Focus On
PREREQUISITE SKILLS

To be successful in this chapter, you'll need to understand these concepts and be able to apply them. Refer to the lesson in parentheses if you need more review before beginning the chapter.

Determine the number of roots of a quadratic equation by examining the graph. (Lesson 6–1)

Graph each equation. Then give the number of real solutions.

1. $x^2 + 3x - 5 = 0$ **2.** $2x^2 + x + 6 = 0$ **3.** $x^2 + 20x + 100 = 0$

Divide polynomials using long division. (Lesson 5–3)

Simplify.

4. $(y^2 - 2y - 30) \div (y + 7)$ **5.** $(3a^2 - 14a - 24) \div (3a + 4)$

Divide polynomials by binomials using synthetic division. (Lesson 5–3)

Use synthetic division to find each quotient.

6. $(x^3 + 2x^2 - 3x + 1) \div (x + 1)$

7. $(3x^3 - x^2 + 5x - 2) \div (x - 2)$

Use discriminants to determine the nature of roots of quadratic equations. (Lesson 6–4)

Find the value of the discriminant and describe the number and nature of the roots (real, imaginary, rational, irrational) of each quadratic equation.

8. $c^2 + 12c + 5 = 0$ **9.** $z^2 + 3z + 4 = 0$

10. $d^2 - 12d + 36 = 0$ **11.** $2x^2 - x - 3 = 0$

Focus On
READING SKILLS

In this chapter, you will learn about ***polynomial functions***. You have already learned that a polynomial is a monomial or sum of monomials. You have also learned that a function is a special type of relation in which each element of the domain is paired with exactly one element from the range. A polynomial function is a function in only one variable. An example is $f(x) = x^4 + 2x^3 - 5x + 9$, where x is the only variable. You will learn techniques for graphing and solving polynomial functions in this chapter.

8-1 Polynomial Functions

***What* YOU'LL LEARN**

- To evaluate polynomial functions, and
- to identify general shapes of the graphs of polynomial functions.

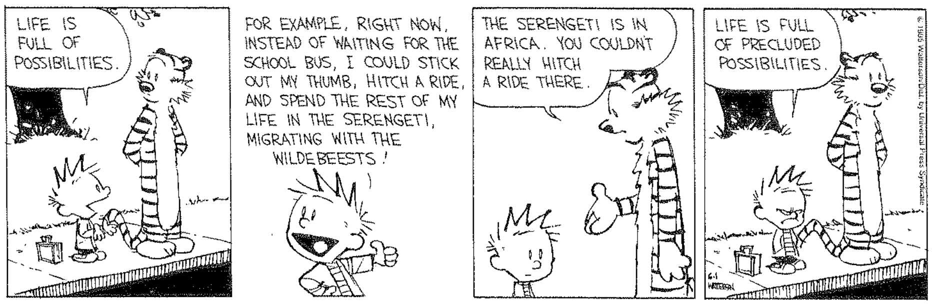
***Why* IT'S IMPORTANT**

You can use polynomial functions to solve problems involving biology and energy.

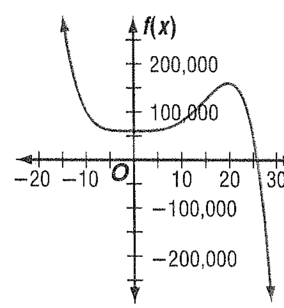
CONNECTION

Biology

Calvin and Hobbes **by Bill Watterson**



Calvin has to give up his ambition of migrating with the wildebeests when he learns that they live on another continent. The Serengeti Plain is a wild game reserve in Africa where herds of wildebeests, or gnus, roam. The population of wildebeests on the Serengeti Plain can be described by the function $f(x) = -0.125x^5 + 3.125x^4 + 58{,}000$, where x represents the number of years since 1990. The graph at the right shows this function. The expression $-0.125x^5 + 3.125x^4 + 58{,}000$ is a **polynomial in one variable.** *You will find the population of wildebeests in Example 2.*



Definition of a Polynomial in One Variable	A polynomial of degree n in one variable x is an expression of the form $a_0x^n + a_1x^{n-1} + \ldots + a_{n-2}x^2 + a_{n-1}x + a_n$, where the coefficients $a_0, a_1, a_2, \ldots, a_n$ represent real numbers, a_0 is not zero, and n represents a nonnegative integer.

In Chapter 2, linear functions, which are degree 1, were identified and graphed. In Chapter 6, quadratic functions, which are degree 2, were identified and graphed. In general, the degree of a polynomial in one variable is determined by the greatest exponent of its variable.

Remember that $4 = 4x^0$ and $x + 8 = x^1 + 8x^0$.

Polynomial	Expression	Degree
Constant	4	0
Linear	$x + 8$	1
Quadratic	$3x^2 + 4x - 3$	2
Cubic	$4x^3 - 5$	3
General	$a_0x^n + a_1x^{n-1} + \ldots + a_{n-2}x^2 + a_{n-1}x + a_n$	n

Example 1 **Determine if each expression is a polynomial in one variable. If so, determine its degree.**

a. $6x^4 + 3x^2 + 4x - 8$

This is a polynomial in one variable, x. The degree is 4.

b. $9x^3y^5 + 2x^2y^6 - 4$

This is not a polynomial in one variable. It contains two variables, x and y.

c. $t^{-3} + 4t^2 - 1$

This is not a polynomial, because the variable has a negative exponent.

d. $5x^7 + 3x^2 + \frac{2}{x}$

This is not a polynomial, because the term $\frac{2}{x}$ cannot be written in the form x^n, where n is a nonnegative integer.

fabulous **FIRSTS**

Joshua Stewart (1983–)

Joshua Stewart became the youngest climber to reach the 19,340-foot summit of Kilimanjaro, when he reached the peak with his father and African guides in 1994 at the age of 11. This mountain, located near the Serengeti Plain in Tanzania, is the highest peak in Africa.

When a polynomial equation is used to represent a function, the function is a **polynomial function.** For example, the equation $f(x) = 4x^2 - 5x + 2$ describes a quadratic polynomial function, and the equation $p(x) = 2x^3 + 4x^2 - 5x + 7$ describes a cubic polynomial function. These and other polynomial functions can be defined by the following general rule.

Definition of a Polynomial Function	**A polynomial function of degree n can be described by an equation of the form $P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-2}x^2 + a_{n-1}x + a_n$, where the coefficients $a_0, a_1, a_2, \ldots, a_{n-1}$, and a_n represent real numbers, a_0 is not zero, and n represents a nonnegative integer.**

If you know an element in the domain of any polynomial function, you can find the corresponding value in the range. Remember that if $f(x)$ is the function and 4 is an element in the domain, the corresponding element in the range is $f(4)$. To find $f(4)$, evaluate the function for $x = 4$.

Example 2 CONNECTION **Biology**

Refer to the application at the beginning of the lesson. Use the polynomial function to estimate the population of wildebeests in 1995.

x represents the number of years since 1990, 1995−1990 or 5.

$f(x) = -0.125x^5 + 3.125x^4 + 58{,}000$

$f(5) = -0.125(5)^5 + 3.125(5)^4 + 58{,}000$ *Replace x with 5.*

$= -390.625 + 1953.125 + 58{,}000$ or $59{,}562.5$ *Evaluate.*

Therefore, in 1995, there were approximately 59,562 wildebeests.

Example 3 **a. Find $p(a + 2)$ if $p(x) = x^3 - 2x + 1$.**

$p(a + 2) = (a + 2)^3 - 2(a + 2) + 1$ *Substitute $a + 2$ for x.*

$= a^3 + 6a^2 + 12a + 8 - 2a - 4 + 1$

$= a^3 + 6a^2 + 10a + 5$

(continued on the next page)

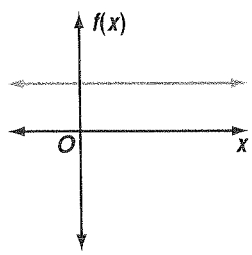
b. Find $-2p(a) + p(a + 1)$ if $p(x) = x^3 + 3x^2 - 5$.

$$-2p(a) + p(a + 1) = [-2(a^3 + 3a^2 - 5)] + [(a + 1)^3 + 3(a + 1)^2 - 5]$$
$$= -2a^3 - 6a^2 + 10 + a^3 + 3a^2 + 3a + 1 + 3(a^2 + 2a + 1) - 5$$
$$= -2a^3 - 6a^2 + 10 + a^3 + 3a^2 + 3a + 1 + 3a^2 + 6a + 3 - 5$$
$$= -a^3 + 9a + 9$$

Remember that the x-coordinate of a point at which the graph crosses the x-axis is called a zero of the function. On the coordinate plane, these zeros are real numbers.

The graphs of several polynomial functions are shown below. Notice how many times the graph of each function intersects the x-axis. In each case, this is the maximum number of real zeros the function may have. How does the degree compare to the maximum number of real zeros?

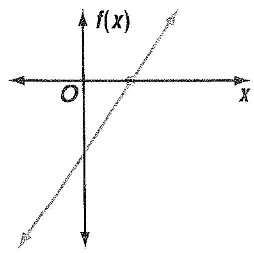
Constant function



$f(x) = 2$

Degree 0

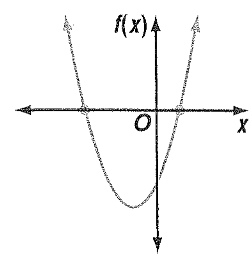
Linear function



$f(x) = \frac{3}{2}x - 3$

Degree 1

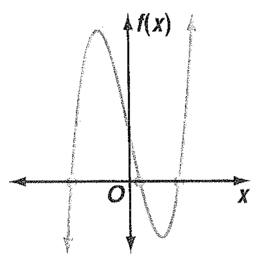
Quadratic function



$f(x) = x^2 + 2x - 3$

Degree 2

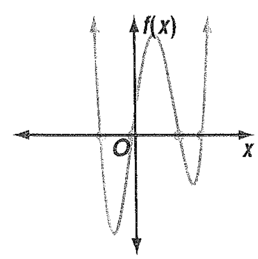
Cubic function



$f(x) = x^3 - 5x + 2$

Degree 3

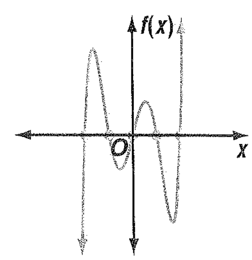
Quartic function



$f(x) = x^4 - 3x^3 - 2x^2 + 7x + 1$

Degree 4

Quintic function



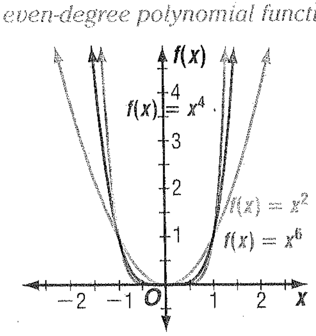
$f(x) = x^5 - 5x^3 + 4x$

Degree 5

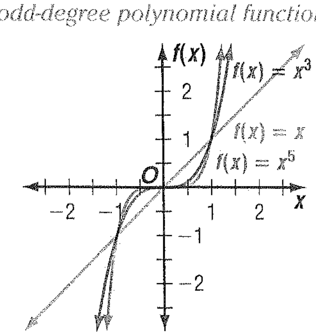
These are the general shapes of the graphs for polynomial functions with degree greater than 0 and positive **leading coefficients**. The leading coefficient is the coefficient of the term with the highest degree. So, for the function $f(x) = 5x^3 + 2x^2 + 8x - 1$, the leading coefficient is 5.

Notice that the simplest polynomial graphs have equations in the form $f(x) = x^n$, where n is a positive number. Note the general shapes of the graphs for even-degree polynomial functions and odd-degree polynomial functions.

even-degree polynomial functions



odd-degree polynomial functions



LOOK BACK

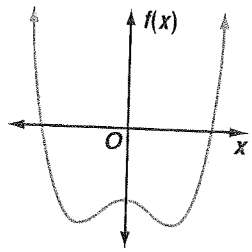
Refer to Lesson 6-1 for a review of roots of quadratic equations.

Note that the even-degree functions are tangent to the x-axis at the origin. When this happens, the function has two zeros or roots that are the same number. For example, $x^2 - 6x + 9 = 0$ can be factored as $(x - 3)(x - 3) = 0$. So 3 is the root of the equation.

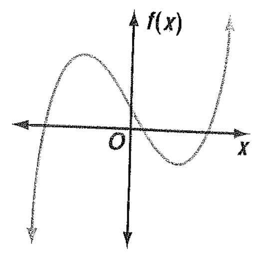
An even-degree function may or may not intersect the x-axis, depending on its location in the coordinate plane. If it does not intersect the x-axis, its roots are all imaginary. An odd-degree function always crosses the x-axis at least once. *Why?*

Example 4 **Determine if each graph represents an odd-degree function or an even-degree function. Then state how many real zeros each function has.**

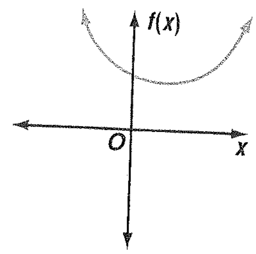
a.



b.

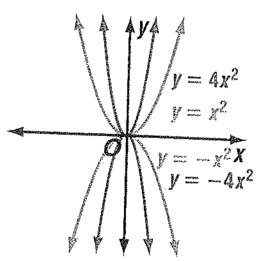


c.

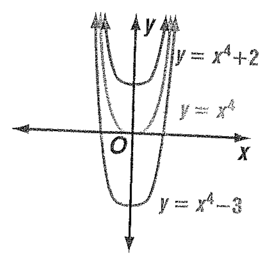


Graph	left-most y values	right-most y values	degree of function	times graph crosses x-axis	number of real zeros
a.	positive	positive	even	2	2
b.	negative	positive	odd	3	3
c.	positive	positive	even	0	0

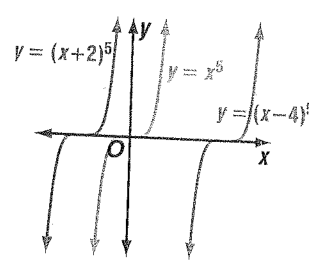
In Chapter 2, you studied families of graphs of linear equations. In Chapter 6, you studied families of parabolas. Some families of graphs of polynomial equations are shown below. The equation below each graph is the equation of the parent graph for that family.



$y = x^2$



$y = x^4$



$y = x^5$

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. Refer to Example 4. What appears to be the degree of each function?
2. **Describe** the characteristics of the graphs of odd-degree and even-degree polynomial functions whose leading coefficients are positive.
3. **State** how many real zeros are possible for each polynomial function.

