $\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx = 5 + (-2) = 3$

(c) $\int_{-1}^{1} [2f(x) + 3h(x)]\,dx = 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx = 2(5) + 3(7)$

(d) Not enough information given. (We cannot assume, for example, th integrating over half the interval would give half the integral!)

(e) Not enough information given. (We have no information about the function h outside the interval $[-1, 1]$.)

(f) Not enough information given (same reason as in (e)).

Example 2 FINDING BOUNDS FOR AN INTEGRAL

Show that the value of $\int_0^1 \sqrt{1 + \cos x}\,dx$ is less than 3/2.

Solution The Max-Min Inequality for definite integrals (Rule 6) says that $\min f \cdot (b - a)$ is a *lower bound* for the value of $\int_a^b f(x)\,dx$ and $\max f \cdot (b - a)$ is an *upper bound*. The maximum value of $\sqrt{1 + \cos}$ on $[0, 1]$ is $\sqrt{2}$, so

$$\int_0^1 \sqrt{1 + \cos x}\,dx \le \sqrt{2} \cdot (1 - 0) = \sqrt{2}.$$

Since $\int_0^1 \sqrt{1 + \cos x}\,dx$ is bounded above by $\sqrt{2}$ (which is 1.414...) less than 3/2.

Average Value of a Function

The *average* of n numbers is the sum of the numbers divided by n. How v we define the average value of an arbitrary function f over a closed int $[a, b]$? As there are infinitely many values to consider, adding them and dividing by infinity is not an option.

Consider, then, what happens if we take a large *sample* of n numbers from regular subintervals of the interval $[a, b]$. One way would be to take some number c_k from each of the n subintervals of length

$$\Delta x = \frac{b-a}{n}.$$

The average of the n sampled values is

$$\frac{f(c_1)+f(c_2)+\cdots+f(c_n)}{n} = \frac{1}{n}\cdot\sum_{k=1}^{n} f(c_k)$$

$$= \frac{\Delta x}{b-a}\sum_{k=1}^{n} f(c_k) \qquad \frac{1}{n} = \frac{\Delta x}{b-a}$$

$$= \frac{1}{b-a}\cdot\sum_{k=1}^{n} f(c_k)\Delta x.$$

Does this last sum look familiar? It is $1/(b-a)$ times a Riemann sum fo f on $[a, b]$. That means that when we consider this averaging process as $n \to \infty$ we find it *has a limit*, namely $1/(b-a)$ times the integral of f over $[a, b]$. W are led by this remarkable fact to the following definition.

Definition Average (Mean) Value

If f is integrable on $[a, b]$, its **average (mean) value** on $[a, b]$ is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example 3 APPLYING THE DEFINITION

Find the average value of $f(x) = 4 - x^2$ on $[0, 3]$. Does f actually take c this value at some point in the given interval?

Solution

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{3-0}\int_0^3 (4-x^2)\,dx$$

$$= \frac{1}{3-0}\cdot 3 \qquad \text{Using NINT}$$

$$= 1$$

The average value of $f(x) = 4 - x^2$ over the interval $[0, 3]$ is 1. The function assumes this value when $4 - x^2 = 1$ or $x = \pm\sqrt{3}$. Since $x = \sqrt{3}$ lies in the interval $[0, 3]$, the function does assume its average value in the given interval (Figure 5.24).

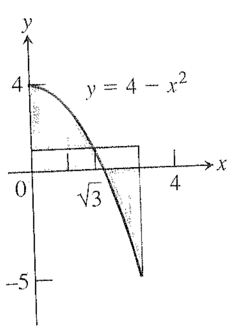


Figure 5.24 The rectangle with base $[0, 3]$ and with height equal to 1 (the average value of the function $f(x) = 4 - x^2$) has area equal to the net area between f and the x-axis from 0 to 3. (Example 3)

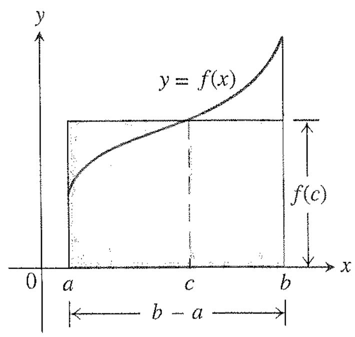


Figure 5.25 The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of f on $[a, b]$. When $f \geq 0$, the area of the shaded rectangle

$$f(c)(b-a) = \int_a^b f(x)\,dx,$$

is the area under the graph of f from a to b.

Mean Value Theorem for Definite Integrals

It was no mere coincidence that the function in Example 3 took on its ave value at some point in the interval. Look at the graph in Figure 5.25 and i ine rectangles with base $(b - a)$ and heights ranging from the minimum (a rectangle too small to give the integral) to the maximum of f (a rectangl large). Somewhere in between there is a "just right" rectangle, and its to will intersect the graph of f if f is continuous. The statement that a contin function on a closed interval *always* assumes its average value at least on the interval is known as the Mean Value Theorem for Definite Integrals.

Theorem 3 The Mean Value Theorem for Definite Integrals

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

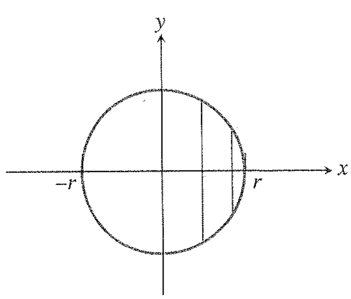


Figure 5.26 Chords perpendicular to the diameter $[-r, r]$ in a circle of radius r centered at the origin. (Exploration 1)

Exploration 1 **How Long Is the Average Chord of a Circle?**

Suppose we have a circle of radius r centered at the origin. We want t know the average length of the chords perpendicular to the diameter $[-r, r]$ on the x-axis.

1. Show that the length of the chord at x is $2\sqrt{r^2 - x^2}$ (Figure 5.26).
2. Set up an integral expression for the average value of $2\sqrt{r^2 - x^2}$ over the interval $[-r, r]$.
3. Evaluate the integral by identifying its value as an area.
4. So, what is the average length of a chord of a circle of radius r?
5. Explain how we can use the Mean Value Theorem for Definite Integrals (Theorem 3) to show that the function assumes the valu in step 4.

Connecting Differential and Integral Calculus

Before we move on to the next section, let us pause for a moment of histo perspective that can help you to appreciate the power of the theorem tha are about to encounter. In Example 3 we used NINT to find the integral, a Section 5.2, Example 2 we were fortunate that we could use our knowled the area of a circle. The area of a circle has been around for a long time NINT has not; so how did people evaluate definite integrals when they c not apply some known area formula? For example, in Exploration 1 of the vious section we used the fact that

$$\int_0^{\pi} \sin x\,dx = 2.$$

Would Newton and Leibniz have known this fact? How?

They did know that *quotients of infinitely small quantities,* as they p could be used to get velocity functions from position functions, and that *of infinitely thin "rectangle areas"* could be used to get position func from velocity functions. In some way, then, there had to be a conne between these two seemingly different processes. Newton and Leibniz able to picture that connection, and it led them to the Fundamental Theore Calculus. Can you picture it? Try Exploration 2.

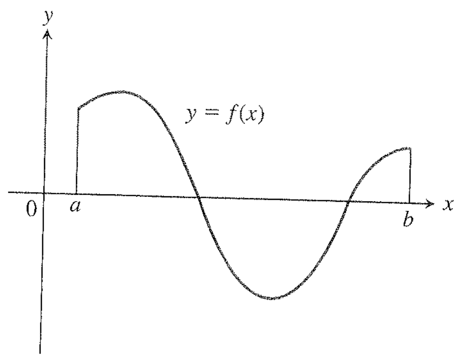


Figure 5.27 The graph of the function in Exploration 2.

Exploration 2 Finding the Derivative of an Integral

Suppose we are given the graph of a continuous function f, as in Figure 5.27. *Work in groups of two or three.*

1. Copy the graph of f onto your own paper. Choose any x greater than a in the interval $[a, b]$ and mark it on the x-axis.
2. Using only *vertical line segments,* shade in the region between the graph of f and the x-axis from a to x. (Some shading might be below the x-axis.)
3. Your shaded region represents a definite integral. Explain why this integral can be written as $\int_a^x f(t)\,dt$. (Why don't we write it as $\int_a^x f(x)\,dx$?)
4. Compare your picture with others produced by your group. Notice how your integral (a real number) depends on which x you chose in the interval $[a, b]$. The integral is therefore a *function of* x on $[a, b]$. Call it F.
5. Recall that $F'(x)$ is the limit of $\Delta F/\Delta x$ as Δx gets smaller and smaller. Represent ΔF in your picture by drawing *one more vertical shading segment* to the right of the last one you drew in step 2. ΔF is the (signed) *area* of your vertical segment.
6. Represent Δx in your picture by moving x to beneath your newly-drawn segment. That small change in Δx is the *thickness* of your vertical segment.
7. What is now the *height* of your vertical segment?
8. Can you see why Newton and Leibniz concluded that $F'(x) = f(x)$?

If all went well in Exploration 2, you concluded that the derivative wi respect to x of the integral of f from a to x is simply f. Specifically,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

This means that the integral is an *antiderivative* of f, a fact we can exploit i the following way.

If F is any antiderivative of f, then

$$\int_a^x f(t)\,dt = F(x) + C$$

for some constant C. Setting x in this equation equal to a gives

$$\int_a^a f(t)\,dt = F(a) + C$$

$$0 = F(a) + C$$

$$C = -F(a).$$

Putting it all together,

$$\int_a^x f(t)\,dt = F(x) - F(a).$$

The implications of this last equation were enormous for the discover calculus. It meant that they could evaluate the definite integral of f from a t number x simply by computing $F(x) - F(a)$, where F *is any antiderivative*

Example 4 FINDING AN INTEGRAL USING ANTIDERIVATIVES

Find $\int_0^{\pi} \sin x\,dx$ using the formula $\int_a^x f(t)\,dt = F(x) - F(a)$.

Solution Since $F(x) = -\cos x$ is an antiderivative of $\sin x$, we have

$$\int_0^{\pi} \sin x\,dx = -\cos(\pi) - (-\cos(0))$$

$$= -(-1) - (-1)$$

$$= 2.$$

This explains how we obtained the value for Exploration 1 of the previ section.

Quick Review 5.3

In Exercises 1–10, find dy/dx.

1. $y = -\cos x$
2. $y = \sin x$
3. $y = \ln(\sec x)$
4. $y = \ln(\sin x)$
5. $y = \ln(\sec x + \tan x)$
6. $y = x \ln x - x$
7. $y = \dfrac{x^{n+1}}{n+1} \quad (n \neq -1)$
8. $y = \dfrac{1}{2^x + 1}$
9. $y = xe^x$
10. $y = \tan^{-1} x$

Section 5.3 Exercises

The exercises in this section are designed to reinforce your understanding of the definite integral from the algebraic and geometric points of view. For this reason, you should not use the numerical integration capability of your calculator (NINT) except perhaps to support an answer.

1. Suppose that f and g are continuous functions and that

$$\int_1^2 f(x)\,dx = -4, \quad \int_1^5 f(x)\,dx = 6, \quad \int_1^5 g(x)\,dx = 8.$$

Use the rules in Table 5.3 to find each integral.

(a) $\int_2^2 g(x)\,dx$ (b) $\int_5^1 g(x)\,dx$

(c) $\int_1^2 3f(x)\,dx$ (d) $\int_2^5 f(x)\,dx$

(e) $\int_1^5 [f(x) - g(x)]\,dx$ (f) $\int_1^5 [4f(x) - g(x)]\,d$

2. Suppose that f and h are continuous functions and that

$$\int_1^9 f(x)\,dx = -1, \quad \int_7^9 f(x)\,dx = 5, \quad \int_7^9 h(x)\,dx = 4.$$

Use the rules in Table 5.3 to find each integral.

(a) $\int_1^9 -2f(x)\,dx$ **(b)** $\int_7^9 [f(x) + h(x)]\,dx$

(c) $\int_7^9 [2f(x) - 3h(x)]\,dx$ **(d)** $\int_9^1 f(x)\,dx$

(e) $\int_1^7 f(x)\,dx$ **(f)** $\int_9^7 [h(x) - f(x)]\,dx$

3. Suppose that $\int_1^2 f(x)\,dx = 5$. Find each integral.

(a) $\int_1^2 f(u)\,du$ **(b)** $\int_1^2 \sqrt{3}\,f(z)\,dz$

(c) $\int_2^1 f(t)\,dt$ **(d)** $\int_1^2 [-f(x)]\,dx$

4. Suppose that $\int_{-3}^0 g(t)\,dt = \sqrt{2}$. Find each integral.

(a) $\int_0^{-3} g(t)\,dt$ **(b)** $\int_{-3}^0 g(u)\,du$

(c) $\int_{-3}^0 [-g(x)]\,dx$ **(d)** $\int_{-3}^0 \frac{g(r)}{\sqrt{2}}\,dr$

5. Suppose that f is continuous and that

$$\int_0^3 f(z)\,dz = 3 \quad \text{and} \quad \int_0^4 f(z)\,dz = 7.$$

Find each integral.

(a) $\int_3^4 f(z)\,dz$ **(b)** $\int_4^3 f(t)\,dt$

6. Suppose that h is continuous and that

$$\int_{-1}^1 h(r)\,dr = 0 \quad \text{and} \quad \int_{-1}^3 h(r)\,dr = 6.$$

Find each integral.

(a) $\int_1^3 h(r)\,dr$ **(b)** $-\int_3^1 h(u)\,du$

In Exercises 7–16, evaluate the integral.

7. $\int_3^1 7\,dx$ **8.** $\int_0^2 5x\,dx$

9. $\int_3^5 \frac{x}{8}\,dx$ **10.** $\int_0^2 (2t - 3)\,dt$

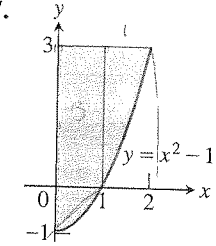
11. $\int_0^{\sqrt{2}} (t - \sqrt{2})\,dt$ **12.** $\int_2^1 \left(1 + \frac{z}{2}\right)dz$

13. $\int_{-1}^1 \frac{dx}{1 + x^2}$ **14.** $\int_{-1/2}^{1/2} \frac{dx}{\sqrt{1 - x^2}}$

15. $\int_0^2 e^x\,dx$ **16.** $\int_0^3 \frac{3\,dx}{x + 1}$

In Exercises 17–20, find the total shaded area.

17.



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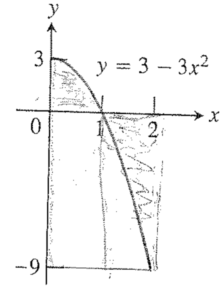

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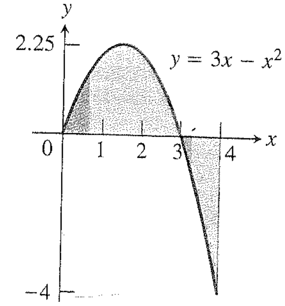

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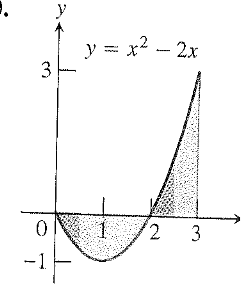


In Exercises 21–24, graph the function over the interval. Then **(a)** integrate the function over the interval and **(b)** find the area of the region between the graph and the x-axis.

21. $y = x^2 - 6x + 8, \quad [0, 3]$

22. $y = -x^2 + 5x - 4, \quad [0, 2]$

23. $y = 2x - x^2, \quad [0, 3]$ **24.** $y = x^2 - 4x, \quad [0, 5]$

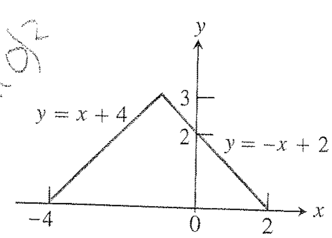
In Exercises 25–28, find the average value of the function on the interval. At what point(s) in the interval does the function assume its average value?

25. $y = x^2 - 1, \quad [0, \sqrt{3}]$ **26.** $y = -\frac{x^2}{2}, \quad [0, 3]$

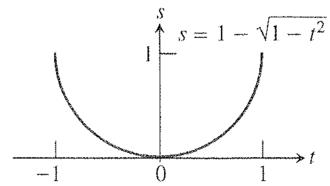
27. $y = -3x^2 - 1, \quad [0, 1]$ **28.** $y = (x - 1)^2, \quad [0, 3]$

In Exercises 29–32, find the average value of the function on the interval without integrating, by appealing to the geometry of the region between the graph and the x-axis.

29. $f(x) = \begin{cases} x + 4, & -4 \le x \le -1, \\ -x + 2, & -1 < x \le 2, \end{cases}$ on $[-4, 2]$



30. $f(t) = 1 - \sqrt{1-t^2}, \quad [-1, 1]$



31. $f(t) = \sin t, \quad [0, 2\pi]$

32. $f(\theta) = \tan\theta, \quad \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

In Exercises 33 and 34, *work in groups of two or three.*

33. Use the Max-Min Inequality to find upper and lower bounds for the value of

$$\int_0^1 \frac{1}{1+x^4}\,dx.$$

34. *(Continuation of Exercise 33)* Use the Max-Min Inequality to find upper and lower bounds for the values of

$$\int_0^{0.5} \frac{1}{1+x^4}\,dx \quad \text{and} \quad \int_{0.5}^{1} \frac{1}{1+x^4}\,dx.$$

Add these to arrive at an improved estimate for

$$\int_0^1 \frac{1}{1+x^4}\,dx.$$

35. Show that the value of $\int_0^1 \sin(x^2)\,dx$ cannot possibly be 2.

36. Show that the value of $\int_0^1 \sqrt{x+8}\,dx$ lies between $2\sqrt{2} \approx 2.8$ and 3.

37. *Integrals of Nonnegative Functions* Use the Max-Min Inequality to show that if f is integrable then

$$f(x) \geq 0 \text{ on } [a, b] \quad \Rightarrow \quad \int_a^b f(x)\,dx \geq 0.$$

38. *Integrals of Nonpositive Functions* Show that if f is integrable then

$$f(x) \leq 0 \text{ on } [a, b] \quad \Rightarrow \quad \int_a^b f(x)\,dx \leq 0.$$

39. Writing to Learn If $av(f)$ really is a typical value of the integrable function $f(x)$ on $[a, b]$, then the number $av(f)$ should have the same integral over $[a, b]$ that f does. Does it? That is, does

$$\int_a^b av(f)\,dx = \int_a^b f(x)\,dx?$$

Give reasons for your answer.

40. Writing to Learn A driver averaged 30 mph on a 15(trip and then returned over the same 150 miles at the ra 50 mph. He figured that his average speed was 40 mph the entire trip.

(a) What was his total distance traveled?

(b) What was his total time spent for the trip?

(c) What was his average speed for the trip?

(d) Explain the error in the driver's reasoning.

41. Writing to Learn A dam released 1000 m^3 of water 10 m^3/min and then released another 1000 m^3 at 20 m What was the average rate at which the water was rele Give reasons for your answer.

42. Use the inequality $\sin x \leq x$, which holds for $x \geq 0$, find an upper bound for the value of $\int_0^1 \sin x\,dx$.

43. The inequality $\sec x \geq 1 + (x^2/2)$ holds on $(-\pi/2, \pi$ Use it to find a lower bound for the value of $\int_0^1 \sec x\, d$

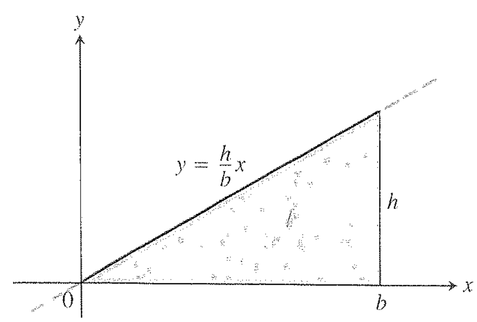
Exploration

44. *Comparing Area Formulas* Consider the region in th first quadrant under the curve $y = (h/b)\,x$ from $x = 0$ $x = b$ (see figure).

(a) Use a geometry formula to calculate the area of the region.

(b) Find all antiderivatives of y.

(c) Use an antiderivative of y to evaluate $\int_0^b y(x)\,dx$.



Extending the Ideas

45. *Graphing Calculator Challenge* If $k > 1$, and if th average value of x^k on $[0, k]$ is k, what is k? Check yo result with a CAS if you have one available.

46. Show that if $F'(x) = G'(x)$ on $[a, b]$, then

$$F(b) - F(a) = G(b) - G(a).$$

5.4 Fundamental Theorem of Calculus

Fundamental Theorem, Part 1 • Graphing the Function $\int_a^x f(t)$ d
Fundamental Theorem, Part 2 • Area Connection • More Applications

Sir Isaac Newton (1642–1727)

Sir Isaac Newton is considered to be one of the most influential mathematicians of all time. Moreover, by the age of 25, he had also made revolutionary advances in optics, physics, and astronomy.

Fundamental Theorem, Part 1

This section presents the discovery by Newton and Leibniz of the aston connection between integration and differentiation. This connection start mathematical development that fueled the scientific revolution for th 200 years, and is still regarded as the most important computational disc in the history of mathematics: The Fundamental Theorem of Calculus.

The Fundamental Theorem comes in two parts, both of which wer viewed in Exploration 2 of the previous section. The first part says th definite integral of a continuous function is a differentiable function of its limit of integration. Moreover, it tells us what that derivative is. The secon says that the definite integral of a continuous function from a to b can be from any one of the function's antiderivatives F as the number $F(b) - F$

Theorem 4 The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a, b]$, then the function

$$F(x) = \int_a^x f(t)\,dt$$

has a derivative at every point x in $[a, b]$, and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Proof The geometric exploration at the end of the previous section con the idea of the proof, but it glossed over the necessary limit arguments. we will be more precise.

Apply the definition of the derivative directly to the function F. That is,

$$\frac{dF}{dx} = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h\to 0}\frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

$$= \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h} \qquad \text{Rules for integrals Section 5.3}$$

$$= \lim_{h\to 0}\left[\frac{1}{h}\int_x^{x+h} f(t)\,dt\right].$$

The expression in brackets in the last line is the average value of f from $x + h$. We know from the Mean Value Theorem for Definite Integrals (T rem 3, Section 5.3) that f, being continuous, takes on its average value at once in the interval; that is,

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c) \quad \text{for some } c \text{ between } x \text{ and } x + h.$$

We can therefore continue our proof, letting $(1/h)\int_x^{x+h} f(t)\,dt = f(c)$,

$$\begin{aligned}\frac{dF}{dx} &= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt \\ &= \lim_{h\to 0} f(c), \quad \text{where } c \text{ lies between } x \text{ and } x + h.\end{aligned}$$

What happens to c as h goes to zero? As $x + h$ gets closer to x, it carri along with it like a bead on a wire, forcing c to approach x. Since f is coı uous, this means that $f(c)$ approaches $f(x)$:

$$\lim_{h\to 0} f(c) = f(x).$$

Putting it all together,

$$\begin{aligned}\frac{dF}{dx} &= \lim_{h\to 0}\frac{F(x+h) - F(x)}{h} && \text{Definition of derivatives} \\ &= \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h} && \text{Rules for integrals} \\ &= \lim_{h\to 0} f(c) \quad \text{for some } c \text{ between } x \text{ and } x + h. \\ &= f(x). && \text{Because } f \text{ is continuous}\end{aligned}$$

This concludes the proof.

It is difficult to overestimate the power of the equation

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x). \quad (1$$

It says that every continuous function f is the derivative of some other func namely $\int_a^x f(t)\,dt$. It says that every continuous function has an antideriva And it says that the processes of integration and differentiation are inverse one another. If any equation deserves to be called the Fundamental Theoreı Calculus, this equation is surely the one.

Example 1 APPLYING THE FUNDAMENTAL THEOREM

Find

$$\frac{d}{dx}\int_{-\pi}^{x} \cos t\,dt \quad \text{and} \quad \frac{d}{dx}\int_0^x \frac{1}{1+t^2}\,dt$$

by using the Fundamental Theorem.

Solution

$$\frac{d}{dx}\int_{-\pi}^{x} \cos t\, dt = \cos x \qquad \text{Eq. 1 with } f(t) = \cos t$$

$$\frac{d}{dx}\int_{0}^{x} \frac{1}{1+t^2}\, dt = \frac{1}{1+x^2}. \qquad \text{Eq. 1 with } f(t) = \frac{1}{1+t^2}$$

Example 2 THE FUNDAMENTAL THEOREM WITH THE CHAI RULE

Find dy/dx if $y = \int_1^{x^2} \cos t\, dt$.