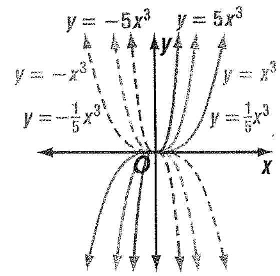
   a. quartic b. linear c. quadratic d. quintic e. cubic

4. **Sketch** the graph of an odd-degree function with a positive leading coefficient and three real roots.

5. **You Decide** Carlos explains to his friend Zach, "The graphs of odd-degree polynomial functions always intersect the x-axis an odd number of times, and the graphs of even-degree functions always intersect the x-axis an even number of times." Zach doesn't believe this is always the case. Who is correct? Give an example to support your answer.

MATH JOURNAL

6. Look at the family of graphs at the right for the function $f(x) = x^3$. Investigate the relationship between the similar functions and their graphs.



Guided Practice

Find the degree of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

7. $4t^3 + 8t^2 + 2t - 1$
8. $9ab^2 + 4ab + 3$
9. $7x^3 - 8x^5 + 8x - 7$

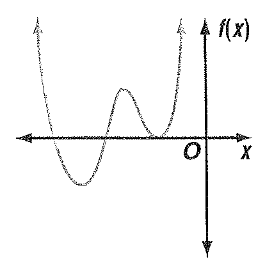
Identify each polynomial function as *linear, quadratic, cubic, quartic,* or *quintic.* State the degree and how many real zeros are possible.

10. $f(x) = 6x^3 + 8x + 7$
11. $f(x) = 3x^5 + 7x^4 - 5x^3 + 6x + 9$
12. $f(x) = 2x + 5$
13. $f(x) = 5x^3 + 6x^2 - 8x^4 - 10x + 17$

Match the polynomial and its functional value.

14. $p(x) = 3x^2 + 4x + 5$ — a. $p(4) = -166$
15. $p(x) = x^4 - 7x^3 + 8x - 6$ — b. $p(5) = 100$
16. $p(x) = 7x^2 - 9x + 10$ — c. $p(-4) = -259$
17. $p(x) = 4x^3 - 2x^2 - 6x + 5$ — d. $p(-2) = 56$

18. Refer to the graph at the right.
    a. Determine whether the degree of the function is even or odd.
    b. How many real zeros does the polynomial function have?



Find $p(2)$ and $p(-1)$ for each function.

19. $p(x) = 2x^2 + 6x - 8$
20. $p(x) = -3x^4 + 1$

Find $f(x + h)$ for each function.

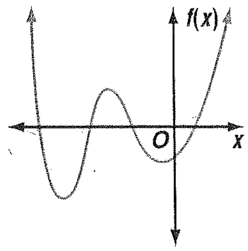
21. $f(x) = 2x - 3$
22. $f(x) = 4x^2$

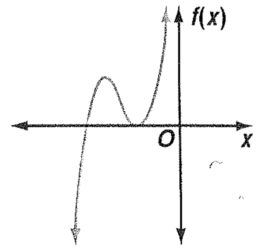
23. **Energy** The power generated by a windmill is a function of the speed of the wind. The approximate power is given by the function $P(s) = \frac{s^3}{1000}$, where s represents the speed of the wind in kilometers per hour. Find the units of power generated by a windmill when the wind speed is 25 kilometers per hour.

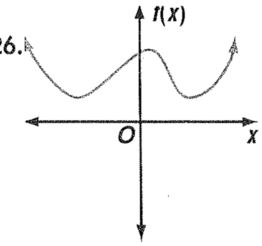
EXERCISES

Practice

Determine whether the degree of the function represented by each graph is even or odd. How many real zeros does each polynomial function have?

24.


25.


26.


Find $p(3)$ and $p(-2)$ for each function.

27. $p(x) = 5x + 6$

28. $p(x) = x^2 - 2x + 1$

29. $p(x) = 2x^3 - x^2 - 3x + 1$

30. $p(x) = x^5 - x^2$

31. $p(x) = -x^4 + 53$

32. $p(x) = x^5 + 5x^4 - 15x^2 - 8$

Find $f(x + h)$ for each function.

33. $f(x) = x + 2$

34. $f(x) = x - 4$

35. $f(x) = 5x^2$

36. $f(x) = x^2 - 2x + 5$

37. $f(x) = 3x^2 + 7$

38. $f(x) = x^3 + x$

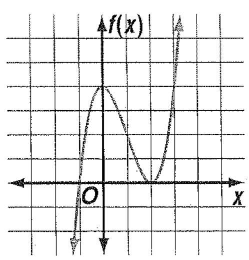
Find $4[p(x)]$ for each function.

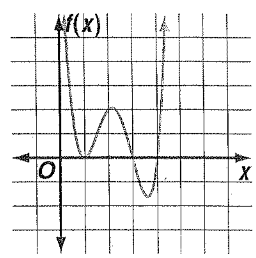
39. $p(x) = x^2 + 5$

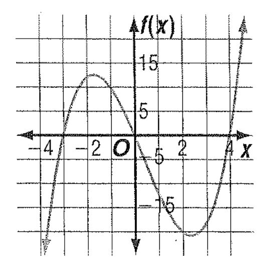
40. $p(x) = 6x^3 - 4x^2 + 2$

41. $p(x) = \frac{x^3}{4} + \frac{x^2}{16} - 2$

Find an equation for each graph.

42.


43.


44.


45. Sketch a graph of a polynomial function $f(x)$ that has the indicated number and type of zeros.
 a. 5 real
 b. 3 real, 2 imaginary
 c. 4 imaginary

Find $2p(a) + p(a - 1)$ for each function.

46. $p(x) = 4x + 1$

47. $p(x) = x^2 + 3$

48. $p(x) = x^2 - 5x + 8$

Find $2[f(x + 3)]$ for each function.

49. $f(x) = 2x + 9$

50. $f(x) = x^2 - 6$

51. $f(x) = x^2 + 3x + 12$

Graphing Calculator

52. Sketch a graph that matches each description. Write an equation for the graph of each function.
 a. a quadratic function with zeros at −1 and 2.
 b. a cubic function with zeros at −2, −1, and 2.
 c. a quartic function with zeros at −2, −1, 1, and 2.
 d. a quintic function with zeros at −2, −1, 0, 1, and 2.

Critical Thinking

53. Although a fourth-degree function can have as many as four real zeros, $P(x) = x^4 + x^2 + 1$ has no real zeros. Can you explain why?

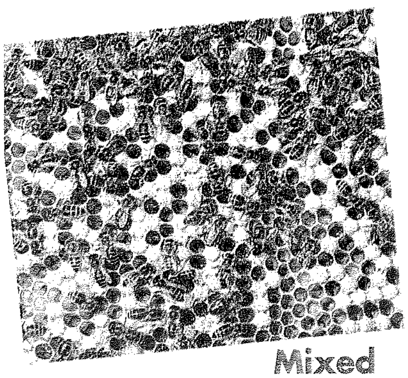
54. The graph of the polynomial function $f(x) = ax(x - 4)(x + 1)$ goes through the point at (5, 15).
    a. Find the value of a.
    b. Sketch the graph of the function.

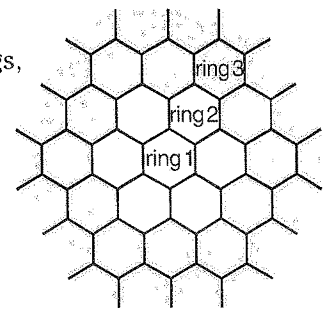
Applications and Problem Solving

Real World

55. **Biology** The intensity of light emitted by a firefly can be determined by the polynomial function $L(t) = 10 + 0.3t + 0.4t^2 - 0.01t^3$, where t is the temperature in Celsius and $L(t)$ is the light intensity in lumens. If the temperature is 30° C, find the light intensity.

56. **Patterns** If you look at a cross section of a honeycomb, you see a pattern of hexagons. This pattern has one hexagon surrounded by six more hexagons. Surrounding these is a third "ring" of 12 hexagons, and so on. Assume that the pattern continues.
    a. Find the number of hexagons in the 4th ring.
    b. Make a table that shows the total number of hexagons in the first ring, the first two rings, the first three rings, and the first four rings.
    c. Show that the polynomial function $h = 3r^2 - 3r + 1$ gives the total number of hexagons when $r = 1, 2, 3$. Identify the domain and range of this function.
    d. Use the equation to find the total number of hexagons in a honeycomb with 12 rings.





Mixed Review

57. **Seismology** Two tracking stations have detected an earthquake. The first station determined that the epicenter was 25 miles away. The second station determined that the epicenter was 42 miles away. If the first station is located at the origin and the second station is 50 miles due east of the first station, where could the epicenter have been? (Lesson 7–7)

58. Write the standard form of the equation $x^2 + 4y^2 = 4$. Graph the equation and state whether the graph is a parabola, a circle, an ellipse, or a hyperbola. (Lesson 7–6)

59. Find two numbers whose difference is −40 and whose product is a minimum. (Lesson 6–4)

60. Solve $\sqrt{n + 12} - \sqrt{n} = 2$. (Lesson 5–8)

61. **ACT Practice** $\sqrt[4]{a^3} \cdot \sqrt[8]{a^2} =$
    A $a^{\frac{3}{16}}$ B $a^{\frac{1}{2}}$ C $a^{\frac{3}{4}}$ D a E $a^{\frac{5}{4}}$

62. Determine the dimensions of the product $A_{4\times2} \cdot B_{2\times3}$. (Lesson 4–3)

63. Find $\begin{bmatrix} -9 & 6 \\ 5 & 19 \end{bmatrix} + \begin{bmatrix} -3 & 18 \\ -4 & 12 \end{bmatrix}$. (Lesson 4–2)

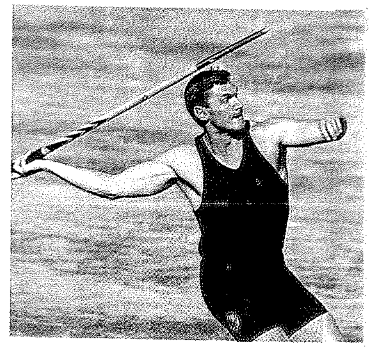
64. In which octant does the point at (7, −2, 9) lie? (Lesson 3–7)

65. Solve the system of equations by using either the substitution or elimination method. (Lesson 3–2)
    $2x - y = 36$
    $3x - \frac{1}{2}y = 26$

66. Evaluate $2|-3x| - 9$ if $x = 5$. (Lesson 1–5)

For Extra Practice, see page 894.

8-2 The Remainder and Factor Theorems



What YOU'LL LEARN

- To find factors of polynomials by using the factor theorem and synthetic division.

Why IT'S IMPORTANT

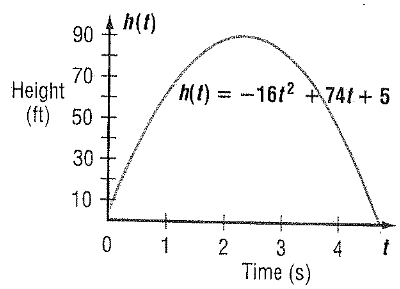
You can use the remainder and factor theorems to find factors of polynomials that model situations in engineering and architecture.

Real World APPLICATION

Sports

A javelin is usually thrown from about shoulder height, around 5 feet off the ground. It is not unusual for the javelin to start off with an upward velocity of about 74 feet per second. Based on this information, one can determine that the height of a javelin t seconds after it is thrown can be described by the function $h(t) = -16t^2 + 74t + 5$, if the effect of air resistance is ignored. The 16 in the function is associated with the strength of the Earth's gravity; if the javelin were thrown on another planet, there would be a different number in the equation.

The graph of $h(t)$ is shown at the right. Notice that when $t = 0$, the value of $h(t)$ is 5. Suppose we find the height of the javelin after 3 seconds.



$h(t) = -16t^2 + 74t + 5$

$h(3) = -16(3)^2 + 74(3) + 5$ *Replace t with 3.*

$= -144 + 222 + 5$

$= 83$

After 3 seconds, the height of the javelin is 83 feet.

Divide the polynomial in the function by $t - 3$, and compare the remainder to $h(3)$.

Method 1: Long Division

$$\begin{array}{r} -16t + 26 \\ t - 3\overline{)-16t^2 + 74t + 5} \\ \underline{-16t^2 + 48t} \\ 26t + 5 \\ \underline{26t - 78} \\ 83 \end{array}$$

Method 2: Synthetic Division

$$\begin{array}{r|rrr} 3 & -16 & 74 & 5 \\ & & -48 & 78 \\ \hline & -16 & 26 & 83 \end{array}$$

LOOK BACK

Refer to Lesson 5-3 for a review of synthetic division.

Notice that the value of $h(3)$ is the same as the remainder when the polynomial is divided by $t - 3$. This illustrates the **remainder theorem.**

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - a$, the remainder is the constant $f(a)$, and

$$\textit{dividend} = \textit{quotient} \cdot \textit{divisor} + \textit{remainder}$$
$$f(x) = q(x) \cdot (x - a) + f(a),$$

where $q(x)$ is a polynomial with degree one less than the degree of $f(x)$.

Example 1 **Let $f(x) = 3x^4 - x^3 + 2x - 6$. Show that $f(2)$ is the remainder when $f(x)$ is divided by $x - 2$.**

Use synthetic division to divide by $x - 2$.

$$\begin{array}{r|rrrr|r} 2 & 3 & -1 & 0 & 2 & -6 \\ & & 6 & 10 & 20 & 44 \\ \hline & 3 & 5 & 10 & 22 & 38 \end{array}$$

Long division could also be used.

The quotient is $3x^3 + 5x^2 + 10x + 22$ with a remainder of 38.

Now find $f(2)$.

$$\begin{aligned} f(2) &= 3(2)^4 - (2)^3 + 2(2) - 6 \\ &= 48 - 8 + 4 - 6 \text{ or } 38 \end{aligned}$$

Thus, $f(2) = 38$, the same number as the remainder after division by $x - 2$.

As illustrated in Example 1, synthetic division can be used to find the value of a function. When synthetic division is used to find the value of a function, it is called **synthetic substitution.** This is a convenient way of finding the value of a function, especially when the degree of the polynomial is greater than 2.

Example 2 **If $f(x) = x^4 + 2x^3 - 10x^2 + 5x - 7$, find $f(8)$.**

Method 1: Synthetic Substitution

When $f(x)$ is divided by $x - 8$, the remainder is $f(8)$.

$$\begin{array}{r|rrrr|r} 8 & 1 & 2 & -10 & 5 & -7 \\ & & 8 & 80 & 560 & 4520 \\ \hline & 1 & 10 & 70 & 565 & 4513 \end{array}$$

The remainder is 4513. Thus, by synthetic substitution, $f(8) = 4513$.