Solution The upper limit of integration is not x but x^2. This makes y composite of

$$y = \int_1^{u} \cos t\, dt \qquad \text{and} \qquad u = x^2.$$

We must therefore apply the Chain Rule when finding dy/dx.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} \\
&= \left(\frac{d}{du}\int_1^{u} \cos t\, dt\right)\cdot\frac{du}{dx} \\
&= \cos u \cdot \frac{du}{dx} \\
&= \cos (x^2)\cdot 2x \\
&= 2x \cos x^2
\end{aligned}$$

Example 3 VARIABLE LOWER LIMITS OF INTEGRATION

Find dy/dx.

(a) $y = \int_x^5 3t \sin t\, dt$ **(b)** $y = \int_{2x}^{x^2} \frac{1}{2+e^t}\, dt$

Solution The rules for integrals set these up for the Fundamental Theo

(a)
$$\begin{aligned}
\frac{d}{dx}\int_x^5 3t \sin t\, dt &= \frac{d}{dx}\left(-\int_5^x 3t \sin t\, dt\right) \\
&= -\frac{d}{dx}\int_5^x 3t \sin t\, dt \\
&= -3x \sin x
\end{aligned}$$

(b)
$$\begin{aligned}
\frac{d}{dx}\int_{2x}^{x^2} \frac{1}{2+e^t}\, dt &= \frac{d}{dx}\left(\int_0^{x^2} \frac{1}{2+e^t}\, dt - \int_0^{2x} \frac{1}{2+e^t}\, dt\right) \\
&= \frac{1}{2+e^{x^2}}\frac{d}{dx}(x^2) - \frac{1}{2+e^{2x}}\frac{d}{dx}(2x) \qquad \text{Chain Ru} \\
&= \frac{1}{2+e^{x^2}}\cdot 2x - \frac{1}{2+e^{2x}}\cdot 2 \\
&= \frac{2x}{2+e^{x^2}} - \frac{2}{2+e^{2x}}
\end{aligned}$$

Example 4 CONSTRUCTING A FUNCTION WITH A GIVEN DERIVATIVE AND VALUE

Find a function $y = f(x)$ with derivative

$$\frac{dy}{dx} = \tan x$$

that satisfies the condition $f(3) = 5$.

Solution The Fundamental Theorem makes it easy to construct a funct with derivative tan x:

$$y = \int_3^x \tan t\, dt.$$

Since $y(3) = 0$, we have only to add 5 to this function to construct on with derivative tan x whose value at $x = 3$ is 5:

$$f(x) = \int_3^x \tan t\, dt + 5.$$

Although the solution to the problem in Example 4 satisfies the required conditions, you might question whether it is in a useful form. many years ago, this form might have posed a computation problem. Inc for such problems much effort has been expended over the centuries tryir find solutions that do not involve integrals. We will see some in Chapt where we will learn (for example) how to write the solution in Example 4

$$y = \ln\left|\frac{\cos 3}{\cos x}\right| + 5.$$

However, now that computers and calculators are capable of evaluating integ the form given in Example 4 is not only useful, but in some ways preferab is certainly easier to find and is always available.

Graphing the Function $\int_a^x f(t)\, dt$

Consider for a moment the two forms of the function we have just l discussing,

$$F(x) = \int_3^x \tan t\, dt + 5 \quad \text{and} \quad F(x) = \ln\left|\frac{\cos 3}{\cos x}\right| + 5.$$

With which expression is it easier to evaluate, say, $F(4)$? From the tim Newton almost to the present, there has been no contest: the expression o right. At least it provides something to compute, and there have always l tables or slide rules or calculators to facilitate that computation. The expres on the left involved at best a tedious summing process and almost certainl increased opportunity for error.

Today we can find $F(4)$ from either expression on the same machine. choice is between NINT (tan x, x, 3, 4) + 5 and ln (abs(cos(3)/cos(4))) Both calculations give 5.415135083 in approximately the same amount of t

We can even use NINT to graph the function. This modest techno feat would have absolutely dazzled the mathematicians of the 18th 19th centuries, who knew how the solutions of differential equations, suc $dy/dx = \tan x$, could be written as integrals, but for whom integrals were c practical use computationally unless they could be written in exact form. S so few integrals could, in fact, be written in exact form, NINT would spared generations of scientists much frustration.

Nevertheless, one must not proceed blindly into the world of c computation. Exploration 1 will demonstrate the need for caution.

Exploration 1 **Graphing NINT f**

Let us use NINT to attempt to graph the function we just discuss

$$F(x) = \int_3^x \tan t\, dt + 5.$$

1. Graph the function $y = F(x)$ in the window $[-10, 10]$ by $[-10, 10]$. You will probably wait a long time and see no grap Break out of the graphing program if necessary.
2. Recall that the graph of the function $y = \tan x$ has vertical asymptotes. Where do they occur on the interval $[-10, 10]$?
3. When attempting to graph the function $F(x) = \int_3^x \tan t\, dt + 5$ on the interval $[-10, 10]$, your grapher begins by trying to find $F(-10)$. Explain why this might cause a problem for your calculator.
4. Set your viewing window so that your calculator graphs only over the domain of the continuous branch of the tangent functi that contains the point $(3, \tan 3)$.
5. What is the domain in step 4? Is it an open interval or a closed interval?
6. What is the domain of $F(x)$? Is it an open interval or a closed interval?
7. Your calculator graphs over the closed interval $[x_{\min}, x_{\max}]$. Fin viewing window that will give you a good look at the graph of and produce the graph on your calculator.
8. Describe the graph of F.

Graphing NINT f

Some graphers can graph the numerical integral $y =$ NINT $(f(x), x, a, x)$ directly as a function of x. Others will require a tool-box program such as the one called NINT-GRAF provided in the *Technology Resource Manual*.

You have probably noticed that your grapher moves slowly when ing NINT. This is because it must compute each value as a limit of s comparatively slow work even for a microprocessor. Here are some w speed up the process:

1. Change the *tolerance* on your grapher. The smaller the tolerance, t more accurate the calculator will try to be when finding the limitin value of each sum (and the longer it will take to do so). The defaul value is usually quite small (like 0.00001), but a value as large as 1 be used for graphing in a typical viewing window.
2. Change the *x-resolution*. The default resolution is 1, which means t the grapher will compute a function value for every vertical column pixels. At resolution 2 it computes only every second value, and so With higher resolutions, some graph smoothness is sacrificed for sp
3. Switch to parametric mode. To graph $y =$ NINT $(f(x), x, a, x)$ in metric mode, let $x(t) = t$ and let $y(t) =$ NINT $(f(t), t, a, t)$. You then control the speed of the grapher by changing the t-step. (Choo a bigger t-step has the same effect as choosing a larger x-resolution.

Exploration 2 **The Effect of Changing a in $\int_a^x f(t)\,dt$**

The first part of the Fundamental Theorem of Calculus asserts that the derivative of $\int_a^x f(t)\,dt$ is $f(x)$, regardless of the value of a.

1. Graph NDER (NINT $(x^2, x, 0, x)$).
2. Graph NDER (NINT $(x^2, x, 5, x)$).
3. Without graphing, tell what the x-intercept of NINT $(x^2, x, 0, x)$ i Explain.
4. Without graphing, tell what the x-intercept of NINT $(x^2, x, 5, x)$ i Explain.
5. How does changing a affect the graph of $y = (d/dx)\int_a^x f(t)\,dt$?
6. How does changing a affect the graph of $y = \int_a^x f(t)\,dt$?

Fundamental Theorem, Part 2

The second part of the Fundamental Theorem of Calculus shows how to uate definite integrals directly from antiderivatives.

Theorem 4 (continued) **The Fundamental Theorem of Calculus, Part 2**

If f is continuous at every point of $[a, b]$, and if F is any antiderivativ of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This part of the Fundamental Theorem is also called the **Integral Evaluation Theorem.**

Proof Part 1 of the Fundamental Theorem tells us that an antiderivativ exists, namely

$$G(x) = \int_a^x f(t)\,dt.$$

Thus, if F is *any* antiderivative of f, then $F(x) = G(x) + C$ for some con C (by Corollary 3 of the Mean Value Theorem for Derivatives, Section 4

Evaluating $F(b) - F(a)$, we have

$$\begin{aligned} F(b) - F(a) &= [G(b) + C] - [G(a) + C] \\ &= G(b) - G(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\ &= \int_a^b f(t)\,dt - 0 \\ &= \int_a^b f(t)\,dt. \end{aligned}$$

At the risk of repeating ourselves: It is difficult to overestimate th of the simple equation

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

It says that any definite integral of any continuous function f can be cal without taking limits, without calculating Riemann sums, and often effort—so long as an antiderivative of f can be found. If you can imagi it was like before this theorem (and before computing machines) approximations by tedious sums were the only alternative for solvin real-world problems, then you can imagine what a miracle calcul thought to be. If any equation deserves to be called the Fundamental Tl of Calculus, this equation is surely the (second) one.

Integral Evaluation Notation

The usual notation for $F(b) - F(a)$ is

$$F(x)\Big]_a^b \quad \text{or} \quad \Big[F(x)\Big]_a^b,$$

depending on whether F has one or more terms. This notation provides a compact "recipe" for the evaluation, allowing us to show the antiderivative in an intermediate step.

Example 5 EVALUATING AN INTEGRAL

Evaluate $\int_{-1}^{3} (x^3 + 1)\,dx$ using an antiderivative.

Solution

Solve Analytically

A simple antiderivative of $x^3 + 1$ is $(x^4/4) + x$. Therefore,

$$\begin{aligned}\int_{-1}^{3}\left(x^3 + 1\right)dx &= \left[\frac{x^4}{4} + x\right]_{-1}^{3}\\ &= \left(\frac{81}{4} + 3\right) - \left(\frac{1}{4} - 1\right)\\ &= 24.\end{aligned}$$

Support Numerically

NINT $(x^3 + 1, x, -1, 3) = 24$.

Area Connection

In Section 5.2 we saw that the definite integral could be interpreted as th area between the graph of a function and the x-axis. We can therefore con areas using antiderivatives, but we must again be careful to distinguish net (in which area below the x-axis is counted as negative) from total area. unmodified word "area" will be taken to mean *total area*.

Example 6 FINDING AREA USING ANTIDERIVATIVES

Find the area of the region between the curve $y = 4 - x^2$, $0 \le x \le 3$, the x-axis.

Solution The curve crosses the x-axis at $x = 2$, partitioning the inter $[0, 3]$ into two subintervals, on each of which $f(x) = 4 - x^2$ will not change sign.

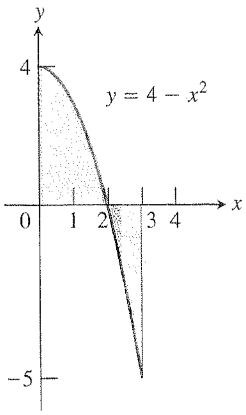


Figure 5.28 The function $f(x) = 4 - x^2$ changes sign only at $x = 2$ on the interval $[0, 3]$. (Example 6)

We can see from the graph (Figure 5.28) that $f(x) > 0$ on $[0, 2)$ and $f(x) < 0$ on $(2, 3]$.

$$\text{Over } [0, 2]\text{:} \quad \int_0^2 (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_0^2 = \frac{16}{3}.$$

$$\text{Over } [2, 3]\text{:} \quad \int_2^3 (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_2^3 = -\frac{7}{3}.$$

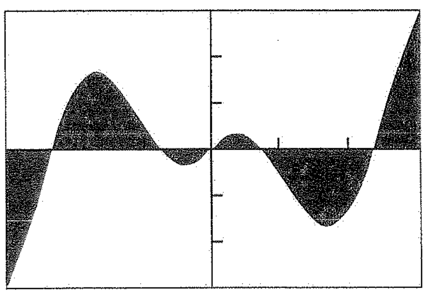
$$\text{The area of the region is } \left|\frac{16}{3}\right| + \left|-\frac{7}{3}\right| = \frac{23}{3}.$$

How to Find Total Area Analytically

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$ analytically,

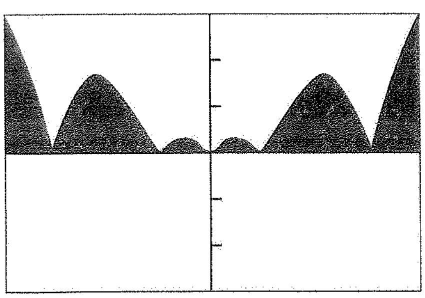
1. partition $[a, b]$ with the zeros of f,
2. integrate f over each subinterval,
3. add the absolute values of the integrals.

We can find area numerically by using NINT to integrate the *abs value* of the function over the given interval. There is no need to partitioi taking absolute values, we automatically reflect the negative portions o graph across the x-axis to count all area as positive (Figure 5.29).



[–3, 3] by [–3, 3]

(a)



[–3, 3] by [–3, 3]

(b)

Figure 5.29 The graphs of (a) $y = x \cos 2x$ and (b) $y = |x \cos 2x|$ over $[-3, 3]$. The shaded regions have the same area.

Example 7 FINDING AREA USING NINT

Find the area of the region between the curve $y = x \cos 2x$ and the x- over the interval $-3 \le x \le 3$ (Figure 5.29).

Solution Rounded to two decimal places, we have

$$\text{NINT}\,(|x \cos 2x|, x, -3, 3) = 5.43.$$

How to Find Total Area Numerically

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$ numerically, evaluate

$$\text{NINT}\,(|f(x)|, x, a, b).$$

More Applications

We close the section with two applications from the world of economics.

Example 8 DETERMINING COST FROM MARGINAL COST

The fixed cost of starting a manufacturing run and producing the first 10 units is \$200. After that, the marginal cost at x units output is

$$\frac{dc}{dx} = \frac{1000}{x^2}.$$

Find the total cost of producing the first 100 units.

Solution If $c(x)$ is the cost of x units, then

$$\underbrace{c(100)}_{\text{Cost of 100 units}} = \underbrace{200}_{\text{Startup and first 10 units}} + \underbrace{c(100) - c(10)}_{\text{Cost of units 11 through 100}}$$

$$= 200 + \int_{10}^{100} \frac{dc}{dx}\,dx \qquad \text{Since } \int_{10}^{100} \frac{dc}{dx}\,dx = c(100) -$$

$$= 200 + \int_{10}^{100} \frac{1000}{x^2}\,dx$$

$$= 200 + 1000\left[-\frac{1}{x}\right]_{10}^{100}$$

$$= 290.$$

The total cost of producing the first 100 units is \$290.

The notion of a function's average value is used in economics to things like average daily inventory. An **inventory function** $I(x)$ gives the ber of radios, shoes, tires, or whatever product a firm has on hand on The average value of I over a time period $a \le x \le b$ is the firm's average inventory for the period.

Definition **Average Daily Inventory**

If $I(x)$ is the number of items on hand on day x, the **average daily inventory** of the items for the period $a \le x \le b$ is

$$av(I) = \frac{1}{b-a}\int_a^b I(x)\,dx.$$

If h is the dollar cost of holding one item per day, the **average daily holding cost** for the period $a \le x \le b$ is $av(I) \cdot h$.

Example 9 FINDING AVERAGE DAILY INVENTORY

Suppose a wholesaler receives a shipment of 1200 cases of boxes of chocolates every 30 days. The chocolate is sold to retailers at a steady rate, and x days after the shipment arrives, the inventory of cases still on hand is $I(x) = 1200 - 40x$. Find the average daily inventory. Als find the average daily holding cost if the cost of holding one case is 3 cents a day.

Figure 5.30 Maintaining a steady inventory to ship to the wholesaler. (Example 9)

Solution The average daily inventory is

$$av(I) = \frac{1}{30-0}\int_0^{30}(1200 - 40x)\,dx$$

$$= \frac{1}{30}\left[1200x - 20x^2\right]_0^{30}$$

$$= 600.$$

The average daily holding cost, then, is $(600)(0.03) = 18$ dollars a day.

Quick Review 5.4

In Exercises 1–10, find dy/dx.

1. $y = \sin(x^2)$

2. $y = (\sin x)^2$

3. $y = \sec^2 x - \tan^2 x$

4. $y = \ln(3x) - \ln(7x)$

5. $y = 2^x$

6. $y = \sqrt{x}$

7. $y = \dfrac{\cos x}{x}$

8. $y = \sin t$ and $x = \cos t$

9. $xy + x = y^2$

10. $dx/dy = 3x$

Section 5.4 Exercises

In Exercises 1–14, evaluate each integral using Part 2 of the Fundamental Theorem. Support your answer with NINT if you are unsure.

1. $\displaystyle\int_{1/2}^{3} \left(2 - \frac{1}{x}\right) dx$

2. $\displaystyle\int_{2}^{-1} 3^x\, dx$

3. $\displaystyle\int_{0}^{1} (x^2 + \sqrt{x})\, dx$

4. $\displaystyle\int_{0}^{5} x^{3/2}\, dx$

5. $\displaystyle\int_{1}^{32} x^{-6/5}\, dx$

6. $\displaystyle\int_{-2}^{-1} \frac{2}{x^2}\, dx$

7. $\displaystyle\int_{0}^{\pi} \sin x\, dx$

8. $\displaystyle\int_{0}^{\pi} (1 + \cos x)\, dx$

9. $\displaystyle\int_{0}^{\pi/3} 2\sec^2\theta\, d\theta$

10. $\displaystyle\int_{\pi/6}^{5\pi/6} \csc^2\theta\, d\theta$

11. $\displaystyle\int_{\pi/4}^{3\pi/4} \csc x \cot x\, dx$

12. $\displaystyle\int_{0}^{\pi/3} 4\sec x \tan x\, dx$

13. $\displaystyle\int_{-1}^{1} (r + 1)^2\, dr$

14. $\displaystyle\int_{0}^{4} \frac{1 - \sqrt{u}}{\sqrt{u}}\, du$

In Exercises 15–18, find the total area of the region between the curve and the x-axis.

15. $y = 2 - x, \quad 0 \le x \le 3$

16. $y = 3x^2 - 3, \quad -2 \le x \le 2$

17. $y = x^3 - 3x^2 + 2x, \quad 0 \le x \le 2$

18. $y = x^3 - 4x, \quad -2 \le x \le 2$

In Exercises 19–24, answer the following questions about the integral.

(a) Writing to Learn Can the Fundamental Theorem of Calculus, Part 2 (Theorem 4) be used to evaluate the integral? Explain.

(b) Does the integral have a value? If so, what is it? Explain.

19. $\displaystyle\int_{-2}^{3} \frac{x^2 - 1}{x + 1}\, dx$

20. $\displaystyle\int_{0}^{5} \frac{9 - x^2}{3x - 9}\, dx$

21. $\displaystyle\int_{0}^{2\pi} \tan x\, dx$

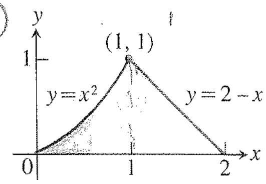
22. $\displaystyle\int_{0}^{2} \frac{x + 1}{x^2 - 1}\, dx$

23. $\displaystyle\int_{-1}^{2} \frac{\sin x}{x}\, dx$

24. $\displaystyle\int_{-2}^{3} \frac{1 - \cos x}{x^2}\, dx$

In Exercises 25–28, find the area of the shaded region.

25.



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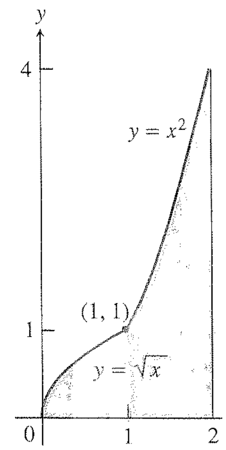

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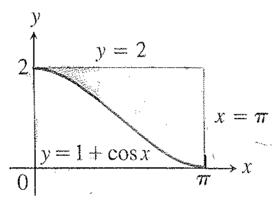

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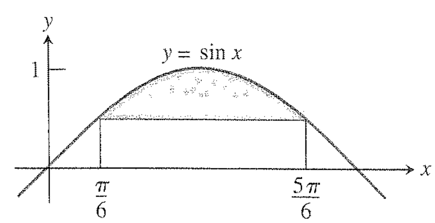


In Exercises 29–34, use NINT to solve the problem.

29. Evaluate $\displaystyle\int_{0}^{10} \frac{1}{3 + 2\sin x}\, dx.$

30. Evaluate $\displaystyle\int_{-0.8}^{0.8} \frac{2x^4 - 1}{x^4 - 1}\, dx.$

31. Find the average value of $\sqrt{\cos x}$ on the interval $[-1,$

32. Find the area of the semielliptical region between the x-axis and the graph of $y = \sqrt{8 - 2x^2}$.

33. For what value of x does $\int_0^x e^{-t^2}\, dt = 0.6$?

34. Find the area of the region in the first quadrant enclose the coordinate axes and the graph of $x^3 + y^3 = 1$.

In Exercises 35 and 36, find K so that

$$\int_a^x f(t)\,dt + K = \int_b^x f(t)\,dt.$$

35. $f(x) = x^2 - 3x + 1;\quad a = -1;\quad b = 2$

36. $f(x) = \sin^2 x;\quad a = 0;\quad b = 2$

In Exercises 37–42, find dy/dx.

37. $y = \int_0^x \sqrt{1+t^2}\,dt$

38. $y = \int_x^1 \frac{1}{t}\,dt,\quad x > 0$

39. $y = \int_0^{\sqrt{x}} \sin(t^2)\,dt$

40. $y = \int_0^{2x} \cos t\,dt$

41. $y = \int_{x^2}^{x^3} \cos(2t)\,dt$

42. $y = \int_{\sin x}^{\cos x} t^2\,dt$

In Exercises 43–46, choose which of the following functions has the given derivative and numerical value. Confirm your answer.

(a) $y = \int_1^x e^{-t^2}dt - 3$

(b) $y = \int_0^x \sec t\,dt + 4$

(c) $y = \int_{-1}^x \sec t\,dt + 4$

(d) $y = \int_\pi^x e^{-t^2}\,dt - 3$

43. $\frac{dy}{dx} = e^{-x^2},\quad y(\pi) = -3$

44. $\frac{dy}{dx} = \sec x,\quad y(-1) = 4$

45. $\frac{dy}{dx} = \sec x,\quad y(0) = 4$

46. $\frac{dy}{dx} = e^{-x^2},\quad y(1) = -3$

47. *Identifying a Zero* For what value of x is $\int_a^x f(t)\,dt$ sure to be zero?

48. Suppose $\int_1^x f(t)\,dt = x^2 - 2x + 1$. Find $f(x)$.

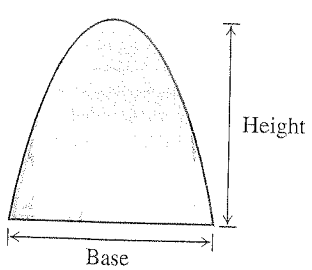
49. *Linearization* Find the linearization of

$$f(x) = 2 + \int_0^x \frac{10}{1+t}\,dt \quad \text{at} \quad x = 0.$$

50. Find $f(4)$ if $\int_0^x f(t)\,dt = x\cos \pi x$.

51. *Finding Area* Show that if k is a positive constant, then the area between the x-axis and one arch of the curve $y = \sin kx$ is always $2/k$.

52. *Archimedes' Area Formula for Parabolas* Archimedes (287–212 B.C.), inventor, military engineer, physicist, and the greatest mathematician of classical times, discovered that the area under a parabolic arch like the one shown here is always two-thirds the base times the height.



(a) Find the area under the parabolic arch

$$y = 6 - x - x^2,\quad -3 \le x \le 2.$$

(b) Find the height of the arch.

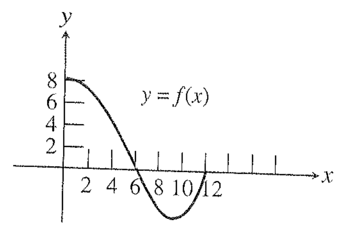
(c) Show that the area is two-thirds the base times the h

In Exercises 53–55, *work in groups of two or three.*

53. Let

$$H(x) = \int_0^x f(t)\,dt,$$

where f is the continuous function with domain [0, 12] graphed here.



(a) Find $H(0)$.

(b) On what interval is H increasing? Explain.

(c) On what interval is the graph of H concave up? Expl

(d) Is $H(12)$ positive or negative? Explain.