Method 2: Direct Substitution

$$\begin{aligned} f(8) &= (8)^4 + 2(8)^3 - 10(8)^2 + 5(8) - 7 \\ &= 4096 + 1024 - 640 + 40 - 7 \text{ or } 4513 \end{aligned}$$

By substitution, $f(8) = 4513$, the same result as by synthetic substitution.

Consider $f(x) = x^4 + x^3 - 13x^2 - 25x - 12$. If $f(x)$ is divided by $x - 4$, then the remainder is 0. Therefore, 4 is a zero of $f(x)$.

$$\begin{array}{r|rrrr|r} 4 & 1 & 1 & -13 & -25 & -12 \\ & & 4 & 20 & 28 & 12 \\ \hline & 1 & 5 & 7 & 3 & 0 \end{array}$$

The quotient of $f(x)$ and $x - 4$ is $x^3 + 5x^2 + 7x + 3$.

Check:

$$\begin{aligned} f(x) &= x^4 + x^3 - 13x^2 - 25x - 12 \\ f(4) &\stackrel{?}{=} (4)^4 + (4)^3 - 13(4)^2 - 25(4) - 12 \\ 0 &\stackrel{?}{=} 256 + 64 - 208 - 100 - 12 \\ 0 &= 0 \quad \checkmark \end{aligned}$$

From the results of the division and by using the remainder theorem, we can make the following statement.

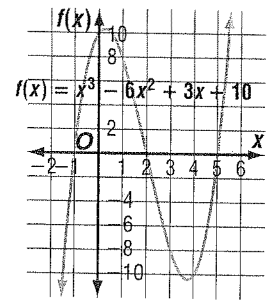
dividend	=	*quotient*	·	*divisor*	+	*remainder*
$x^4 + x^3 - 13x^2 - 25x - 12$	=	$(x^3 + 5x^2 + 7x + 3)$	·	$(x - 4)$	+	0

Since the remainder is 0, $x - 4$ is a factor of $x^4 + x^3 - 13x^2 - 25x - 12$. This illustrates the **factor theorem,** which is a special case of the remainder theorem.

The Factor Theorem

The binomial $x - a$ is a factor of the polynomial $f(x)$ if and only if $f(a) = 0$.

Suppose you wanted to find the zeros of $f(x) = x^3 - 6x^2 + 3x + 10$. From the graph at the right, you can see that the graph crosses the x-axis at -1, 2, and 5. These are the zeros of the function. Using these zeros and the zero product property, we can express the polynomial in factored form.



$$f(x) = (x + 1)(x - 2)(x - 5)$$

Many polynomial functions are not easily graphed and once graphed, the exact zeros are often difficult to determine. The factor theorem can help you find all the factors of a polynomial. Suppose we wanted to determine if $x - 3$ is a factor of $x^3 + 5x^2 - 12x - 36$ and, if it is, what the other factors are.

Let $f(x) = x^3 + 5x^2 - 12x - 36$. The binomial $x - 3$ is a factor of the polynomial if 3 is a zero. Use the factor theorem.

3	1	5	−12	−36
		3	24	36
	1	8	12	0

Since the remainder is 0, $x - 3$ is a factor of the polynomial. Further, since $x - 3$ is a factor of the polynomial, it follows that the remainder is 0.

When you divide a polynomial by one of its binomial factors, the quotient is called a **depressed polynomial.** The polynomial $x^3 + 5x^2 - 12x - 36$ can be factored as $(x - 3)(x^2 + 8x + 12)$. The polynomial $x^2 + 8x + 12$ is the depressed polynomial, which also may be factorable.

$x^2 + 8x + 12 = (x + 2)(x + 6)$

So, $x^3 + 5x^2 - 12x - 36 = (x - 3)(x + 2)(x + 6)$.

Example 3

Real World APPLICATION

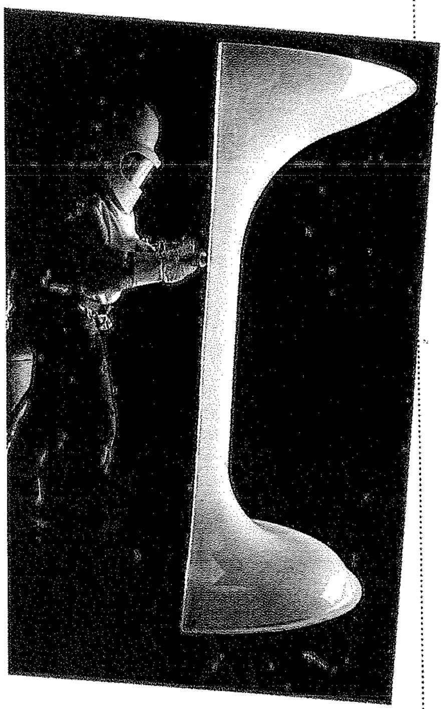
Engineering

When a certain type of plastic is cut into sections, the length of each section determines the strength of the plastic. The function $f(x) = x^4 - 14x^3 + 69x^2 - 140x + 100$ can describe the relative strength of a section of length x feet. Sections of plastic x feet long, where $f(x) = 0$, are extremely weak. After testing the plastic, engineers discovered that sections 2 feet long and 5 feet long were extremely weak.

a. Show that $x - 2$ and $x - 5$ are factors of the polynomial function.

b. Find other lengths of plastic that are extremely weak, if they exist.

(continued on the next page)



a.

2	1	−14	69	−140	100
		2	−24	90	−100
	1	−12	45	−50	0

The remainder is 0, so $x - 2$ is a factor of $x^4 - 14x^3 + 69x^2 - 140x + 100$.

So, $x^4 - 14x^3 + 69x^2 - 140x + 100 = (x^3 - 12x^2 + 45x - 50)(x - 2)$.

5	1	−12	45	−50
		5	−35	50
	1	−7	10	0

The remainder is 0, so $(x - 5)$ is a factor of $x^4 - 14x^3 + 69x^2 - 140x + 100$.

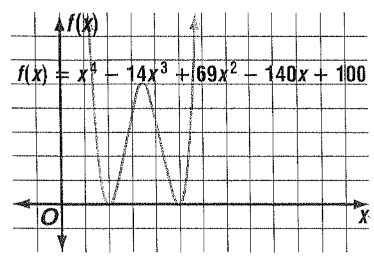
So, $x^4 - 14x^3 + 69x^2 - 140x + 100 = (x - 2)(x - 5)(x^2 - 7x + 10)$.

b. Are there other values of x at which $f(x) = 0$? If so, sections of plastic with these lengths will be extremely weak.

Factor the depressed polynomial, if possible.

$x^2 - 7x + 10 = (x - 2)(x - 5)$

So, $x^4 - 14x^3 + 69x^2 - 140x + 100 = (x - 2)(x - 5)(x - 2)(x - 5)$ or $(x - 2)^2 (x - 5)^2$.



The graph of the polynomial function touches the x-axis at 2 and 5. Thus, the only lengths of plastic that are extremely weak are 2 feet and 5 feet long.

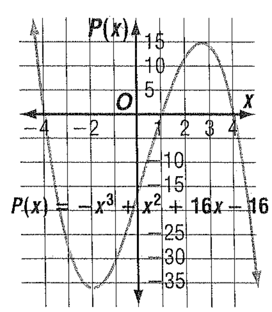
CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. If the divisor is a factor of a polynomial, then what is the remainder after division?
2. **You Decide** Jack tells Mayuko that if $x - 6$ is a factor of a polynomial $f(x)$, then $f(6) = 0$. Mayuko argues that if $x - 6$ is a factor of $f(x)$, then $f(-6) = 0$. Who is correct? Explain.
3. **a. State** the zeros of the polynomial $P(x)$ whose graph is shown at the right.

   b. Describe what $P(x) + 100$ would look like. How many real zeros would it have?



4. **State** the degree of each polynomial. Then state the degree of the depressed polynomial that would result from dividing the polynomial by one of its binomial factors.

   a. $4x^4 + 3x^3 - 5x^2 + 8$

   b. $5x^2 + 7x^5 - 8x - 2$

Math Journal

5. **Assess Yourself** A depressed polynomial of degree 2 is the result of synthetic substitution. What options do you have to find the remaining factors? Which would you choose to use first? Explain.

Guided Practice

Divide using synthetic division and write your answer in the form *dividend = quotient · divisor + remainder.* Is the binomial a factor of the polynomial?

6. $(x^3 - 4x^2 + 2x - 6) \div (x - 4)$

7. $(x^4 - 16) \div (x - 2)$

Use synthetic substitution to find $g(2)$ and $g(-1)$ for each function.

8. $g(x) = x^3 - 5x + 2$

9. $g(x) = x^4 - 6x - 8$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

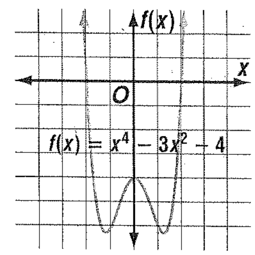
10. $x^3 + 2x^2 - x - 2; x - 1$

11. $x^3 - 6x^2 + 11x - 6; x - 2$

12. $2x^3 + 7x^2 - 53x - 28; (x + 7)$

13. $x^4 + 2x^3 + 2x^2 - 2x - 3; x + 1$

14. Use the graph of the polynomial function at the right to determine at least one binomial factor of the polynomial. Then find all factors of the polynomial.



15. Use synthetic substitution to show that $x - 8$ is a factor of $x^3 - 4x^2 - 29x - 24$. Then find any remaining factors.

EXERCISES

Practice

Divide using synthetic division and write your answer in the form *dividend = quotient · divisor + remainder.* Is the binomial a factor of the polynomial?

16. $(x^3 - 6x^2 + 2x - 4) \div (x - 2)$

17. $(x^3 - 2x^2 - 5x + 6) \div (x - 3)$

18. $(2x^3 + 8x^2 - 3x - 1) \div (x - 2)$

19. $(x^3 + 27) \div (x + 3)$

20. $(2x^3 + x^2 - 8x + 16) \div (x + 4)$

21. $(6x^3 + 9x^2 - 6x + 2) \div (x + 2)$

22. $(x^3 - 64) \div (x - 4)$

23. $(4x^4 - 2x^2 + x + 1) \div (x - 1)$

Use synthetic substitution to find $f(2)$ and $f(-1)$ for each function.

24. $f(x) = x^3 - 2x^2 - x + 1$

25. $f(x) = 2x^2 - 8x + 6$

26. $f(x) = x^3 + 2x^2 - 3x + 1$

27. $f(x) = x^3 - 8x^2 - 2x + 5$

28. $f(x) = 5x^4 - 6x^2 + 2$

29. $f(x) = 3x^4 + x^3 - 2x^2 + x + 12$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

30. $x^3 - x^2 - 5x - 3; x + 1$

31. $x^3 - 3x + 2; x - 1$

32. $x^3 + x^2 - 16x - 16; x - 4$

33. $6x^3 - 25x^2 + 2x + 8; 3x - 2$

34. $2x^3 + 17x^2 + 23x - 42; 2x + 7$

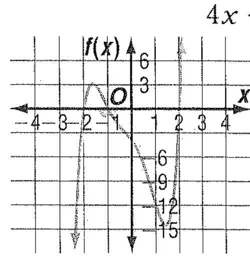
35. $x^4 + 2x^3 - 8x - 16; x + 2$

36. $8x^4 + 32x^3 + x + 4; 2x + 1$

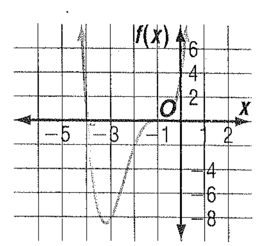
37. $16x^5 - 32x^4 - 81x + 162; x - 2$

Use the graph of each polynomial function to determine at least one binomial factor of the polynomial. Then find all of the factors.

38. $f(x) = x^5 + x^4 - 3x^3 - 3x^2 - 4x - 4$



39. $f(x) = x^4 + 7x^3 + 15x^2 + 13x + 4$



Find values for k so that each remainder is 3.

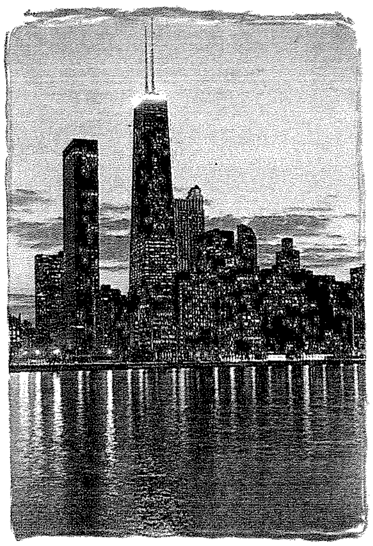
40. $(x^2 - x + k) \div (x - 1)$
41. $(x^2 + kx - 17) \div (x - 2)$
42. $(x^3 + 4x^2 + x + k) \div (x + 2)$
43. $(x^2 + 5x + 7) \div (x + k)$

Critical Thinking

44. Consider the function $f(x) = x^3 + 2x^2 - 5x - 6$.
   a. Use synthetic substitution to find $f(-4)$, $f(-2)$, $f(0)$, $f(2)$, and $f(4)$.
   b. On a coordinate plane, graph the ordered pairs of the form $(x, f(x))$ you found and connect them to make a smooth curve.
   c. How many times does the graph cross the x-axis? Does this agree with what you learned in Lesson 8-1 about graphs of polynomial functions?

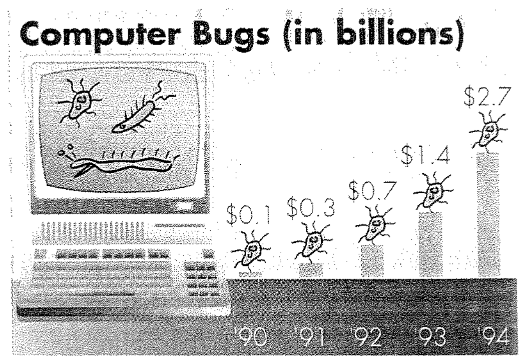
Real World

Applications and Problem Solving



45. **Architecture** According to the *Guinness Book of Records,* the 1454-foot Sears Tower in Chicago, Illinois, is the tallest office building in the world. It has 110 floors and 18 elevators. The elevators traveling from one floor to the next do not travel at a constant speed. Suppose the speed of an elevator in feet per second is given by the function $F(t) = -0.5t^4 + 4t^3 - 12t^2 + 16t$, where t is the time in seconds.
   a. Find the speed of the elevator at 1, 2, and 3 seconds.
   b. It takes 4 seconds for the elevator to go from one floor to the next. Use synthetic substitution to find $f(4)$. Explain what this means.