(e) Where does H achieve its maximum value? Explain.

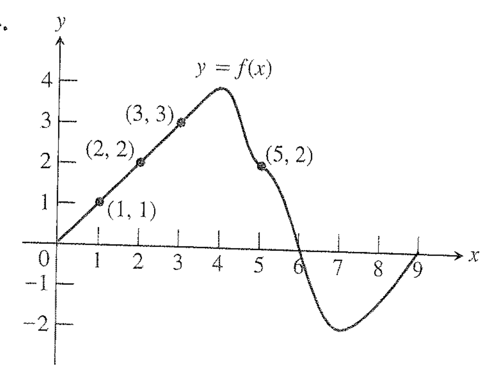
(f) Where does H achieve its minimum value? Explain.

In Exercises 54 and 55, f is the differentiable function whose graph is shown in the figure. The position at time t (sec) of a particle moving along a coordinate axis is

$$s = \int_0^t f(x)\,dx$$

meters. Use the graph to answer the questions. Give reasons f your answers.

54.



(a) What is the particle's velocity at time $t = 5$?

(b) Is the acceleration of the particle at time $t = 5$ positi or negative?

(c) What is the particle's position at time $t = 3$?

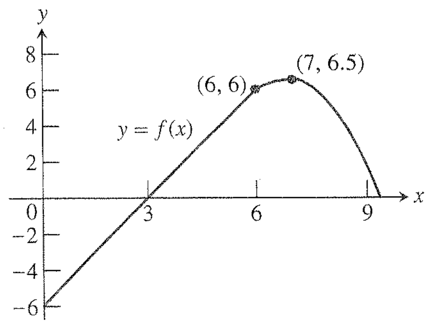
(d) At what time during the first 9 sec does s have its larg value?

(e) Approximately when is the acceleration zero?

(f) When is the particle moving toward the origin? away from the origin?

(g) On which side of the origin does the particle lie at time $t = 9$?

55.



(a) What is the particle's velocity at time $t = 3$?

(b) Is the acceleration of the particle at time $t = 3$ positive or negative?

(c) What is the particle's position at time $t = 3$?

(d) When does the particle pass through the origin?

(e) Approximately when is the acceleration zero?

(f) When is the particle moving toward the origin? away from the origin?

(g) On which side of the origin does the particle lie at time $t = 9$?

Exploration

56. *The Sine Integral Function* The sine integral function

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt$$

is one of the many useful functions in engineering that are defined as integrals. Although the notation does not show it, the function being integrated is

$$f(t) = \begin{cases} \dfrac{\sin t}{t}, & t \neq 0 \\ 1, & t = 0, \end{cases}$$

the continuous extension of $(\sin t)/t$ to the origin.

(a) Show that $\text{Si}(x)$ is an odd function of x.

(b) What is the value of $\text{Si}(0)$?

(c) Find the values of x at which $\text{Si}(x)$ has a local extreme value.

(d) Use NINT to graph $\text{Si}(x)$.

57. *Cost from Marginal Cost* The marginal cost of printi poster when x posters have been printed is

$$\frac{dc}{dx} = \frac{1}{2\sqrt{x}}$$

dollars. Find

(a) $c(100) - c(1)$, the cost of printing posters 2 to 100.

(b) $c(400) - c(100)$, the cost of printing posters 101 tc

58. *Revenue from Marginal Revenue* Suppose that a company's marginal revenue from the manufacture and of eggbeaters is

$$\frac{dr}{dx} = 2 - \frac{2}{(x+1)^2},$$

where r is measured in thousands of dollars and x in thou sands of units. How much money should the company ex from a production run of $x = 3$ thousand eggbeaters? To out, integrate the marginal revenue from $x = 0$ to $x =$

59. *Average Daily Holding Cost* Solon Container receive 450 drums of plastic pellets every 30 days. The invento function (drums on hand as a function of days) is $I(x) = 450 - x^2/2$.

(a) Find the average daily inventory.

(b) If the holding cost for one drum is \$0.02 per day, fi the average daily holding cost.

60. Suppose that f has a negative derivative for all values o and that $f(1) = 0$. Which of the following statements ı be true of the function

$$h(x) = \int_0^x f(t)\,dt?$$

Give reasons for your answers.

(a) h is a twice-differentiable function of x.

(b) h and dh/dx are both continuous.

(c) The graph of h has a horizontal tangent at $x = 1$.

(d) h has a local maximum at $x = 1$.

(e) h has a local minimum at $x = 1$.

(f) The graph of h has an inflection point at $x = 1$.

(g) The graph of dh/dx crosses the x-axis at $x = 1$.

Extending the Ideas

61. **Writing to Learn** If f is an odd continuous function, a graphical argument to explain why $\int_0^x f(t)\,dt$ is eve

62. **Writing to Learn** If f is an even continuous function a graphical argument to explain why $\int_0^x f(t)\,dt$ is odd

63. **Writing to Learn** Explain why we can conclude fror Exercises 61 and 62 that every even continuous functio the derivative of an odd continuous function and vice v

64. Give a convincing argument that the equation

$$\int_0^x \frac{\sin t}{t}\,dt = 1$$

has exactly one solution. Give its approximate value.

5.5 Trapezoidal Rule

Trapezoidal Approximations • Other Algorithms • Error Analys

Trapezoidal Approximations

You probably noticed in Section 5.1 that MRAM was generally more effi in approximating integrals than either LRAM or RRAM, even though all RAM approximations approached the same limit. All three RAM approx tions, however, depend on the areas of rectangles. Are there other geom shapes with known areas that can do the job more efficiently? The answ yes, and the most obvious one is the trapezoid.

As shown in Figure 5.31, if $[a, b]$ is partitioned into n subinterva equal length $h = (b - a)/n$, the graph of f on $[a, b]$ can be approximate a straight line segment over each subinterval.

The region between the curve and the x-axis is then approximated by trapezoids, the area of each trapezoid being the length of its horizontal " tude" times the average of its two vertical "bases." That is,

$$\int_a^b f(x)\,dx \approx h \cdot \frac{y_0 + y_1}{2} + h \cdot \frac{y_1 + y_2}{2} + \cdots + h \cdot \frac{y_{n-1} + y_n}{2}$$

$$= h\left(\frac{y_0}{2} + y_1 + y_2 + \cdots + y_{n-1} + \frac{y_n}{2}\right)$$

$$= \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right),$$

where

$$y_0 = f(a), \quad y_1 = f(x_1), \quad \ldots, \quad y_{n-1} = f(x_{n-1}), \quad y_n = f(b).$$

This is algebraically equivalent to finding the numerical average of LRAM RRAM; indeed, that is how some texts define the Trapezoidal Rule.

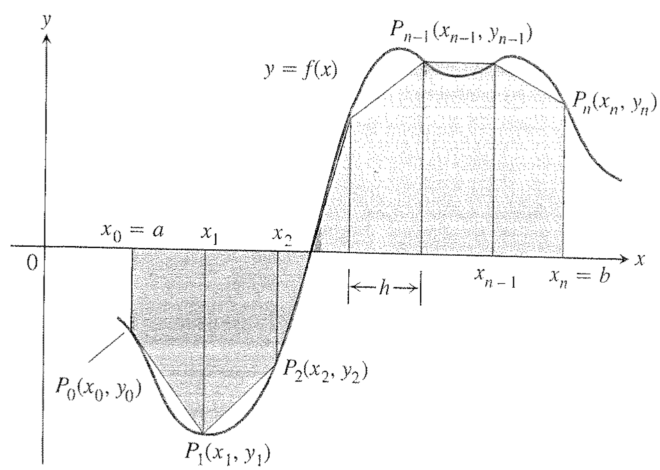


Figure 5.31 The trapezoidal rule approximates short stretches of the curve $y = f$ with line segments. To approximate the integral of f from a to b, we add the "signe areas of the trapezoids made by joining the ends of the segments to the x-axis.

The Trapezoidal Rule

To approximate $\int_a^b f(x)\,dx$, use

$$T = \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right),$$

where $[a, b]$ is partitioned into n subintervals of equal length $h = (b - a)/n$.

Equivalently,

$$T = \frac{\text{LRAM}_n + \text{RRAM}_n}{2},$$

where LRAM_n and RRAM_n are the Riemann sums using the left and right endpoints, respectively, for f for the partition.

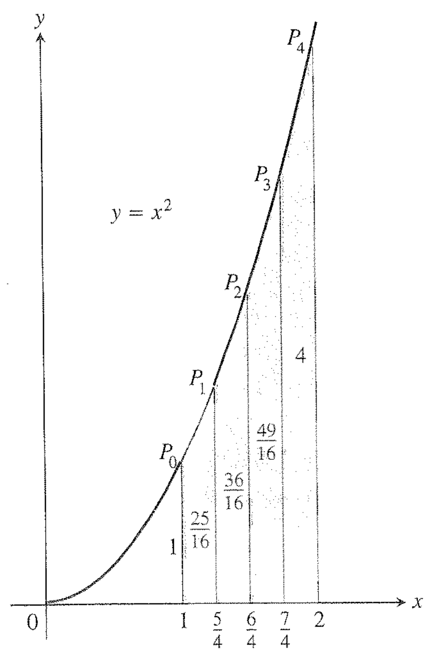


Figure 5.32 The trapezoidal approximation of the area under the graph of $y = x^2$ from $x = 1$ to $x = 2$ is a slight overestimate. (Example 1)

Table 5.4

x	$y = x^2$
1	1
$\frac{5}{4}$	$\frac{25}{16}$
$\frac{6}{4}$	$\frac{36}{16}$
$\frac{7}{4}$	$\frac{49}{16}$
2	4

Example 1 APPLYING THE TRAPEZOIDAL RULE

Use the Trapezoidal Rule with $n = 4$ to estimate $\int_1^2 x^2\,dx$. Compare estimate with the value of NINT $(x^2, x, 1, 2)$ and with the exact value

Solution Partition $[1, 2]$ into four subintervals of equal length (Figure 5 Then evaluate $y = x^2$ at each partition point (Table 5.4).

Using these y values, $n = 4$, and $h = (2 - 1)/4 = 1/4$ in the Trapez Rule, we have

$$T = \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right)$$

$$= \frac{1}{8}\left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right)$$

$$= \frac{75}{32} = 2.34375.$$

The value of NINT $(x^2, x, 1, 2)$ is 2.333333333.

The exact value of the integral is

$$\int_1^2 x^2\,dx = \left.\frac{x^3}{3}\right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

The T approximation overestimates the integral by about half a percent c true value of 7/3. The percentage error is $(2.34375 - 7/3)/(7/3) \approx 0.00$

We could have predicted that the Trapezoidal Rule would overestima integral in Example 1 by considering the geometry of the graph in Figure Since the parabola is concave *up,* the approximating segments lie abov curve, giving each trapezoid slightly more area than the corresponding under the curve. In Figure 5.31 we see that the straight segments lie *und* curve on those intervals where the curve is concave *down,* causin Trapezoidal Rule to *underestimate* the integral on those intervals. The pretation of "area" changes where the curve lies below the x-axis but it i the case that the higher y-values give the greater signed area. So we can a say that T overestimates the integral where the graph is concave up and u estimates the integral where the graph is concave down.

Example 2 AVERAGING TEMPERATURES

An observer measures the outside temperature every hour from noon midnight, recording the temperatures in the following table.

Time	N	1	2	3	4	5	6	7	8	9	10	11
Temp	63	65	66	68	70	69	68	68	65	64	62	58

What was the average temperature for the 12-hour period?

Solution We are looking for the average value of a continuous func (temperature) for which we know values at discrete times that are on apart. We need to find

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

without having a formula for $f(x)$. The integral, however, can be apr imated by the Trapezoidal Rule, using the temperatures in the table a function values at the points of a 12-subinterval partition of the 12-h interval (making $h = 1$).

$$T = \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{11} + y_{12}\right)$$

$$= \frac{1}{2}\left(63 + 2 \cdot 65 + 2 \cdot 66 + \cdots + 2 \cdot 58 + 55\right)$$

$$= 782$$

Using T to approximate $\int_a^b f(x)\,dx$, we have

$$av(f) \approx \frac{1}{b-a} \cdot T = \frac{1}{12} \cdot 782 \approx 65.17.$$

Rounding to be consistent with the data given, we estimate the averag temperature as 65 degrees.