46. **Technology** The graph shows how the cost of computer viruses in the U.S. has drastically increased since 1990. The function $C(x) = 0.03x^3 - 0.02x^2 + 0.2x + 0.1$ estimates this cost over the interval shown, with $x = 0$ representing the year 1990. Estimate the cost of computer viruses in the year 2005.



Source: National Computer Security Association

Mixed Review

47. **Art** Joyce Jackson purchases works of art for an art gallery. Two years ago, she bought a painting for $20,000, and last year, she bought one for $35,000. If these paintings appreciate at 14% per year, how much are the two paintings worth now? (Lesson 8–1)

48. Find the coordinates of the vertices and foci and the slopes of the asymptotes for the hyperbola given by the equation $\frac{y^2}{81} - \frac{x^2}{25} = 1$. (Lesson 7–5)

49. Is $(4, -4)$ a solution to the quadratic inequality $-y \leq -x^2 + 5x$? (Lesson 6–7)

50. Solve $2x^2 - 5x + 4 = 0$ by using the quadratic formula. (Lesson 6–4)

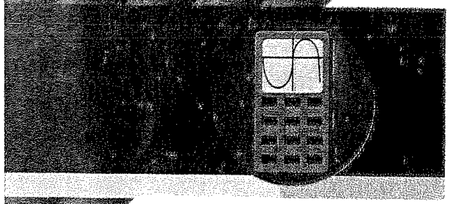
51. Find $(6x^3 - 5x^2 - 12x - 4) \div (3x + 2)$. (Lesson 5–3)

52. **SAT Practice** **Grid-in** If $\frac{2}{p} - \frac{4}{p^2} = -\frac{2}{p^3}$, then what is the value of p?

53. Solve $\begin{bmatrix} 2x \\ y + 1 \end{bmatrix} = \begin{bmatrix} y \\ 3 \end{bmatrix}$ for x and y. (Lesson 4–1)

54. If $a = -1$, $b = 7$, $c = 4$, and $-a^3b^2 + 2ab + 3d \geq \frac{1}{2}c^3$, solve for d. (Lesson 1–6)

For Extra Practice, see page 894.



8–3A Graphing Technology Polynomial Functions

A Preview of Lesson 8–3

You can use a graphing calculator to graph polynomial functions and approximate the real zeros of the function. When using a calculator to approximate zeros, it is important to view a complete graph of the function before zooming in on a certain point. Otherwise, zeros may be overlooked because they were not in the viewing window. Remember that a complete graph of a function shows all the characteristics of the graph such as all x- and y-intercepts, relative maximum and minimum points, and the end behavior of the graph.

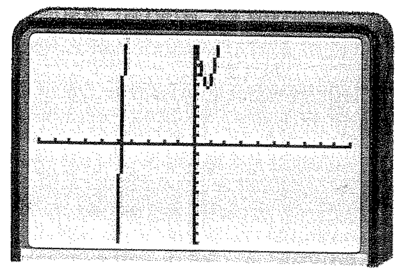
Example 1 **Use a graphing calculator to obtain a complete graph of $f(x) = 2x^3 + 6x^2 - 14x + 12$. Then approximate each real zero to the nearest hundredth.**

Let's try graphing in the standard viewing window.

Enter: [Y=] 2 [X,T,θ,n] [^] 3 [+]

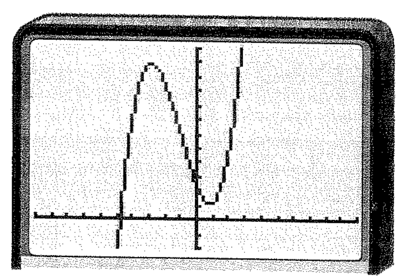
6 [X,T,θ,n] [x^2] [−] 14

[X,T,θ,n] [+] 12 [ZOOM] 6



We see that this viewing window does not contain a complete graph. Change the viewing window to $[-10, 10]$ by $[-10, 60]$ with a scale factor of 1 for the x-axis and 5 for the y-axis.

This window can accommodate the complete graph.



According to the graph, there is one x-intercept (real zero) for this function. There are three zeros for any third-degree polynomial, so two of the zeros for this function must be imaginary. Use ZOOM, TRACE, or ROOT to approximate the real zero.

LOOK BACK

Refer to Lesson 6-1A for information on using the automatic ROOT feature.

The only real zero is approximately -4.74.

When using a graphing calculator to approximate real zeros, it is helpful to know that a function with degree n has at most n real zeros. Thus, a function with degree 5 has at most five real zeros. If you can see five x-intercepts in the viewing window, you know you have found all of the zeros and that they are all real. However, if there are fewer than five x-intercepts, there are duplicate real zeros or the zeros are not all real. Complex or imaginary zeros occur in conjugate pairs, so a fifth-degree function may have one, three, or five real zeros.

Example 2 **Use a graphing calculator to obtain a complete graph of $f(x) = 3x^5 - 5x^4 - 2x^3 + x^2 - 6x + 8$. Then approximate each real zero to the nearest hundredth.**

First, try graphing in the standard viewing window.

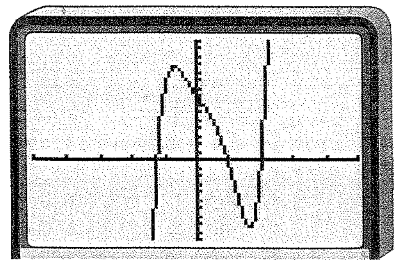
Enter: 3 [X,T,θ,n] [^] 5 [–] 5 [X,T,θ,n]

[^] 4 [–] 2 [X,T,θ,n] [^] 3 [+]

[X,T,θ,n] [x^2] [–] 6 [X,T,θ,n] [+]

8 [ZOOM] 6

The standard viewing window does not accommodate the complete graph. The view shown at the right uses the window [−5, 5] by [−10, 15].



According to the graph, there are three real zeros for this function. Use ZOOM, TRACE, or ROOT to approximate the real zeros. They are approximately −1.24, 0.93, and 2.

EXERCISES

Use a graphing calculator to obtain a complete graph of each polynomial function. Describe your viewing window and state the number of real zeros.

1. $f(x) = 2x^3 - 3x^2 - 12x + 17$
2. $f(x) = 3x^4 - 8x^3 - 35x^2 + 72x + 47$
3. $j(x) = 0.1x^4 + x^3 - x^2 + 3x + 18$
4. $g(x) = x^5 - 4x^4 + 2x^3 - 7x + 15$

Graph each function so that a complete graph is shown. Then approximate each of the real zeros to the nearest hundredth.

5. $f(x) = x^4 - 3x^2 - 6x - 2$
6. $h(x) = 2x^5 + 3x - 2$
7. $c(x) = 3x^{13} + 4x^3 + 2$
8. $m(x) = 2x^8 + 4x^2 + 1$
9. $p(x) = 8x^5 - 20x^3 + 73x^2 + 28x - 4$
10. $f(x) = x^5 + x^4 - 8x^3 - 10x^2 + 7x - 4$

8-3 Graphing Polynomial Functions and Approximating Zeros

What YOU'LL LEARN

- To approximate the real zeros of polynomial functions,
- to find maxima and minima of polynomial functions, and
- to graph polynomial functions.

Why IT'S IMPORTANT

You can graph polynomial functions to solve problems involving geometry and physical fitness.

CAREER CHOICES

Industrial designers develop and design manufactured products and their packaging. They combine artistic talent with research on product use to create the most appealing and functional design.

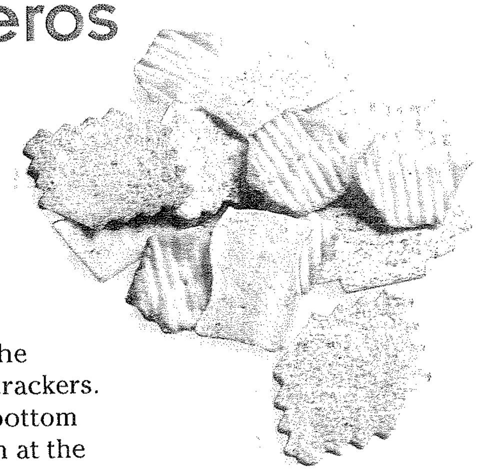
A bachelor's degree is required and training in computer-aided design (CAD) is very helpful.

For more information, contact:

Industrial Designers Society of America
1142 E Walker Rd.
Great Falls, VA 22066

INTEGRATION

Geometry



In her geometry class, Hillary has a project in which she must create cracker shapes that form a tessellation in the same manner as a certain brand of crackers. For her art class, she must design a special packaging box for the crackers. In order to best display the tessellation, the bottom of the box will be a square, and it will be open at the top so that the crackers can be seen through cellophane.

Hillary is to use a 108 square-inch sheet of a special metallic paper to cover the sides and bottom of the box. What will be the dimensions of the box if it is to hold a maximum volume of crackers? You can use a graph to solve this problem.

First, write polynomial equations to describe the surface area and volume of the box. Let x represent the length of the side of the square on the bottom, and let h represent the height of the box.

$$\underbrace{\text{Surface Area}} = \underbrace{\text{area of the base}} + \underbrace{\text{area of the four sides}}$$

$$SA = x^2 + 4xh$$

$$108 = x^2 + 4xh \quad \text{Replace } SA \text{ with } 108.$$

$$\underbrace{\text{Volume}} = \underbrace{\text{area of the base}} \cdot \underbrace{\text{height}}$$

$$V = x^2 \cdot h$$

In order to find the volume by graphing, we need to express the volume in terms of one variable. First, solve the surface area formula for h in terms of x.

$$x^2 + 4xh = 108$$

$$4xh = 108 - x^2$$

$$h = \frac{108 - x^2}{4x}$$

Then, substitute the value of h in the formula for volume.

$$V = x^2 h$$

$$= x^2\left(\frac{108 - x^2}{4x}\right) \quad \text{Replace } h \text{ with } \frac{108 - x^2}{4x}.$$

$$= \frac{108x^2 - x^4}{4x} \quad \text{Simplify.}$$

$$= 27x - \frac{x^3}{4}$$

Let $V(x) = 27x - \frac{x^3}{4}$.

Make a table of values.

x	$V(x)$
−11	**35.75**
−10	**−20**
−9	−60.75
−8	−88
−7	−103.25
−6	−108
−5	−103.75
−4	−92

A zero is between $x = -11$ and $x = -10$.

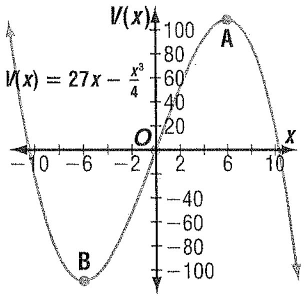
x	$V(x)$
−3	−74.25
−2	−52
−1	−26.75
0	**0**
1	26.75
2	52
3	74.25
4	92

A zero is at $x = 0$.

x	$V(x)$
5	103.75
6	108
7	103.25
8	88
9	60.75
10	**20**
11	**−35.75**
12	−108

Greatest value for $V(x)$ (at $x = 6$)

A zero is between $x = 10$ and $x = 11$.



Sketch the graph of the function $V(x)$ by connecting those points with a smooth curve. The graph will cross the x-axis somewhere between the pairs of x values where the corresponding $V(x)$ values change sign. Since the x-intercepts are zeros of the function, there is a zero between each pair of these x values. This strategy is called the **location principle.**

The Location Principle	**Suppose $y = f(x)$ represents a polynomial function and a and b are two numbers such that $f(a) < 0$ and $f(b) > 0$. Then the function has at least one real zero between a and b.**

The plurals of maximum and minimum are maxima and minima.

The graph above shows the shape of the graph of a general third-degree polynomial function. Point A on the graph is a **relative maximum** of the cubic function, since no other nearby points have a greater y-coordinate. Likewise, point B is a **relative minimum,** since no other nearby points have a lesser y-coordinate. You can also see from the tables of values where there is a relative maximum and a relative minimum. You can use this information to help graph functions that have imaginary zeros.

You can use the coordinates of the relative maximum to determine the point at which the box has the maximum volume. In the tables of values, the point at (6, 108) appears to have the greatest y-coordinate. To check whether it is truly the relative maximum, compute the y values for an x value on either side of this point.

Find $V(5.9)$ and $V(6.1)$.

$$V(x) = 27x - \frac{x^3}{4}$$

$$V(5.9) = 27(5.9) - \frac{5.9^3}{4}$$
$$\approx 107.96$$

$$V(x) = 27x - \frac{x^3}{4}$$

$$V(6.1) = 27(6.1) - \frac{6.1^3}{4}$$
$$\approx 107.95$$

Since both y values are less than the y value for the maximum, the point at (6, 108) is a relative maximum. So, in order for the box to have a maximum volume, the side of the box has to be 6 inches long. The box would have a maximum volume of 108 cubic inches.

To determine the height of the box, substitute 6 for x in the surface area equation.

$$h = \frac{108 - x^2}{4x}$$

$$= \frac{108 - 6^2}{4(6)} \quad \textit{Replace x with 6.}$$

$$= 3$$

The box must be 3 inches tall to have maximum volume. Thus, the dimensions of Hillary's cracker box are 6 inches by 6 inches by 3 inches.

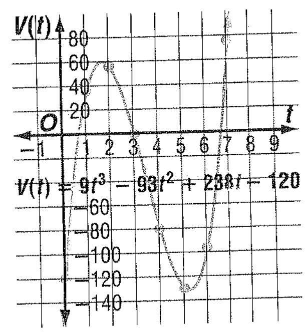
Example 1

CONNECTION
Physics

Under certain conditions, the velocity of an object as a function of time is described by the function $V(t) = 9t^3 - 93t^2 + 238t - 120$. Approximate the zeros of $V(t)$ to the nearest tenth and draw the graph.

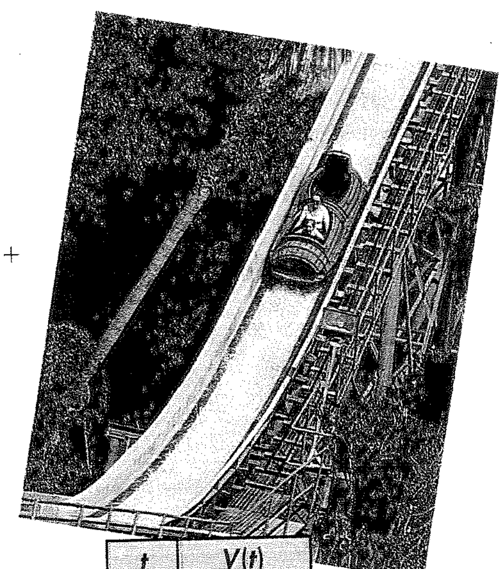
Evaluate the function for several successive values of t to locate the zeros. Then plot the points and connect them to form a smooth graph.

t	$V(t)$	
0	**−120**	*zero between t = 0 and t = 1*
1	**34**	
2	56	
3	**0**	*zero at t = 3*
4	−80	
5	−130	
6	**−96**	*zero between t = 6 and t = 7*
7	**76**	



One zero lies between 0 and 1. Another zero is 3. A third zero lies between 6 and 7.

To approximate the zeros to the nearest tenth, you have to repeat the process of evaluating $V(t) = 9t^3 - 93t^2 + 238t - 120$ for successive values of t expressed in tenths, as we did in the application at the beginning of the lesson. Using a scientific calculator will help find these values more easily.



To evaluate $V(0.5)$, do the following.

Enter: 9 [×] .5 [y^x] 3 [−] 93 [×] .5 [x^2] [+] 238 [×] .5 [−] 120 [=] −23.125

t	$V(t)$	
0.5	−23.125	
0.6	−8.736	*zero*
0.7	4.117	
6.5	−30.625	
6.6	−12.816	*zero*
6.7	6.697	

Following this procedure for the rest of the values in the chart, you will find that the zeros approximated to the nearest tenth are 0.7 and 6.7.

Example 2 **Graph $f(x) = x^3 - 5x^2 + 3x + 12$.**

In order to graph the function, you need to find several points and then connect them to make a smooth curve. Since $f(x)$ is a third-degree polynomial function, it will have 3 or 1 real zeros. Also, its left-most points will have negative values for y, and its right-most points will have positive values for y.

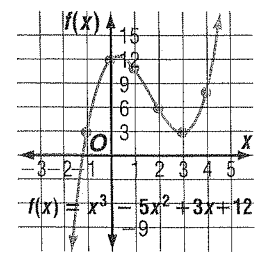
Make a table and evaluate several successive values of x to locate the zeros and to find the relative maximum and relative minimum.

x	f(x)
−2	−22
−1	3
0	**12**
0.5	**12.375**
1	**11**
2	6
2.9	**3.039**
3	**3**
4	**8**

zero between $x = -2$ and $x = -1$

indicates a relative maximum

indicates a relative minimum



The function has one relative maximum and one relative minimum. The values of $f(0.5)$ and $f(2.9)$ were calculated to approximate the maximum and minimum more closely. There is a zero between −2 and −1. *Use a graphing calculator to check the graph.*

A graphing calculator can be helpful in finding the relative maximum and relative minimum of a function.

EXPLORATION — GRAPHING CALCULATORS

To find the relative maximum and relative minimum of $f(x) = x^3 - 6x^2 + 6x + 5$, press [Y=] and enter the equation. Then press [ZOOM] 6. The graph appears to have a relative minimum between 3 and 4 and a relative maximum between 0 and 1. To find the actual relative minimum, follow these steps.

Enter: [MATH] 6 [2nd] [Y-VARS] 1 [ENTER] [,] [X,T,θ,n] [,] 3 [,] 4 [)] [ENTER] 3.414213414

Thus, there is a relative minimum at $x \approx 3.41$.

Your Turn

a. Find the y-coordinate of the relative minimum to the nearest hundredth.

b. Find the coordinates of the relative maximum of the function to the nearest hundredth. (*Hint*: Use the fMax feature by pressing [MATH] 7.)

c. Graph the function $f(x) = x^3 + x^2 - 7x - 3$, and find the relative maximum and relative minimum to the nearest hundredth.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **State** the greatest number of relative minima that are possible for each condition.
   a. a third-degree polynomial with a positive leading coefficient
   b. a third-degree polynomial with a negative leading coefficient
   c. a fourth-degree polynomial with a positive leading coefficient
   d. a fourth-degree polynomial with a negative leading coefficient
2. Refer to the application at the beginning of the lesson. Why did we not choose one of the negative zeros for the volume of the box?
3. **Sketch** a graph of each polynomial.
   a. even-degree polynomial function with one relative maximum and two relative minima
   b. odd-degree polynomial function with one relative maximum and one relative minimum; the leading coefficient is negative
   c. even-degree polynomial function with four relative maxima and three relative minima
   d. odd-degree polynomial function with three relative maxima and three relative minima; the left-most points are negative
4. Consider the function $f(x) = x^4 - 8x^2 + 10$.
   a. Evaluate $f(x)$ for successive integers between -4 and 4 inclusive.
   b. Between what successive integers do the zeros appear? Approximate those zeros to the nearest tenth.
   c. State the ranges of x values where the values of $f(x)$ are negative and ranges where the values of $f(x)$ are positive.
   d. State the relative maximum(s) and relative minimum(s).
   e. Graph the function.

Guided Practice

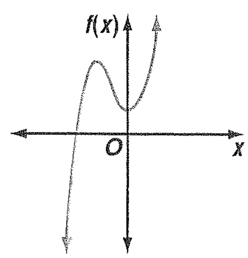
Approximate the real zeros of each function to the nearest tenth.

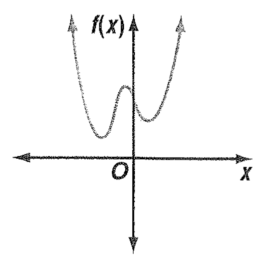
5. $f(x) = x^3 - x^2 + 1$
6. $g(x) = x^4 + 3x^3 - 5$

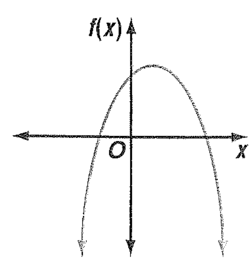
Graph each function.

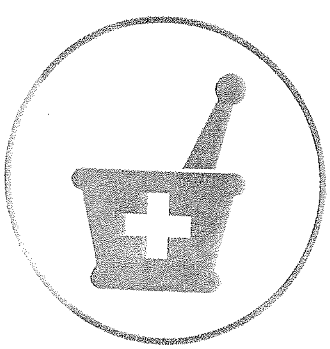
7. $f(x) = x^3$
8. $f(x) = x^3 - x^2 - 4x + 4$
9. $f(x) = -3x^3 + 20x^2 - 36x + 16$
10. $f(x) = x^4 - 7x^2 + x + 5$

State whether each graph is of odd degree or even degree. State the number of relative minima and relative maxima.

11.


12.


13.




14. **Pharmacy** A syringe is to deliver an injection of 2 cubic centimeters of medication. If the plunger is pulled out two centimeters to have the proper dosage, approximate the radius of the inside of the syringe to the nearest hundredth of a centimeter. Use the formula for the volume of a cylinder, $V = \pi r^2 h$.

EXERCISES

Practice

Approximate the real zeros of each function to the nearest tenth.

15. $f(x) = x^3 - 2x^2 + 6$
16. $h(x) = 2x^5 + 3x - 2$
17. $r(x) = x^5 - 6$
18. $g(x) = x^3 + 1$
19. $f(x) = x^4 + 2x^3 - x^2 - 3$
20. $p(x) = x^3 + 2x^2 - 3x - 5$
21. $n(x) = 3x^3 - 16x^2 + 12x + 6$
22. $h(x) = x^4 - 4x^2 + 2$

Graph each function.

23. $f(x) = 4x^6$
24. $f(x) = 3x^5$
25. $f(x) = x^3 - x$
26. $f(x) = -x^3 - 4x^2$
27. $f(x) = x^3 + 5$
28. $f(x) = x^4 - 81$
29. $f(x) = 15x^3 - 16x^2 - x + 2$
30. $f(x) = x^4 - 10x^2 + 9$
31. $f(x) = -x^4 + x^3 + 8x^2 - 3$
32. $f(x) = x^3 - x^2 - 8x + 12$

Approximate the real zeros of each function to the nearest tenth. Then use the functional values to graph the function.

33. $r(x) = x^5 + 4x^4 - x^3 - 9x^2 + 3$
34. $g(x) = x^4 - 9x^3 + 25x^2 - 24x + 6$
35. $h(x) = x^3 - 3x^2 + 2$
36. $f(x) = x^3 + 5x^2 - 9$
37. $f(x) = x^4 + 7x + 1$
38. $p(x) = x^5 + x^4 - 2x^3 + 1$

Graphing Calculator

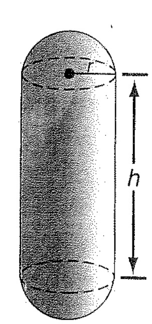
39. a. Graph $y = x^2(x - 2)(x + 3)$ and $y = 4x^2(x - 2)(x + 3)$.
    b. Compare and contrast the graphs.
40. Find the relative maxima and relative minima of each function.
    a. $f(x) = x^3 - 4x^2 + 8$
    b. $f(x) = x^3 + 3x^2 - 12x$

Critical Thinking

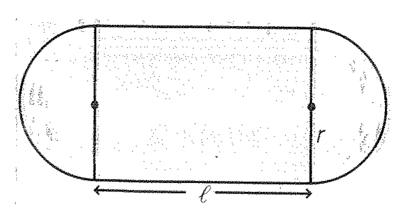
41. Study the graphs for Exercises 23–32. Write a statement comparing the graphs of functions of even degree with those of functions of odd degree.

Applications and Problem Solving

42. **Geometry** A function that represents the volume of a pyramid with a height of the same measure as the side of its square base is $V(s) = \frac{1}{3}s^3$.
    a. Graph the function.
    b. Find the zeros of the function.
    c. Find the maximum and minimum of the function.
    d. Make a conjecture about how all of this data relates.
43. **Aerospace Engineering** The space shuttle has an external tank for the fuel that the main engines need for the launch. This tank is shaped like a capsule, a cylinder with a hemispherical dome at either end. The cylindrical part of the tank has a volume of 1170 cubic meters and a height of 17 meters more than the radius of the tank. What are the dimensions of the tank to the nearest tenth of a meter? (*Hint:* Use the formula for the volume of a cylinder.)



44. **Physical Fitness** An indoor running track is being built at a physical fitness center. It will consist of a rectangular region with a semicircle on each end. If the perimeter of the room is to be a 200-meter running track, find the dimensions that will make the area of the rectangular region as large as possible.



Mixed Review

45. **Business** The Energy Booster Company keeps their stock of Health Aid in a rectangular tank with sides that measure $(x - 1)$ cm, $(x + 3)$ cm, and $(x - 2)$ cm. Suppose they would like to bottle their Health Aid in $x - 3$ containers of the same size. How many cubic centimeters of liquid will remain unbottled? (Lesson 8–2)

46. Find the coordinates of the center and foci, and the lengths of the major and minor axes of the ellipse whose equation is $\frac{x^2}{4} + \frac{y^2}{25} = 1$. (Lesson 7–4)

47. Name the coordinates of the vertex and the equation of the axis of symmetry for the graph of $y = (x - 3)^2 - 11$. (Lesson 6–6)

48. Solve $x^2 - 20x = -75$ by factoring. (Lesson 6–2)

49. Solve $-3y^2 = 18$. (Lesson 5–9)

50. **SAT Practice** There are a gallons of water available to fill a tank. After b gallons have been pumped, then, in terms of a and b, what percent of the water has been pumped?

A $\frac{100b}{a}\%$ B $\frac{a}{100b}\%$ C $\frac{100a}{b}\%$

D $\frac{a}{100(a-b)}\%$ E $\frac{100(a-b)}{a}\%$

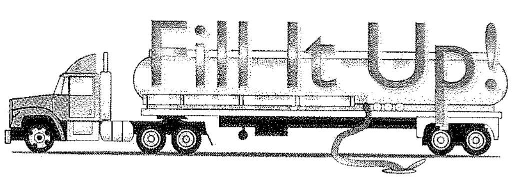
51. Graph the line that passes through $(-2, 1)$ and is perpendicular to $5 + 3x = -2y$. (Lesson 2–3)

52. Simplify $\sqrt{64} \div \sqrt{4}$. (Lesson 1–1)

For Extra Practice, see page 895.

WORKING ON THE In·ves·ti·ga·tion

Refer to the Investigation on pages 474–475.



Select a coffee can that is a different size from the one you have been using. Repeat the experiment with the new coffee can using the same 50-mL increments. Record your measurements and data in a new chart like the one on page 475.

1 What is the capacity of the new coffee can?

2 How much water is in the can when the can is one-half full, one-fourth full, and three-fourths full? Make a note of these numbers.

3 Complete the entire chart. Were your calculations for the capacity of the can when it is one-half full, one-fourth full, and three-fourths full correct?

4 Make a drawing to illustrate the measurement scale of a dipstick for this coffee can. Describe the similarities and differences between the measurements of the two coffee can experiments. How are the scales alike and how are they different?

5 Use the information in your chart to make a table of ordered pairs (x, y), where x represents the amount of water in the coffee can and y represents the percent of a full can. Graph the ordered pairs. Describe the graph.

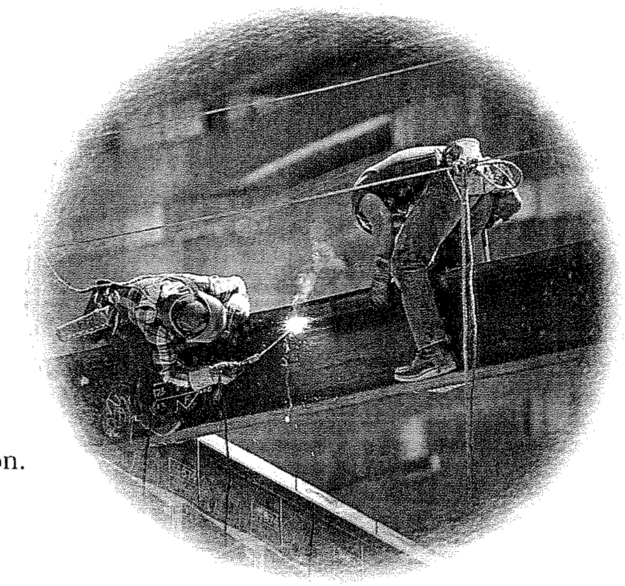
6 Use the data from the first experiment to make another graph like the one described above. Describe the graph. What are the similarities or differences between the two experiments in terms of the graphs you have drawn?

Add the results of your work to your Investigation Folder.

8–3B Graphing Technology
Modeling Real-World Data

An Extension of Lesson 8–3

As we saw in Lesson 2-5B, you can use a graphing calculator to generate a scatter plot of data points and then determine the linear equation for the graph that best fits the plotted points. You can also use a graphing calculator to model data whose curve of best fit is a polynomial function.



Example

The table at the right shows how much time it takes each eight-hour work day to pay one day's worth of taxes. Draw a scatter plot and curve of best fit that shows how the year is related to hours worked.

Year	Hours: Minutes
1930	0:52
1940	1:29
1950	2:02
1960	2:20
1970	2:32
1980	2:40
1990	2:45

First, convert hours in the table to minutes.