Other Algorithms

LRAM, MRAM, RRAM, and the Trapezoidal Rule all give reasonable ap imations to the integral of a continuous function over a closed interva Trapezoidal Rule is more efficient, giving a better approximation for sma ues of n, which makes it a faster algorithm for numerical integration.

Indeed, the only shortcoming of the Trapezoidal Rule seems to be depends on approximating curved arcs with straight segments. You might that an algorithm that approximates the curve with *curved* pieces wou even more efficient (and hence faster for machines), and you would be All we need to do is find a geometric figure with a straight base, straight and a curved top that has a known area. You might not know one, bu ancient Greeks did; it is one of the things they knew about parabolas.

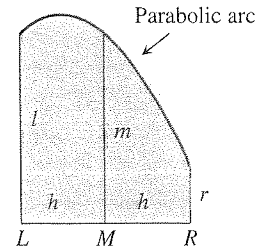


Figure 5.33 The area under the parabolic arc can be computed from the length of the base LR and the lengths of the altitudes constructed at L, R and midpoint M. (Exploration 1)

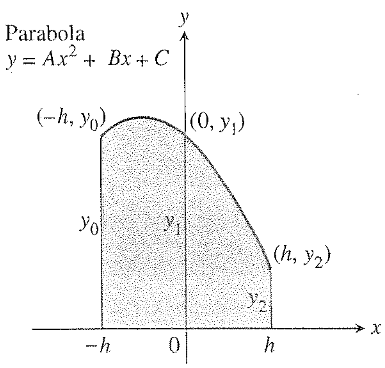


Figure 5.34 A convenient coordinatization of Figure 5.33. The parabola has equation $y = Ax^2 + Bx + C$, and the midpoint of the base is at the origin. (Exploration 1)

Exploration 1 Area Under a Parabolic Arc

The area A_P of a figure having a horizontal base, vertical sides, and a parabolic top (Figure 5.33) can be computed by the formula

$$A_P = \frac{h}{3}\left(l + 4m + r\right),$$

where h is half the length of the base, l and r are the lengths of the left and right sides, and m is the altitude at the midpoint of the base. This formula, once a profound discovery of ancient geometers, is readily verified today with calculus.

1. Coordinatize Figure 5.33 by centering the base at the origin, as shown in Figure 5.34. Let $y = Ax^2 + Bx + C$ be the equation of the parabola. Using this equation, show that $y_0 = Ah^2 - Bh + C$, $y_1 = C$, and $y_2 = Ah^2 + Bh + C$.
2. Show that $y_0 + 4y_1 + y_2 = 2Ah^2 + 6C$.
3. Integrate to show that the area A_P is
$$\frac{h}{3}(2Ah^2 + 6C).$$
4. Combine these results to derive the formula
$$A_P = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right).$$

This last formula leads to an efficient rule for approximating inte numerically. Partition the interval of integration into an even number of s tervals, apply the formula for A_P to successive interval pairs, and ad results. This algorithm is known as Simpson's Rule.

Simpson's Rule

To approximate $\int_a^b f(x)\,dx$, use

$$S = \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right),$$

where $[a, b]$ is partitioned into an *even* number n of subintervals of equal length $h = (b - a)/n$.

What's in a Name?

The formula that underlies Simpson's Rule (see Exploration 1) was discovered long before Thomas Simpson (1720–1761) was born. Just as Pythagoras did not discover the Pythagorean Theorem, Simpson did not discover Simpson's Rule. It is another of history's beautiful quirks that one of the ablest mathematicians of eighteenth-century England is remembered not for his successful textbooks and his contributions to mathematical analysis, but for a rule that was never his, that he never laid claim to, and that bears his name only because he happened to mention it in one of his books.

Example 3 APPLYING SIMPSON'S RULE

Use Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4\,dx$.

Solution Partition $[0, 2]$ into four subintervals and evaluate $y = 5x^4$ the partition points (Table 5.5).

Table 5.5

x	$y = 5x^4$
0	0
$\frac{1}{2}$	$\frac{5}{16}$
1	5
$\frac{3}{2}$	$\frac{405}{16}$
2	80

Then apply Simpson's Rule with $n = 4$ and $h = 1/2$:

$$\begin{aligned} S &= \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right) \\ &= \frac{1}{6}\left(0 + 4\left(\frac{5}{16}\right) + 2\left(5\right) + 4\left(\frac{405}{16}\right) + 80\right) \\ &= \frac{385}{12}. \end{aligned}$$

This estimate differs from the exact value (32) by only 1/12, a perce error of less than three-tenths of one percent—and this was with jus 4 subintervals.

There are still other algorithms for approximating definite integral of them involving fancy numerical analysis designed to make the calcı more efficient for high-speed computers. Some are kept secret by the nies that design the machines. In any case, we will not deal with them

Error Analysis

After finding that the trapezoidal approximation in Example 1 overesı the integral, we pointed out that this could have been predicted from tl cavity of the curve we were approximating.

Knowing something about the error in an approximation is more tl an interesting sidelight. Despite what your years of classroom experienc have suggested, exact answers are not always easy to find in mathemati fortunate that for all *practical* purposes exact answers are also rarely nec (For example, a carpenter who computes the need for a board of $\sqrt{34}$ feet will happily settle for an approximation when cutting the boa

Suppose that an exact answer really can *not* be found, but that wı that an approximation within 0.001 unit is good enough. How can we t our approximation is within 0.001 if we do not know the exact answer? where knowing something about the error is critical.

Since the Trapezoidal Rule approximates curves with straight l seems reasonable that the error depends on how "curvy" the graph i suggests that the error depends on the second derivative. It is also appar the error depends on the length h of the subintervals. It can be shown th is continuous the error in the trapezoidal approximation, denoted E_T, s. the inequality

$$|E_T| \leq \frac{b-a}{12} h^2 M_{f''},$$

where $[a, b]$ is the interval of integration, h is the length of each subir and $M_{f''}$ is the maximum value of $|f''|$ on $[a, b]$.

It can also be shown that the error E_S in Simpson's Rule depends oı the *fourth* derivative. It satisfies the inequality

$$|E_S| \leq \frac{b-a}{180} h^4 M_{f^{(4)}},$$

where $[a, b]$ is the interval of integration, h is the length of each subinterv $M_{f^{(4)}}$ is the maximum value of $|f^{(4)}|$ on $[a, b]$, provided that $f^{(4)}$ is continı

For comparison's sake, if all the assumptions hold, we have the follc *error bounds.*

Error Bounds

If T and S represent the approximations to $\int_a^b f(x)\,dx$ given by the Trapezoidal Rule and Simpson's Rule, respectively, then the errors E_T and E_S satisfy

$$|E_T| \le \frac{b-a}{12} h^2 M_{f''} \quad \text{and} \quad |E_S| \le \frac{b-a}{180} h^4 M_{f^{(4)}}.$$

If we disregard possible differences in magnitude between $M_{f''}$ and we notice immediately that $(b-a)/180$ is one-fifteenth the size of $(b - a$ giving S an obvious advantage over T as an approximation. That, howe almost insignificant when compared to the fact that the trapezoid error var the *square* of h, while Simpson's error varies as the *fourth power* (Remember that h is already a small number in most partitions.)

Table 5.6 shows T and S values for approximations of $\int_1^2 1/x\,dx$ usin ious values of n. Notice how Simpson's Rule dramatically improves ov Trapezoidal Rule. In particular, notice that when we double the value (thereby halving the value of h), the T error is divided by 2 *squared,* whi S error is divided by 2 *to the fourth.*

Table 5.6 Trapezoidal Rule Approximations (T_n) and Simpsor Rule Approximations (S_n) of ln 2 = $\int_1^2 (1/x)\,dx$

n	T_n	|Error| less than ...	S_n	|Error| less thai
10	.6937714032	.0006242227	.6931502307	.000003050
20	.6933033818	.0001562013	.6931473747	.000000194
30	.6932166154	.0000694349	.6931472190	.000000038
40	.6931862400	.0000390595	.6931471927	.000000012
50	.6931721793	.0000249988	.6931471856	.000000005
100	.6931534305	.0000062500	.6931471809	.000000000

This has a dramatic effect as h gets very small. The Simpson appro tion for $n = 50$ rounds accurately to seven places, and for $n = 100$ agr nine decimal places (billionths)!

We close by showing you the values (Table 5.7) we found for $\int_1^5 (\sin x$ by six different calculator methods. The exact value of this integral to six mal places is 0.603848, so both Simpson's method with 50 subinterva NINT give results accurate to at least six places (millionths).

Table 5.7 Approximations of $\int_1^5 (\sin x)/x\,dx$

Method	Subintervals	Value
LRAM	50	0.6453898
RRAM	50	0.5627293
MRAM	50	0.6037425
TRAP	50	0.6040595
SIMP	50	0.6038481
NINT	Tol = 0.00001	0.6038482

Quick Review 5.5

In Exercises 1–10, tell whether the curve is concave up or concave down on the given interval.

1. $y = \cos x$ on $[-1, 1]$

2. $y = x^4 - 12x - 5$ on $[8, 17]$

3. $y = 4x^3 - 3x^2 + 6$ on $[-8, 0]$

4. $y = \sin(x/2)$ on $[48\pi, 50\pi]$

5. $y = e^{2x}$ on $[-5, 5]$

6. $y = \ln x$ on $[100, 200]$

7. $y = \dfrac{1}{x}$ on $[3, 6]$

8. $y = \csc x$ on $[0, \pi]$

9. $y = 10^{10} - 10x^{10}$ on $[10, 10^{10}]$

10. $y = \sin x - \cos x$ on $[1, 2]$

Section 5.5 Exercises

In Exercises 1–6, **(a)** use the Trapezoidal Rule with $n = 4$ to approximate the value of the integral. **(b)** Use the concavity of the function to predict whether the approximation is an overestimate or an underestimate. Finally, **(c)** find the integral's exact value to check your answer.

1. $\int_0^2 x\,dx$

2. $\int_0^2 x^2\,dx$

3. $\int_0^2 x^3\,dx$

4. $\int_1^2 \frac{1}{x}\,dx$

5. $\int_0^4 \sqrt{x}\,dx$

6. $\int_0^{\pi} \sin x\,dx$

7. *Volume of Water in a Swimming Pool* A rectangular swimming pool is 30 ft wide and 50 ft long. The table below shows the depth $h(x)$ of the water at 5-ft intervals from one end of the pool to the other. Estimate the volume of water in the pool using the Trapezoidal Rule with $n = 10$, applied to the integral

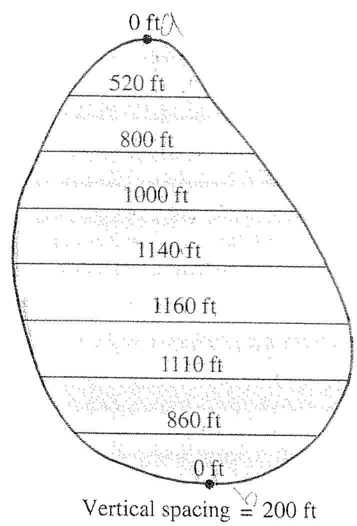
$$V = \int_0^{50} 30 \cdot h(x)\,dx.$$

Position (ft) x	Depth (ft) $h(x)$	Position (ft) x	Depth (ft) $h(x)$
0	6.0	30	11.5
5	8.2	35	11.9
10	9.1	40	12.3
15	9.9	45	12.7
20	10.5	50	13.0
25	11.0		

8. *Stocking a Fish Pond* As the fish and game warden of your township, you are responsible for stocking the town pond with fish before the fishing season. The average dept of the pond is 20 feet. Using a scaled map, you measure distances across the pond at 200-foot intervals, as shown i the diagram.

(a) Use the Trapezoidal Rule to estimate the volume of the pond.

(b) You plan to start the season with one fish per 1000 cubic feet. You intend to have at least 25% of the opening day's fish population left at the end of the season. What is the maximum number of licenses the town can sel if the average seasonal catch is 20 fish per license?



9. *Ford® Mustang Cobra™* The accompanying table shows time-to-speed data for a 1994 Ford Mustang Cobra accelerating from rest to 130 mph. How far had the Mustang traveled by the time it reached this speed? (Use trapezoids to estimate the area under the velocity curve, but be careful: the time intervals vary in length.)