Year	Number of Minutes
1930	52
1940	89
1950	122
1960	140
1970	152
1980	160
1990	165

Next, set the window parameters. The values of the data suggest that you should use a viewing window [1920, 2000] by [0, 200] with a scale factor of 5 for the x-axis and 10 for the y-axis.

Then enter the data. Press STAT 1 to display lists for storing data. (If old data has been previously stored, clear the lists.) The years will be entered into list L1.

Enter: 1930 ENTER 1940 ENTER 1950 ENTER . . . 1990 ENTER

Use the ▶ key to move the cursor to column L2 to enter the number of minutes worked to pay taxes.

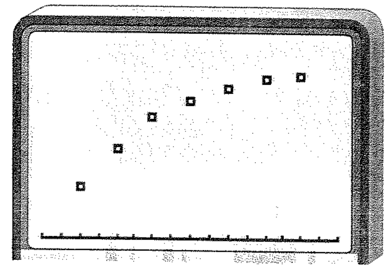
Enter: 52 ENTER 89 ENTER 122 ENTER . . . 165 ENTER

TECHNOLOGY Tip

To clear a list, highlight the list heading and press CLEAR ENTER.

The Plot 1 setting should be a scatter plot using lists L1 and L2.

Now draw the scatter plot.



Enter: 2nd [STAT PLOT] 1 [ENTER] [GRAPH]

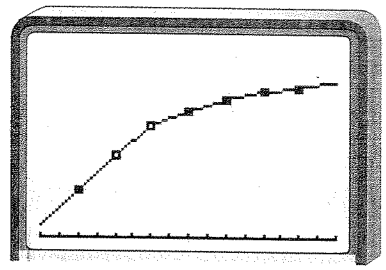
Next, compute and graph the equation of the curve of best fit. Try a cubic curve for this equation.

Enter: [STAT] [▶] 7 [2nd] [L1] [,]

[2nd] [L2] [ENTER] [Y=]

[VARS] 5 [▶] [▶] 7 [GRAPH]

The ZOOM IN feature allows you to move a cursor along the graph or scatter plot and read the coordinates of the points. Press ZOOM 2 and any of the arrow keys to observe what happens.



Approximately how many minutes should you expect to work each day to pay taxes in the year 2000?

In the year 2000, you should expect to work approximately 168.29 minutes each day to pay taxes.

EXERCISES

When an earthquake occurs, seismic waves are detected thousands of kilometers away from the epicenter within a matter of minutes. The table at the right gives the travel time of a primary seismic wave and the corresponding distance from the epicenter for several minutes.

Travel Time (min)	Distance (km)
1	400
2	800
5	2500
7	3900
10	6250
12	8400
13	10,000

1. Use a graphing calculator to draw a scatter plot for the data. State the viewing windows and scale factors that you used.
2. Calculate and graph curves of best fit that show how travel time is related to the distance. Try LinReg, QuadReg, CubicReg, and QuartReg.

3. Write the equation for the curve you think best fits the data. Describe the fit of the graph to the data.
4. Based on the graph of a QuartReg curve, how far away from the epicenter will the wave be felt $8\frac{1}{2}$ minutes after the quake occurs? About how far are you from the epicenter if you feel the wave 15 minutes after the quake?

8-4 Roots and Zeros

What YOU'LL LEARN

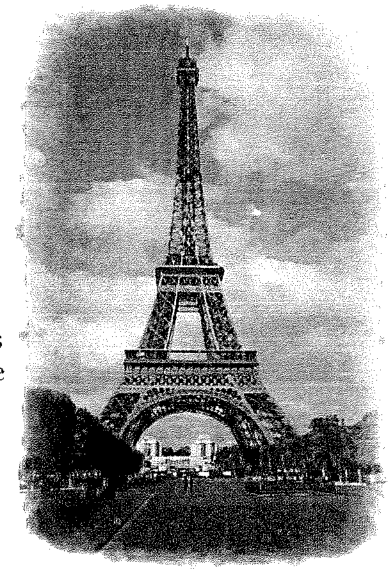
- To find the number and type of zeros of a polynomial function.

Why IT'S IMPORTANT

You can qualify the number of roots for polynomials that model situations in marketing and physiology.

Real World APPLICATION

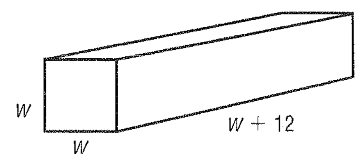
Marketing



Mrs. Botti's French class is selling long-stemmed roses to raise money for their trip to France. The students are making boxes in which they will deliver the flowers. They want the boxes to have square ends (width and height the same), but the length should be 12 inches longer than the width so the very long flowers will fit. They want the volume of each box to be 256 cubic inches, so that each box can hold enough moistened packing material to keep the roses fresh. Find the dimensions for a box that satisfy all these requirements.

We can find the dimensions of the box by writing a polynomial equation. Then we can use the factor theorem and synthetic substitution. First define each dimension of the box in terms of the width w.

w = width = height $\quad$ $w + 12$ = length



Then let $V(w)$ be the function defining the volume.

$V(w) = w \cdot w \cdot (w + 12)$ *volume = ℓwh*

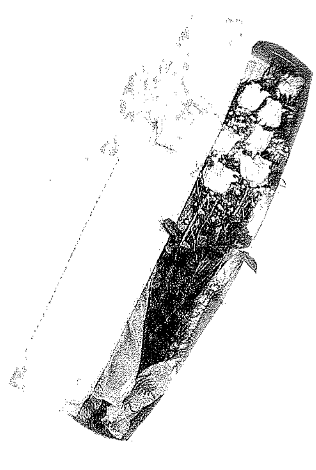
$256 = w^3 + 12w^2$ *Substitute 256 for V(w) and multiply.*

$0 = w^3 + 12w^2 - 256$ *Subtract 256 from each side.*

Study the chart below. We will use a shortened form of synthetic substitution for several values of w to search for the solutions to $0 = w^3 + 12w^2 - 256$. The values for w are in the first column of the chart. Beside each value is the last line of the synthetic substitution. Recall that the first three numbers are the coefficients of the depressed polynomial. The last number in each row is the remainder.

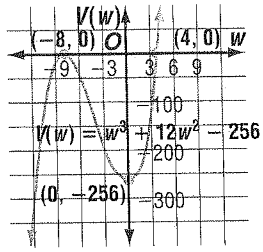
w	1	12	0	−256
1	1	13	13	−243
2	1	14	28	−200
3	1	15	45	−121
4	1	16	64	0
5	1	17	85	169

A remainder of 0 occurs when $w = 4$. This means that $w - 4$ is a factor of the polynomial. The depressed polynomial is $w^2 + 16w + 64$.



The polynomial $w^3 + 12w^2 - 256$ can be factored as $(w - 4)(w^2 + 16w + 64)$. The trinomial $w^2 + 16w + 64$ can be further factored as $(w + 8)(w + 8)$, or $(w + 8)^2$. Thus, the solutions of the equation $0 = (w - 4)(w + 8)^2$ are $w = 4$ and $w = -8$. Since negative widths are not possible when designing a box, the width and height of the box should be 4 inches, and the length should be 4 + 12 or 16 inches. *Do these dimensions produce the correct volume?*

In Chapter 6, you learned that a *zero* of a function $f(x)$ is any value a such that $f(a) = 0$. This zero is also a *root*, or *solution*, of the equation formed when $f(x) = 0$. When the function is graphed, the real zeros of the function will be the x-intercepts of the graph.



In the equation $w^3 + 12w^2 - 256 = 0$, the roots are 4 and -8. The graph of $V(w) = w^3 + 12w^2 - 256$ touches or crosses the horizontal axis at those two points.

When you solve a polynomial equation with degree greater than zero, it may have one or more real roots, or no real roots (the roots are imaginary). Since real numbers and imaginary numbers both belong to the set of complex numbers, all polynomial equations with degree greater than zero have at least one root in the set of complex numbers. This is the **fundamental theorem of algebra.**

Fundamental Theorem of Algebra	**Every polynomial equation with degree greater than zero has at least one root in the set of complex numbers.**

The following corollary of the fundamental theorem of algebra is an even more powerful tool for problem solving.

Corollary	**A polynomial equation of the form $P(x) = 0$ of degree n with complex coefficients has exactly n roots in the set of complex numbers.**

Notice that $w^3 + 12w^2 - 256 = 0$ appears to have only two roots, even though it is a third-degree equation. However, remember that the factored version of the equation was $(w - 4)(w + 8)^2 = 0$. The fact that $w + 8$ appears twice among the factors of $w^3 + 12w^2 - 256$ means -8 is a "double root." Since it is understood that -8 is counted twice among the roots of the equation, we know that $w^3 + 12w^2 - 256 = 0$ really has three roots: -8, -8, and 4. Thus, this third-degree polynomial equation has three roots, which verifies the corollary above. The roots of a polynomial equation may be different real numbers or they may be complex. For example, $x^3 + x = 0$ has three roots: 0, $\boldsymbol{i}$, and $-\boldsymbol{i}$. *You should verify this by factoring.*

Many problems can be solved by using any one of a number of different strategies. Sometimes it takes more than one strategy to solve a problem.

Example 1 **Find all roots of $0 = x^3 + 3x^2 - 10x - 24$.**

We can find the roots of the equation by combining some of the strategies that you have previously learned. Let's *list some possibilities* for roots and then eliminate those that are not roots. Suppose we begin with integral values from -4 to 4 and use the shortened form of synthetic substitution. Because this polynomial function has degree 3, the equation has three roots. However, some of them may be imaginary.

The related function of the equation is $f(x) = x^3 + 3x^2 - 10x - 24$. You can use either synthetic substitution or a scientific calculator to find $f(a)$ quickly. Find $f(-4)$.

Method 1: Synthetic Substitution

-4	1	3	-10	-24
		-4	4	24
	1	-1	-6	0

(continued on the next page)

Method 2: Scientific Calculator

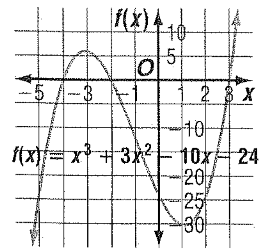
Enter: 4 [+/−] [STO▸] [X] 1 [+] 3 [=] −1

[X] [RCL] [+] 10 [+/−] [=] −6

[X] [RCL] [+] 24 [+/−] [=] 0

The display shown after each [=] gives the second, third, and fourth coefficients of the depressed polynomial. To evaluate $f(x)$ for other values for x, simply change the first number entered in the series of keystrokes shown above.

x	1	3	−10	−24
−4	1	−1	−6	**0**
−3	1	0	−10	6
−2	1	1	−12	**0**
−1	1	2	−12	−12
0	1	3	−10	−24
1	1	4	−6	−30
2	1	5	0	−24
3	1	6	8	**0**



LOOK BACK

Refer to Lesson 6-4 for information on imaginary roots.

The zeros occur at $x = -4$, $x = -2$, and $x = 3$. The graph of the function verifies that there are three real roots.

Remember when you solved a quadratic equation like $x^2 + 9 = 0$, there were always two imaginary roots. In this case, $3i$ and $-3i$ are the roots. These numbers are a *conjugate pair*. In any polynomial function, if an imaginary number is a zero of that function, its conjugate is also a zero. This is called the **complex conjugates theorem.**

Complex Conjugates Theorem	**Suppose a and b are real numbers with $b \neq 0$. If $a + bi$ is a zero of a polynomial function, then $a - bi$ is also a zero of the function.**

Example 2 **Find all zeros of $f(x) = x^3 - 5x^2 - 7x + 51$ if $4 - i$ is one zero of $f(x)$.**

Since $4 - i$ is a zero, $4 + i$ is also a zero, according to the complex conjugates theorem. So, both $x - (4 - i)$ and $x - (4 + i)$ are factors of the polynomial $x^3 - 5x^2 - 7x - 51$.

$f(x) = [x - (4 - i)][x - (4 + i)](\underline{\ ?\ })$

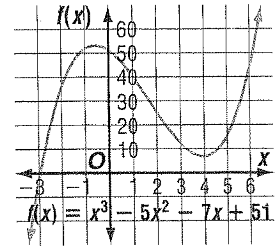
$= [x^2 - (4 + i)x - (4 - i)x + (4 - i)(4 + i)](\underline{\ ?\ })$ *Multiply.*

$= (x^2 - 4x - ix - 4x + ix + 16 - i^2)(\underline{\ ?\ })$ *Simplify.*

$= (x^2 - 8x + 17)(\underline{\ ?\ })$ *Remember that $-i^2 = 1$.*

Since $f(x)$ has degree 3, there are three factors. Use division to find the third factor.

$$
\begin{array}{r}
x + 3 \\
x^2 - 8x + 17 \overline{) x^3 - 5x^2 - 7x + 51} \\
\underline{x^3 - 8x^2 + 17x} \quad \text{Subtract.} \\
3x^2 - 24x + 51 \\
\underline{3x^2 - 24x + 51} \quad \text{Subtract.} \\
0
\end{array}
$$



Therefore, $f(x) = (x^2 - 8x + 17)(x + 3)$. Since $x + 3$ is also a factor, -3 is also a zero. The three zeros are $4 - i$, $4 + i$, and -3. The graph verifies the nature of the zeros.

French mathematician René Descartes made more discoveries about zeros of polynomial functions. His rule of signs is given below.

Descartes' Rule of Signs	**If $P(x)$ is a polynomial function whose terms are arranged in descending powers of the variable,** • **the number of positive real zeros of $P(x)$ is the same as the number of changes in sign of the coefficients of the terms, or is less than this by an even number, and** • **the number of negative real zeros of $P(x)$ is the same as the number of changes in sign of the coefficients of the terms of $P(-x)$, or is less than this by an even number.**

Example 3

State the number of positive and negative real zeros for $p(x) = 4x^5 + 3x^4 - 2x^3 + 5x^2 - 6x + 1$.

Use Descartes' rule of signs. Count the number of changes in sign for the coefficients of $p(x)$.

$$p(x) = 4x^5 + 3x^4 - 2x^3 + 5x^2 - 6x + 1$$

4 → 3 → −2 → 5 → −6 → 1

no, *yes*, *yes*, *yes*, *yes*

Since there are four sign changes, there are either 4, 2, or 0 positive real zeros.

Find $p(-x)$ and count the number of changes in signs for its coefficients.

$$p(-x) = 4(-x)^5 + 3(-x)^4 - 2(-x)^3 + 5(-x)^2 - 6(-x) + 1$$
$$= -4x^5 + 3x^4 + 2x^3 + 5x^2 + 6x + 1$$

yes, *no*, *no*, *no*, *no*

Since there is one sign change, there is exactly 1 negative real zero.

Thus, the function $p(x)$ has either 4, 2, or 0 positive real zeros and exactly 1 negative real zero.

Using a graphing calculator or sketching the graph may help in determining the nature of the zeros of a function.

In Example 3, since $p(x)$ has degree 5, it has five zeros. Using the information in the example, you can make a chart of the possible combinations of real and imaginary zeros.

Number of Positive Real Zeros	Number of Negative Real Zeros	Number of Imaginary Zeros
4	1	0
2	1	2
0	1	4

$4 + 1 + 0 = 5$
$2 + 1 + 2 = 5$
$0 + 1 + 4 = 5$

Example 4 **Write the polynomial function of least degree with integral coefficients whose zeros include 7 and $3 + 2i$.**

If $3 + 2i$ is a zero, then $3 - 2i$ is also a zero. *Why?*

Use the zero product property to write a polynomial equation that has these zeros, 7, $3 + 2i$, and $3 - 2i$, as roots.

$0 = (x - 7)[x - (3 + 2i)][x - (3 - 2i)]$ *Why is $x - 7$ a factor?*

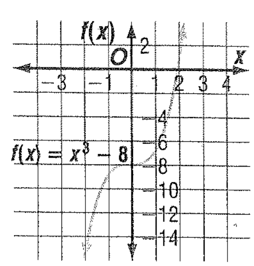
So, $f(x) = (x - 7)[x - (3 + 2i)][x - (3 - 2i)]$
$= (x - 7)[x^2 - (3 - 2i)x - (3 + 2i)x + (3 + 2i)(3 - 2i)]$
$= (x - 7)(x^2 - 3x + 2ix - 3x - 2ix + 9 - 4i^2)$
$= (x - 7)(x^2 - 6x + 13)$
$= x^3 - 13x^2 + 55x - 91$

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **Write** an example of each function. List the possibilities for its zeros.
   a. quadratic b. cubic c. quartic
2. **Describe** the complex conjugate theorem. Use $6 + 7i$ as an example.
3. **State** the zeros of a polynomial function if $(x + 6)$ and $[x - (5 + i)]$ are factors of the polynomial.
4. a. **Write** a polynomial function $p(x)$ whose coefficients have four sign changes.
   b. Find the number of sign changes that $p(-x)$ has.
   c. Describe the nature of the zeros.
5. The graph of $f(x) = x^3 - 8$ is shown at the right.
   a. Describe the nature of the zeros.
   b. Find the zeros of the function.



6. A classmate has been out ill and missed learning Descartes' rule of signs. Write an example and explain Descartes' rule of signs to your fellow classmate.

Guided Practice

Find $f(-x)$ for each function given.

7. $f(x) = 6x^4 - 3x^3 + 5x^2 - x + 2$ 8. $f(x) = x^7 - x^3 + 2x - 1$

State the number of positive real zeros, negative real zeros, and imaginary zeros for each function.

9. $f(x) = x^3 - 6x^2 + 1$

10. $f(x) = x^4 + 5x^3 + 2x^2 - 7x - 9$

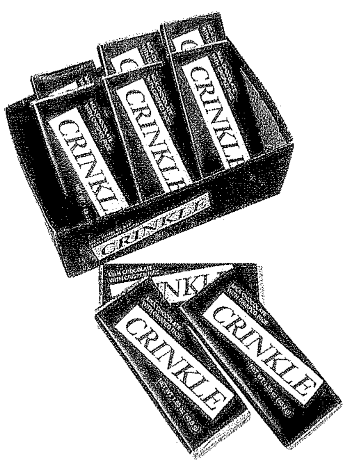
Given a function and one of its zeros, find all of the zeros of the function.

11. $h(x) = x^3 - 6x^2 + 10x - 8; 4$

12. $g(x) = x^3 + 6x^2 + 21x + 26; -2$

13. $f(x) = x^3 + 7x^2 + 25x + 175; 5i$

14. $p(x) = x^4 - 9x^3 + 24x^2 - 6x - 40; 3 - i$



Write the polynomial function of least degree with integral coefficients that has the given zeros.

15. $-4, 1, 5$

16. $9, 1 + 2i$

17. **Manufacturing** The volume of a candy carton is 120 in^3. To hold the correct number of candy bars, the carton must be 3 inches longer than it is wide. The height is 2 inches less than the width. Find the dimensions of the carton.

EXERCISES

Practice

State the number of positive real zeros, negative real zeros, and imaginary zeros for each function.

18. $f(x) = 5x^3 + 8x^2 - 4x + 3$

19. $g(x) = x^4 + x^3 + 2x^2 - 3x - 1$

20. $h(x) = 4x^3 - 6x^2 + 8x - 5$

21. $f(x) = x^4 - 9$

22. $r(x) = x^5 - x^3 - x + 1$

23. $g(x) = x^{14} + x^{10} - x^9 + x - 1$

24. $p(x) = x^5 - 6x^4 - 3x^3 + 7x^2 - 8x + 1$

25. $f(x) = x^{10} - x^8 + x^6 - x^4 + x^2 - 1$

Given a function and one of its zeros, find all of the zeros of the function.

26. $p(x) = x^3 + 2x^2 - 3x + 20; -4$

27. $f(x) = x^3 - 4x^2 + 6x - 4; 2$

28. $v(x) = x^3 - 3x^2 + 4x - 12; 2i$

29. $h(x) = 4x^4 + 17x^2 + 4; 2i$

30. $g(x) = 2x^3 - x^2 + 28x + 51; -\frac{3}{2}$

31. $q(x) = 2x^3 - 17x^2 + 90x - 41; \frac{1}{2}$

32. $f(x) = x^3 - 3x^2 + 9x + 13; 2 + 3i$

33. $r(x) = x^4 - 6x^3 + 12x^2 + 6x - 13; 3 + 2i$

34. $h(x) = x^4 - 15x^3 + 70x^2 - 70x - 156; 5 - i$

Write the polynomial function of least degree with integral coefficients that has the given zeros.

35. $-2, 1, 3$

36. $2, 4i$

37. $4i, 3, -3$

38. $3, 1 + i$

39. $2i, 3i, 1$

40. $6, 2 + 2i$

Critical Thinking

41. If $f(x) = x^3 + kx^2 - 7x - 15$, find the value of k so that $-2 - i$ is a zero of $f(x)$.

42. Suppose a fifth-degree polynomial has exactly two x-intercepts. Describe the nature of the roots of the function. Sketch some examples to support your reasoning.

Applications and Problem Solving

Real World

43. **Medicine** Doctors can measure cardiac output in patients at high risk for a heart attack by monitoring the concentration of dye injected into a vein near the heart. A normal heart's dye concentration is approximated by $d(x) = -0.006x^4 - 0.140x^3 - 0.053x^2 + 1.79x$, where x is the time in seconds.
    a. Find all real zeros by graphing. Then verify them by using synthetic division.
    b. Which root makes sense for an answer to this problem? Why?

44. **Physiology** During a five-second respiratory cycle, the volume of air in liters in the human lungs can be described by the function $A(t) = 0.1729t + 0.1522t^2 - 0.0374t^3$, where t is the time in seconds. Find the volume of air held by the lungs at 3 seconds.

Mixed Review

45. Graph $f(x) = x^3 - 5x + 7$. (Lesson 8–3)

46. Find the center and radius of a circle whose equation is $x^2 + (y - 3)^2 - 4x - 77 = 0$. (Lesson 7–3)

47. Write a quadratic equation that has roots 3 and -5. (Lesson 6–5)

48. **SAT Practice** If $k^2n = 16^2$ and n is an odd integer, then k could be divisible by all of the following EXCEPT
    A 2 B 4 C 8 D 12 E 16

49. **Design** Marco is designing a new dartboard. The center of the board is defined by the inequality $|x| + |y| \leq 2$. Draw the graph of this inequality to see what Marco's new dartboard will look like. (Lesson 3–4)

50. Name which ordered pairs, $(7, -3)$, $(-4, -1)$, or $(12, -6)$, satisfy $-2|x| - 5y < 3$. (Lesson 2–7)

51. Find the value of $f(12)$ when $f(x) = \frac{19}{23 - x}$. (Lesson 2–1)

52. Evaluate $-4|-5x| + 17$ if $x = 2$. (Lesson 1–5)

For Extra Practice, see page 895.

SELF TEST

Find each value if $p(x) = 4x^3 - 3x^2 + 2x - 5$. (Lesson 8–1)

1. $p(a^2)$
2. $p(x + 1)$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. (Lesson 8–2)

3. $x^3 + x^2 - 24x + 36;\ x - 3$
4. $2x^3 + 13x^2 + x - 70;\ x - 2$

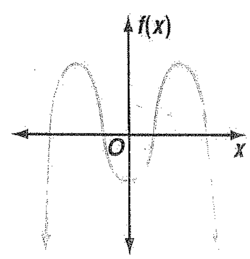
Graph each function. (Lesson 8–3)

5. $g(x) = x^5 - 5$
6. $h(x) = x^3 - x^2 + 4$

State the number of positive real zeros, negative real zeros, and imaginary zeros for each function. (Lesson 8–4)

7. $f(x) = x^3 + 8x^2 - 7x + 10$
8. $f(x) = 6x^4 + 18x^3 + 4x - 9$

9. Determine whether the degree of the function represented by the graph at the right is even or odd. How many real zeros does the polynomial function have? (Lesson 8–1)



10. **Manufacturing** The height of a certain juice can is 4 times the radius of the top of the can. Determine the dimensions of the can if the volume is approximately 17.89 cubic inches. (*Hint*: The formula for the volume of a right circular cylinder is $V = \pi r^2 h$.) (Lesson 8–4)

8-5 Rational Zero Theorem

What YOU'LL LEARN

- To identify all possible rational zeros of a polynomial function by using the rational zero theorem, and
- to find zeros of polynomial functions.

Why IT'S IMPORTANT

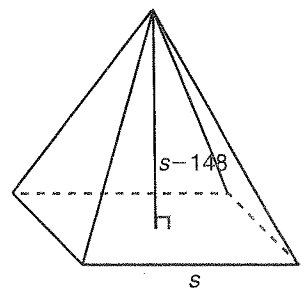
You can use the rational zero theorem to find zeros of polynomials that model situations in finance and food production.

Real World APPLICATION

Architecture

The largest pyramid in the United States is the Luxor Hotel and Casino in Las Vegas, Nevada. The volume of this unique hotel and casino is 28,933,800, or about 29 million cubic feet. The height of the pyramid is 148 feet less than the length of the building. The base of the building is square. What are the dimensions of this building?

The formula for the volume of a pyramid is $V = \frac{1}{3}Bh$, where B represents the area of the base and h represents the height. Let's set up an equation to find the dimensions of the pyramid. Let s represent the length of one side of the base of the pyramid. Then the height is $s - 148$.



$$V = \frac{1}{3}Bh$$

$$28{,}933{,}800 = \frac{1}{3}(s^2)(s - 148)$$

$$86{,}801{,}400 = s^2(s - 148) \quad \textit{Multiply each side by 3.}$$

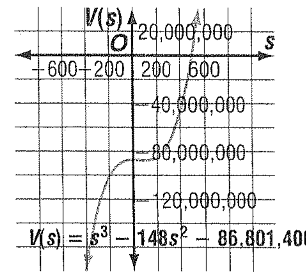
$$86{,}801{,}400 = s^3 - 148s^2 \quad \textit{Distributive property}$$

$$0 = s^3 - 148s^2 - 86{,}801{,}400$$

We could use synthetic substitution to test possible zeros. But the numbers are so large that we might have to test hundreds of possible zeros before we find one. In situations like this, the **rational zero theorem** can give us some direction in testing possible zeros. This theorem and a corollary are stated below.

Rational Zero Theorem	**Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ represent a polynomial function with integral coefficients. If $\frac{p}{q}$ is a rational number in simplest form and is a zero of $y = f(x)$, then p is a factor of a_n and q is a factor of a_0.**
Corollary (Integral Zero Theorem)	**If the coefficients of a polynomial function are integers such that $a_0 = 1$ and $a_n \neq 0$, any rational zeros of the function must be factors of a_n.**

Let $V(s) = s^3 - 148s^2 - 86{,}801{,}400$ be the related function for $0 = s^3 - 148s^2 - 86{,}801{,}400$. All coefficients are integers, $a_0 = 1$, and $a_n = 86{,}801{,}400$. The graph of $V(x)$ is shown at the right.



According to the integral zero theorem, any rational zeros must be factors of 86,801,400.

$$86{,}801{,}400 = 2^3 \times 3^2 \times 5^2 \times 7 \times 83^2$$

So the possible zeros in this case are ± 1 through ± 10, ± 12, ± 14, ± 15, $\pm 18, \ldots, \pm 175$, ± 300, ± 450, ± 489, ± 498, and so on, to $\pm 86{,}801{,}400$.

According to Descartes' rule of signs, there will be only one positive real zero and no negative real zeros. The graph of this function crosses the x-axis one time. We can use synthetic substitution to test for possible zeros, and we can stop testing when we find the first zero. Let's make a chart. Since $s - 148 = h$ and h must be positive, we need to consider only values for s that are greater than 148.

s	1	−148	0	−86,801,400
175	1	27	4725	−85,974,525
300	1	152	45,600	−73,121,400
450	1	302	135,900	−25,646,400
498	1	350	174,300	**0**

One zero is 498. Thus, $s - 498$ is a factor of the polynomial, and 498 is a root of the equation. The dimensions are 498 feet by 498 feet by $498 - 148$ or 350 feet. *Verify the dimensions by substituting them into the formula for the volume of a pyramid.*

Example 1 **List all possible rational zeros of $f(x) = 3x^3 + 9x^2 + x - 10$, and state whether they are positive or negative.**

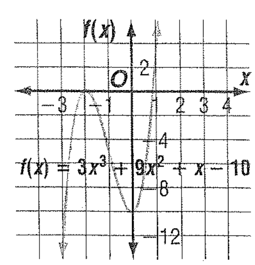
Since $a_0 \neq 1$, we cannot use the integral zero theorem. If $\frac{p}{q}$ is a rational root, then p is a factor of -10 and q is a factor of 3. The possible values of p are ± 1, ± 2, ± 5, and ± 10. The possible values of q are ± 1 and ± 3. So all the possible rational zeros are as follows.

$\pm 1, \pm 2, \pm 5, \pm 10, \pm \frac{1}{3}, \pm \frac{5}{3}$, and $\pm \frac{10}{3}$

Now use Descartes' rule of signs.

$f(x) = 3x^3 + 9x^2 + x - 10$

Since there is one sign change, there is one positive real zero.



$f(x) = -3x^3 + 9x^2 - x - 10$

Since there are two sign changes, there are two or no negative real zeros.