Speed Change	Time (sec)
Zero to 30 mph	2.2
40 mph	3.2
50 mph	4.5
60 mph	5.9
70 mph	7.8
80 mph	10.2
90 mph	12.7
100 mph	16.0
110 mph	20.6
120 mph	26.2
130 mph	37.1

Source: Car and Driver, April 1994.

10. Consider the integral $\int_{-1}^{3}(x^3 - 2x)\,dx$.

(a) Use Simpson's Rule with $n = 4$ to approximate its value.

(b) Find the exact value of the integral. What is the error, $|E_S|$?

(c) Explain how you could have predicted what you found in (b) from knowing the error-bound formula.

(d) Writing to Learn Is it possible to make a general statement about using Simpson's Rule to approximate integrals of cubic polynomials? Explain.

11. Writing to Learn In Example 2 (before rounding) we found the average temperature to be 65.17 degrees when we used the integral approximation, yet the average of the 13 discrete temperatures is only 64.69 degrees. Considering the shape of the temperature curve, explain why you would expect the average of the 13 discrete temperatures to be less than the average value of the temperature function on the entire interval.

12. *(Continuation of Exercise 11)*

(a) In the Trapezoidal Rule, every function value in the sum is doubled except for the two endpoint values. Show that if you double the endpoint values, you get 70.08 for the average temperature.

(b) Explain why it makes more sense to not double the endpoint values if we are interested in the average temperature over the entire 12-hour period.

In Exercises 13–16, use a calculator program to find the Simpson's Rule approximations with $n = 50$ and $n = 100$.

13. $\int_{-1}^{1} 2\sqrt{1 - x^2}\,dx$ The exact value is π.

14. $\int_{0}^{1} \sqrt{1 + x^4}\,dx$ An integral that came up in Newton's research

15. $\int_{0}^{\pi/2} \frac{\sin x}{x}\,dx$

16. $\int_{0}^{\pi/2} \sin(x^2)\,dx$ An integral associated with the diffraction of light

17. Consider the integral $\int_0^{\pi} \sin x\,dx$.

(a) Use a calculator program to find the Trapezoidal R approximations for $n = 10$, 100, and 1000.

(b) Record the errors with as many decimal places of accuracy as you can.

(c) What pattern do you see?

(d) Writing to Learn Explain how the error bound f accounts for the pattern.

18. *(Continuation of Exercise 17)* Repeat Exercise 17 with Simpson's Rule and E_S.

Explorations

In Exercises 19 and 20, *work in groups of two or three.*

19. Consider the integral $\int_{-1}^{1} \sin(x^2)\,dx$.

(a) Find f'' for $f(x) = \sin(x^2)$.

(b) Graph $y = f''(x)$ in the viewing window $[-1, 1]$ b $[-3, 3]$.

(c) Explain why the graph in (b) suggests that $|f''(x)|$ for $-1 \le x \le 1$.

(d) Show that the error estimate for the Trapezoidal R this case becomes

$$|E_T| \le \frac{h^2}{2}.$$

(e) Show that the Trapezoidal Rule error will be less t equal to 0.01 if $h \le 0.1$.

(f) How large must n be for $h \le 0.1$?

20. Consider the integral $\int_{-1}^{1} \sin(x^2)\,dx$.

(a) Find $f^{(4)}$ for $f(x) = \sin(x^2)$. (You may want to check your work with a CAS if you have one available

(b) Graph $y = f^{(4)}(x)$ in the viewing window $[-1, 1]$ $[-30, 10]$.

(c) Explain why the graph in (b) suggests that $|f^{(4)}(x)| \le 30$ for $-1 \le x \le 1$.

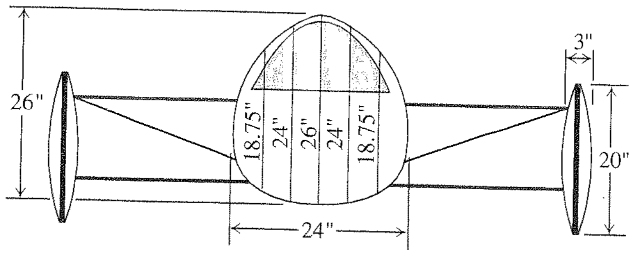
(d) Show that the error estimate for Simpson's Rule in this case becomes

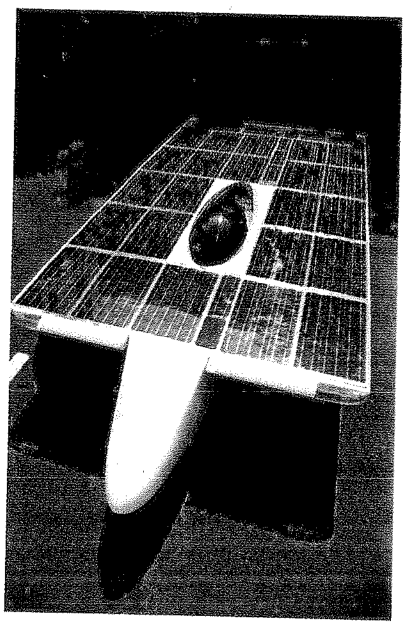
$$|E_S| \le \frac{h^4}{3}.$$

(e) Show that the Simpson's Rule error will be less th or equal to 0.01 if $h \le 0.4$.

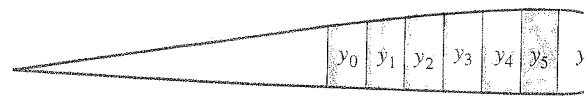
(f) How large must n be for $h \le 0.4$?

21. *Aerodynamic Drag* A vehicle's aerodynamic drag is determined in part by its cross section area, so, all other things being equal, engineers try to make this area as small as possible. Use Simpson's Rule to estimate the cross section area of the body of James Worden's solar-powered Solectria® automobile at M.I.T. from the diagram below.





22. *Wing Design* The design of a new airplane requires a gasoline tank of constant cross section area in each wi A scale drawing of a cross section is shown here. The t must hold 5000 lb of gasoline, which has a density of 42 lb/ft^3. Estimate the length of the tank.



$y_0 = 1.5$ ft, $y_1 = 1.6$ ft, $y_2 = 1.8$ ft, $y_3 = 1.9$ ft,
$y_4 = 2.0$ ft, $y_5 = y_6 = 2.1$ ft Horizontal spacing = 1 ft

Extending the Ideas

23. Using the definitions, prove that, in general,

$$T_n = \frac{\text{LRAM}_n + \text{RRAM}_n}{2}.$$

24. Using the definitions, prove that, in general,

$$S_{2n} = \frac{\text{MRAM}_n + 2T_{2n}}{3}.$$

Chapter 5 Key Terms

area under a curve (p. 262)
average daily holding cost (p. 285)
average daily inventory (p. 285)
average value (p. 271)
bounded function (p. 266)
cardiac output (p. 252)
characteristic function of the rationals (p. 266)
definite integral (p. 260)
differential calculus (p. 247)
dummy variable (p. 262)
error bounds (p. 294)
Fundamental Theorem of Calculus (p. 277)
integrable function (p. 260)
integral calculus (p. 247)
Integral Evaluation Theorem (p. 282)
integral of f from a to b (p. 261)
integral sign (p. 261)
integrand (p. 261)
inventory function (p. 285)
lower bound (p. 270)
lower limit of integration (p. 261)
LRAM (p. 249)
mean value (p. 271)
Mean Value Theorem for Definite Integrals (p. 272)
MRAM (p. 249)
net area (p. 264)
NINT (p. 265)
norm of a partition (p. 260)
partition (p. 259)
Rectangular Approximation Method (RAM) (p. 249)
regular partition (p. 260)
Riemann sum (p. 258)
RRAM (p. 249)
sigma notation (p. 258)
Simpson's Rule (p. 292)
subinterval (p. 259)
total area (p. 283)
Trapezoidal Rule (p. 289)
upper bound (p. 270)
upper limit of integration (p. 261)
variable of integration (p. 261)

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Chapter 5 Review Exercises

Exercises 1–6 refer to the region R in the first quadrant enclosed by the x-axis and the graph of the function $y = 4x - x^3$.

1. Sketch R and partition it into four subregions, each with a base of length $\Delta x = 1/2$.

2. Sketch the rectangles and compute (by hand) the area for the LRAM_4 approximation.

3. Sketch the rectangles and compute (by hand) the area for the MRAM_4 approximation.

4. Sketch the rectangles and compute (by hand) the area for the RRAM_4 approximation.

5. Sketch the trapezoids and compute (by hand) the area for the T_4 approximation.

6. Find the exact area of R by using the Fundamental Theorem of Calculus.

7. Use a calculator program to compute the RAM approximations in the following table for the area under the graph of $y = 1/x$ from $x = 1$ to $x = 5$.

n	LRAM_n	MRAM_n	RRAM_n
10			
20			
30			
50			
100			
1000			

8. *(Continuation of Exercise 7)* Use the Fundamental Theorem of Calculus to determine the value to which the sums in the table are converging.

9. Suppose

$$\int_{-2}^{2} f(x)\,dx = 4, \quad \int_{2}^{5} f(x)\,dx = 3, \quad \int_{-2}^{5} g(x)\,dx = 2.$$

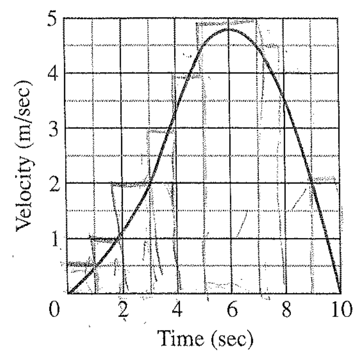
Which of the following statements are true, and which, if any, are false?

(a) $\int_{5}^{2} f(x)\,dx = -3$ **(b)** $\int_{-2}^{5} [f(x) + g(x)]\,dx = 9$

(c) $f(x) \le g(x)$ on the interval $-2 \le x \le 5$

10. The region under one arch of the curve $y = \sin x$ is revolved around the x-axis to form a solid. **(a)** Use the method of Example 3, Section 5.1, to set up a Riemann sum that approximates the volume of the solid. **(b)** Find the volume using NINT.

11. The accompanying graph shows the velocity (m/sec) of body moving along the s-axis during the time interval f $t = 0$ to $t = 10$ sec. **(a)** About how far did the body tr during those 10 seconds?



(b) Sketch a graph of position (s) as a function of time for $0 \le t \le 10$, assuming $s(0) = 0$.

12. The interval $[0, 10]$ is partitioned into n subintervals ol length $\Delta x = 10/n$. We form the following Riemann s choosing each c_k in the k^{th} subinterval. Write the limit $n \to \infty$ of each Riemann sum as a definite integral.

(a) $\sum_{k=1}^{n} (c_k)^3 \Delta x$ **(b)** $\sum_{k=1}^{n} c_k(\sin c_k)\Delta$

(c) $\sum_{k=1}^{n} c_k(3c_k - 2)^2 \Delta x$ **(d)** $\sum_{k=1}^{n} (1 + c_k^2)^-$

(e) $\sum_{k=1}^{n} \pi(9 - \sin^2(\pi c_k/10))\Delta x$

In Exercises 13 and 14, find the total area between the curv the x-axis.

13. $y = 4 - x, \quad 0 \le x \le 6$ **14.** $y = \cos x, \quad 0 \le x \le$

In Exercises 15–24, evaluate the integral analytically by usin Integral Evaluation Theorem (Part 2 of the Fundamental The Theorem 4).

15. $\int_{-2}^{2} 5\,dx$ **16.** $\int_{2}^{5} 4x\,dx$

17. $\int_{0}^{\pi/4} \cos x\,dx$ **18.** $\int_{-1}^{1} (3x^2 - 4x + 7)$

19. $\int_{0}^{1} (8s^3 - 12s^2 + 5)\,ds$ **20.** $\int_{1}^{2} \frac{4}{x^2}\,dx$

21. $\int_{1}^{27} y^{-4/3}\,dy$ **22.** $\int_{1}^{4} \frac{dt}{t\sqrt{t}}$

23. $\int_{0}^{\pi/3} \sec^2\theta\,d\theta$ **24.** $\int_{1}^{e} (1/x)\,dx$

In Exercises 25–29, evaluate the integral.

25. $\displaystyle\int_0^1 \frac{36}{(2x+1)^3}\,dx$

26. $\displaystyle\int_1^2 \left(x + \frac{1}{x^2}\right) dx$

27. $\displaystyle\int_{-\pi/3}^{0} \sec x \tan x\,dx$

28. $\displaystyle\int_{-1}^{1} 2x \sin(1 - x^2)\,dx$

29. $\displaystyle\int_0^2 \frac{2}{y+1}\,dy$

In Exercises 30–32, evaluate the integral by interpreting it as area and using formulas from geometry.

30. $\displaystyle\int_0^2 \sqrt{4 - x^2}\,dx$

31. $\displaystyle\int_{-4}^{8} |x|\,dx$

32. $\displaystyle\int_{-8}^{8} 2\sqrt{64 - x^2}\,dx$

33. *Oil Consumption on Pathfinder Island* A diesel generator runs continuously, consuming oil at a gradually increasing rate until it must be temporarily shut down to have the filters replaced.

Day	Oil Consumption Rate (liters/hour)
Sun	0.019
Mon	0.020
Tue	0.021
Wed	0.023
Thu	0.025
Fri	0.028
Sat	0.031
Sun	0.035

(a) Give an upper estimate and a lower estimate for the amount of oil consumed by the generator during that week.

(b) Use the Trapezoidal Rule to estimate the amount of oil consumed by the generator during that week.

34. *Rubber-Band–Powered Sled* A sled powered by a wound rubber band moves along a track until friction and the unwinding of the rubber band gradually slow it to a stop. A speedometer in the sled monitors its speed, which is recorded at 3-second intervals during the 27-second run.

Time (sec)	Speed (ft/sec)
0	5.30
3	5.25
6	5.04
9	4.71
12	4.25
15	3.66
18	2.94
21	2.09
24	1.11
27	0

(a) Give an upper estimate and a lower estimate f distance traveled by the sled.

(b) Use the Trapezoidal Rule to estimate the dista traveled by the sled.

35. Writing to Learn Your friend knows how to co integrals but never could understand what differen "dx" makes, claiming that it is irrelevant. How wc explain to your friend why it is necessary?

36. The function

$$f(x) = \begin{cases} x^2, & x \ge 0 \\ x - 2, & x < 0 \end{cases}$$

is discontinuous at 0, but integrable on $[-4, 4]$. Fi $\int_{-4}^{4} f(x)\,dx$.

37. Show that $0 \le \int_0^1 \sqrt{1 + \sin^2 x}\,dx \le \sqrt{2}$.

38. Find the average value of

(a) $y = \sqrt{x}$ over the interval $[0, 4]$.

(b) $y = a\sqrt{x}$ over the interval $[0, a]$.

In Exercises 39–42, find dy/dx.

39. $\displaystyle y = \int_2^x \sqrt{2 + \cos^3 t}\,dt$

40. $\displaystyle y = \int_2^{7x^2} \sqrt{2 + c}$

41. $\displaystyle y = \int_x^1 \frac{6}{3 + t^4}\,dt$

42. $\displaystyle y = \int_x^{2x} \frac{1}{t^2 + 1}$

43. *Printing Costs* Including start-up costs, it costs a p \$50 to print 25 copies of a newsletter, after which the marginal cost at x copies is

$$\frac{dc}{dx} = \frac{2}{\sqrt{x}} \text{ dollars per copy.}$$

Find the total cost of printing 2500 newsletters.

44. *Average Daily Inventory* Rich Wholesale Foods, ε manufacturer of cookies, stores its cases of cookies air-conditioned warehouse for shipment every 14 da Rich tries to keep 600 cases on reserve to meet occa peaks in demand, so a typical 14-day inventory func $I(t) = 600 + 600t,\ 0 \le t \le 14$. The holding cost fc case is 4¢ per day. Find Rich's average daily inventc average daily holding cost.

45. Solve for x: $\int_0^x (t^3 - 2t + 3)\,dt = 4$.

46. Suppose $f(x)$ has a positive derivative for all values o and that $f(1) = 0$. Which of the following statement be true of

$$g(x) = \int_0^x f(t)\,dt?$$

(a) g is a differentiable function of x.

(b) g is a continuous function of x.

(c) The graph of g has a horizontal tangent line at $x =$

(d) g has a local maximum at $x = 1$.

(e) g has a local minimum at $x = 1$.

(f) The graph of g has an inflection point at $x = 1$.

(g) The graph of dg/dx crosses the x-axis at $x = 1$.

47. Suppose $F(x)$ is an antiderivative of $f(x) = \sqrt{1 + x^4}$. Express $\int_0^1 \sqrt{1 + x^4}\, dx$ in terms of F.

48. Express the function $y(x)$ with

$$\frac{dy}{dx} = \frac{\sin x}{x} \quad \text{and} \quad y(5) = 3$$

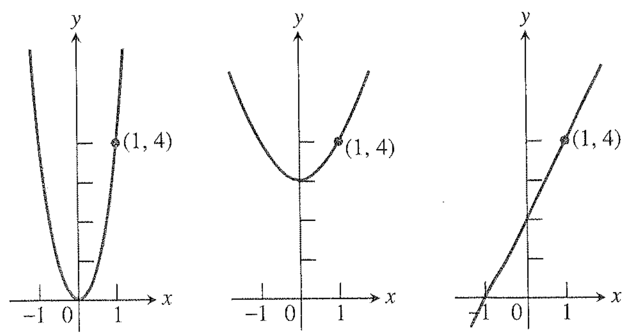
as a definite integral.

49. Show that $y = x^2 + \int_1^x 1/t\, dt + 1$ satisfies both of the following conditions:

i. $y'' = 2 - \dfrac{1}{x^2}$

ii. $y = 2$ and $y' = 3$ when $x = 1$.

50. Writing to Learn Which of the following is the graph of the function whose derivative is $dy/dx = 2x$ and whose value at $x = 1$ is 4? Explain your answer.



(a) (b) (c)

51. *Fuel Efficiency* An automobile computer gives a digital readout of fuel consumption in gallons per hour. During a trip, a passenger recorded the fuel consumption every 5 minutes for a full hour of travel.

time	gal/h	time	gal/h
0	2.5	35	2.5
5	2.4	40	2.4
10	2.3	45	2.3
15	2.4	50	2.4
20	2.4	55	2.4
25	2.5	60	2.3
30	2.6		

(a) Use the Trapezoidal Rule to approximate the total fuel consumption during the hour.

(b) If the automobile covered 60 miles in the hour, what was its fuel efficiency (in miles per gallon) for that portion of the trip?

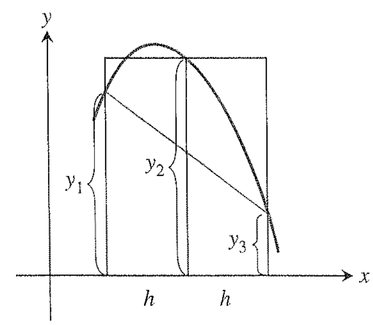
52. *Skydiving* Skydivers A and B are in a helicopter hoveri at 6400 feet. Skydiver A jumps and descends for 4 sec before opening her parachute. The helicopter then climb 7000 feet and hovers there. Forty-five seconds after A le the aircraft, B jumps and descends for 13 sec before ope her parachute. Both skydivers descend at 16 ft/sec with parachutes open. Assume that the skydivers fall freely (\ acceleration -32 ft/sec^2) before their parachutes open.

(a) At what altitude does A's parachute open?

(b) At what altitude does B's parachute open?

(c) Which skydiver lands first?

53. *Relating Simpson's Rule, MRAM, and T* The figure below shows an interval of length $2h$ with a trapezoid, a midpoint rectangle, and a parabolic region on it.



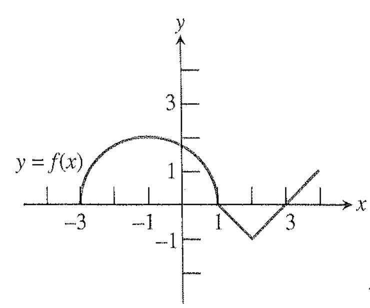
(a) Show that the area of the trapezoid plus twice the ar of the rectangle equals

$$h(y_1 + 4y_2 + y_3).$$

(b) Use the result in (a) to prove that

$$S_{2n} = \frac{2 \cdot \text{MRAM}_n + T_n}{3}.$$

54. The graph of a function f consists of a semicircle and tv line segments as shown below.



Let $g(x) = \int_1^x f(t)\, dt$.

(a) Find $g(1)$. **(b)** Find $g(3)$. **(c)** Find $g(-1)$

(d) Find all values of x on the open interval $(-3, 4)$ at \ g has a relative maximum.

(e) Write an equation for the line tangent to the graph o g at $x = -1$.

(f) Find the x-coordinate of each point of inflection of graph of g on the open interval $(-3, 4)$.

(g) Find the range of g.

55. What is the total area under the curve $y = e^{-x^2/2}$?

The graph approaches the x-axis as an asymptote both to the left and the right, but quickly enough so that the total area is a finite number. In fact,

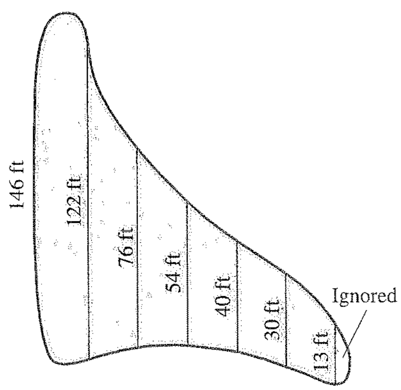
$$\text{NINT}\,(e^{-x^2/2},\ x,\ -10,\ 10)$$

computes all but a negligible amount of the area.

(a) Find this number on your calculator. Verify that NINT $(e^{-x^2/2}, x, -20, 20)$ does not increase the number enough for the calculator to distinguish the difference.

(b) This area has an interesting relationship to π. Perform various (simple) algebraic operations on the number to discover what it is.

56. *Filling a Swamp* A town wants to drain and fill the small polluted swamp shown below. The swamp averages 5 ft deep. About how many cubic yards of dirt will it take to fill the area after the swamp is drained?



Horizontal spacing = 20 ft

57. *Household Electricity* We model the voltage V in our homes with the sine function

$$V = V_{max} \sin(120\,\pi t),$$

which expresses V in volts as a function of time t in seconds. The function runs through 60 cycles each second. The number V_{max} is the *peak voltage.*

To measure the voltage effectively, we use an instrument that measures the square root of the average value of the square of the voltage over a 1-second interval:

$$V_{rms} = \sqrt{(V^2)_{av}}\,.$$

The subscript "rms" stands for "root mean square." It turns out that

$$V_{rms} = \frac{V_{max}}{\sqrt{2}}. \qquad (1)$$

The familiar phrase "115 volts ac" means that the rms voltage is 115. The peak voltage, obtained from Equation 1 as $V_{max} = 115\sqrt{2}$, is about 163 volts.

(a) Find the average value of V^2 over a 1-sec interval. Then find V_{rms}, and verify Equation 1.

(b) The circuit that runs your electric stove is rated 240 volts rms. What is the peak value of the allowable voltage?

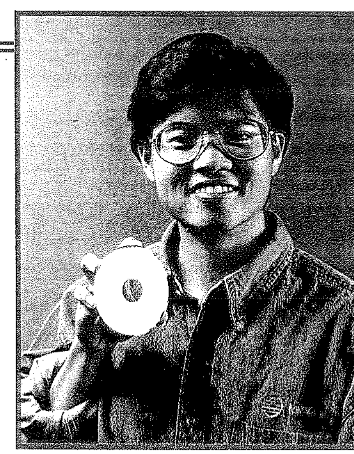
Calculus at Work

I have a degree in Mechanical Engineering with a minor in Psychology. I am a Research and Development Engineer at Komag, which designs and manufactures hard disks in Santa Clara, California. My job is to test the durability and reliability of the disks, measuring the rest friction between the read/write heads and the disk surface, which is called "Contact-Start-Stop" testing.

I use calculus to evaluate the moment of inertia of different disk stacks, which consist of disks on a spindle, separated by spacer rings. Because the rings vary in size as well as material, the mass of each ring must be determined. For such problems, I refer to my college calculus textbook and its tables of summations and integrals. For instance, I use:

$$\text{Moment of Inertia} = \left[\sum_{i=1}^{n} M_i L_i^{\,2}\right] + \frac{1}{3} M_{rod} L_{rod}^{\,2}$$

where i = components 1 to n; M_i = mass of component i such as the disk and/or ring stack; L_i = distance of component i from a reference point; M_{rod} = mass of the spindle that rotates; L_{rod} = length of the spindle.



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