The graph of the function shown above verifies that there is one positive real zero and two negative real zeros. *Note that the two negative real zeros are the same number.*

Example 2

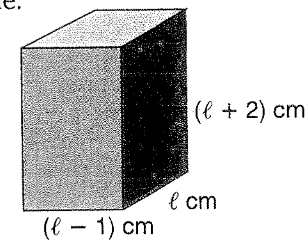
INTEGRATION

Geometry

The volume of a rectangular solid is 1430 cubic centimeters. The width is 1 centimeter less than the length, and the height is 2 centimeters greater than the length. Find the dimensions of the solid.

Explore Read the problem and define the variable.

Let ℓ represent the length of the solid.



Plan Write an equation.

Volume = length × width × height

$V = \ell(\ell - 1)(\ell + 2)$

Solve $V = \ell(\ell - 1)(\ell + 2)$

$1430 = \ell^3 + \ell^2 - 2\ell$ *Replace V with 1430.*

$0 = \ell^3 + \ell^2 - 2\ell - 1430$ *Subtract 1430 from each side.*

Possible rational zeros are ±1, ±2, ±5, ±10, ±11, and ±13. Since measures must be positive and according to Descartes' rule of signs there is one positive real zero, we can stop testing possible zeros when we find the first one. Let's make a table and test each possible rational zero.

$\frac{p}{q}$	1	1	−2	−1430
1	1	2	0	−1430
2	1	3	4	−1422
5	1	6	28	−1290
10	1	11	108	−350
11	1	12	130	**0**

One zero is 11. The other dimensions are 11 − 1 or 10 cm and 11 + 2 or 13 cm.

Examine Check to see if the dimensions are correct.

$10 \times 11 \times 13 = 1430$ ✓

You have learned many rules to help you determine the number and characteristics of the zeros of a function. Example 3 shows how many of them can be used.

Example 3

Find all zeros of $f(x) = 4x^4 - 13x^3 - 13x^2 + 28x - 6$.

- From the corollary to the fundamental theorem of algebra, we know there are exactly 4 complex roots.
- According to Descartes' rule of signs, there are either 3 or 1 positive real zeros and exactly 1 negative real zero.
- According to the rational zero theorem, the possible rational zeros are $\pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}, \pm1, \pm\frac{3}{2}, \pm2, \pm3$, and ±6.
- Use synthetic substitution and a chart to find at least one zero.

$\frac{p}{q}$	4	−13	−13	28	−6	
$\frac{1}{4}$	4	−12	−16	24	0	*One zero is $\frac{1}{4}$.*

(continued on the next page)

The depressed polynomial after division by $x - \frac{1}{4}$ is $4x^3 - 12x^2 - 16x + 24$. Now use a synthetic division chart with this polynomial.

x	4	−12	−16	24
$\frac{1}{2}$	4	−10	−21	$\frac{27}{2}$
$\frac{3}{4}$	4	−9	$-\frac{91}{4}$	$\frac{111}{16}$
1	4	−8	−24	**0**

Another zero is 1.

The new depressed polynomial is $4x^2 - 8x - 24$. Use the quadratic formula to find other possible zeros.

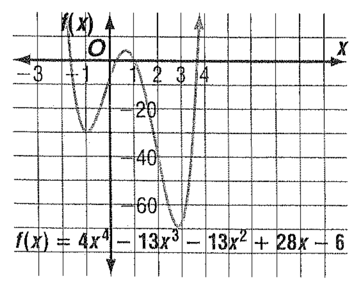
$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(4)(-24)}}{2(4)} \quad a = 4,\ b = 8,\ c = -24$$

$$= \frac{8 \pm \sqrt{448}}{8} \text{ or } 1 \pm \sqrt{7}$$

The zeros are $\frac{1}{4}$, 1, $1 + \sqrt{7}$, and $1 - \sqrt{7}$.

The approximate values of the irrational zeros are 3.65 and −1.65. So, there are 3 positive real zeros and 1 negative zero.

The graph of the function shown at the right crosses the x-axis 4 times, confirming that there are 4 real roots.



CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. a. **Explain** when you can use the integral zero theorem to determine possible rational zeros for a polynomial function.
   b. Why is it helpful to use the rational zero theorem while finding the zeros of a polynomial function?
2. Refer to Example 2. When testing possible zeros, would starting with a number greater than 1 have made more sense? Explain.
3. **Write** a polynomial function with four possible rational zeros.
4. a. **Explain** why there cannot be three positive zeros for $p(x) = x^3 + 4x^2 - 3x + 2$.
   b. **Explain** why there cannot be four positive zeros for $p(x) = x^3 + 2x^2 + 3x + 1$.
5. Refer to the application at the beginning of the lesson. List all of the possible roots between 0 and 100.
6. Write a polynomial function that has possible rational zeros of ±1, ±3, $\pm\frac{1}{2}$, and $\pm\frac{3}{2}$.

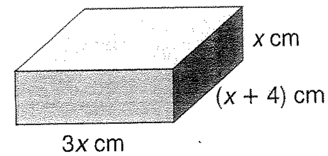
Guided Practice

List all of the possible rational zeros for each function.

7. $h(x) = x^3 + 8x + 6$
8. $d(x) = 6x^3 + 6x^2 - 15x - 2$

Find all of the rational zeros for each function.

9. $f(x) = x^3 - x^2 - 34x - 56$
10. $p(x) = x^3 - 3x - 2$
11. $g(x) = x^4 - 3x^3 + x^2 - 3x$
12. $h(x) = 6x^3 + 11x^2 - 3x - 2$
13. Find all of the zeros of $h(x) = 9x^5 - 94x^3 + 27x^2 + 40x - 12$.
14. Write a polynomial function of least degree that has zeros -3, 2, and 5.
15. **Geometry** The volume of the figure at the right is 384 cm^3. Find the dimensions.



EXERCISES

Practice

List all of the possible rational zeros for each function.

16. $f(x) = x^3 + 6x + 2$
17. $p(x) = x^4 - 10$
18. $n(x) = x^5 + 6x^3 - 12x + 18$
19. $p(x) = 3x^3 - 5x^2 - 11x + 3$
20. $f(x) = 3x^4 + 15$
21. $h(x) = 9x^6 - 5x^3 + 27$

Find all of the rational zeros for each function.

22. $p(x) = x^3 - 5x^2 - 22x + 56$
23. $f(x) = x^3 + x^2 - 80x - 300$
24. $g(x) = x^4 - 3x^3 - 53x^2 - 9x$
25. $h(x) = 2x^3 - 11x^2 + 12x + 9$
26. $f(x) = 2x^5 - x^4 - 2x + 1$
27. $p(x) = x^4 + 10x^3 + 33x^2 + 38x + 8$
28. $n(x) = x^4 + x^2 - 2$
29. $t(x) = x^4 - 13x^2 + 36$
30. $h(x) = x^4 - 3x^3 - 5x^2 + 3x + 4$
31. $p(x) = x^3 + 3x^2 - 25x + 21$
32. $f(x) = x^5 - 6x^3 + 8x$
33. $g(x) = 48x^4 - 52x^3 + 13x - 3$

Find all of the zeros of each function.

34. $f(x) = 6x^3 + 5x^2 - 9x + 2$
35. $p(x) = 6x^4 + 22x^3 + 11x^2 - 38x - 40$
36. $g(x) = 5x^4 - 29x^3 + 55x^2 - 28x$
37. $p(x) = x^5 - 2x^4 - 12x^3 - 12x^2 - 13x - 10$

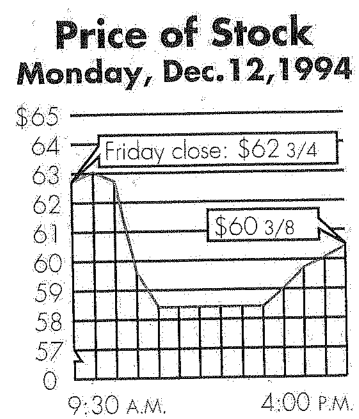
Critical Thinking

38. Suppose k and $2k$ are zeros of $f(x) = x^3 + 4x^2 + 9kx - 90$. Find k and all three zeros of $f(x)$.

Applications and Problem Solving

Real World

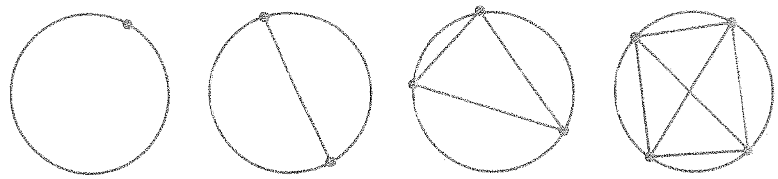
39. **Stock Market** In a recent year, a research lab discovered a flaw in a computer chip that could have caused an error as often as once every 24 days. The computer company's stock was affected on the day the flaw was discovered, as shown in the graph. The function $f(x) = -0.002x^4 + 0.05x^3 - 0.3x^2 - 0.4x + 63$ can be used to model the stock prices at time x, where $x = 0$ represents 9:30 A.M., $x = 1$ represents 10:00 A.M., and so on.
   a. Use $f(x)$ to estimate the price of the stock at 2:30 P.M.
   b. Compare this value to an estimate of the value from the graph.



Source: *Bloomberg Business News*

40. **Food Production** I.C. Dreams makes ice cream cones. The volume of each cone is about 5.24 cubic inches, and the height is 4 inches more than the radius of the opening of the cone. Find the dimensions of the cone. Use the formula for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$.

41. **Patterns** The diagrams below show the number of regions formed by connecting the points on a circle.



The number of regions formed by connecting n points of a circle can be described by the function $f(n) = \frac{1}{24}(n^4 - 6n^3 + 23n^2 - 18n + 24)$.

a. Find the number of regions formed by connecting 5 points of a circle. Draw a diagram to verify your solution.

b. How many points would you have to connect to form 99 regions?

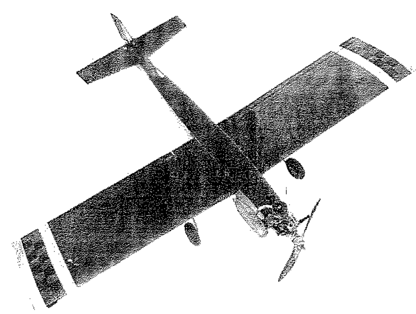
Mixed Review

42. Write a polynomial function of least degree with integral coefficients that has -2 and $2 + 3i$ as zeros. (Lesson 8–4)

43. Write $2y^2 = 14x$ in the form $x = a(y - k)^2 + h$. (Lesson 7–2)

44. Solve $c^2 - 9c - 58 = -7c + 5$ by factoring. (Lesson 6–2)

45. **Physics** A model airplane is fixed on a string so that it flies around in a circle. The designers of the plane want to find the time it takes for the airplane to make a complete circle. They know that the formula $F_c = m\left(\frac{4\pi^2 r}{T^2}\right)$ describes the force required to keep the airplane going in a circle, m represents the mass of the plane, r represents the radius of the circle, and T represents the time for a revolution. Solve the formula for T. Write the answer in simplest radical form. (Lesson 5–8)



46. Use augmented matrices to solve the system of equations. (Lesson 4–7)

$5x - 7y + z = 29$

$-2x - 3y + 5z = 20$

$x - 9y + 3z = 13$

47. **ACT Practice** What is the midpoint of the line segment whose endpoints are represented on the coordinate grid by the points $(-5, -3)$ and $(-1, 4)$?

A $\left(-3, -\frac{1}{2}\right)$ **B** $\left(-3, \frac{1}{2}\right)$ **C** $\left(-2, \frac{7}{2}\right)$ **D** $\left(-2, \frac{1}{2}\right)$ **E** $\left(\frac{1}{2}, -3\right)$

48. Given the function $f(x, y) = 9x - 3y$, find $f(-3, 7)$. (Lesson 3–6)

49. Marcia and Roberto want to build a ramp that they can use while rollerblading. If they want the ramp to have a base of 8 feet and slope of $\frac{1}{2}$, how tall will their ramp be? (Lesson 2–3)

For Extra Practice, see page 895.

50. Margie is 6 years older than Max. Moira is 19 years younger than Margie. If Max is 17, how old is Moira? (Lesson 1–4)

8-6 Using Quadratic Techniques to Solve Polynomial Equations

What YOU'LL LEARN

- To solve nonquadratic equations by using quadratic techniques.

Why IT'S IMPORTANT

You can solve polynomial equations that model situations in finance and geometry.

Real World APPLICATION

Finance

On his seventeenth birthday, Montel received \$100. On his eighteenth birthday, he received \$150. One year ago, on his nineteenth birthday, he received \$200. Montel put his birthday money into an account paying 6% interest, compounded annually, and did not withdraw or add any additional money. Determine the amount of money currently in his account.

We can use the formula for compound interest, $A = P(1 + r)^t$, where P is the original amount of money deposited, r is the interest rate (written as a decimal), and t is the number of years invested. The amount of money currently in his account is the sum of the amounts he received on his last three birthdays, plus interest.

The interest rate is 6%, so $r = 0.06$. Let $x = 1 + r$ or 1.06, and let $T(x)$ represent the total amount of money currently in the account. Find $T(1.06)$.

Total = *money from 17th birthday* + *money from 18th birthday* + *money from 19th birthday*

$$T(x) = 100x^3 + 150x^2 + 200x$$

$$T(1.06) = 100(1.06)^3 + 150(1.06)^2 + 200(1.06)$$ *Replace x with 1.06.*

$= \$499.64$ The amount of money in Montel's account is \$499.64.

LOOK BACK

Refer to Lessons 6-1 through 6-4 for information on quadratic functions.

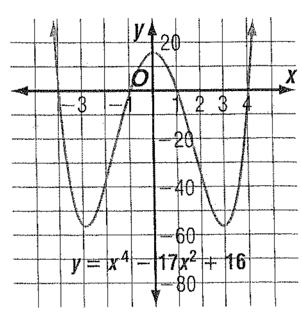
Note that the polynomial function contains a factor that is a quadratic since $T(x) = 100x^3 + 150x^2 + 200x$ or $50x(2x^2 + 3x + 4)$.

In some cases, we can rewrite polynomial equations and use quadratic techniques to solve them. For example, $x^4 - 38x^2 + 72 = 0$ can be written as $(x^2)^2 - 38(x^2) + 72 = 0$. Equations that can be written in the form $a\,[f(x)]^2 + b[f(x)] + c = 0$ are said to be in **quadratic form.**

Definition of Quadratic Form	For any numbers a, b, and c, except $a = 0$, an equation that can be written as $a[f(x)]^2 + b[f(x)] + c = 0$, where $f(x)$ is some expression in x, is in quadratic form.

Example 1 **Solve $x^4 - 17x^2 + 16 = 0$.**

The graph of $y = x^4 - 17x^2 + 16$ crosses the x-axis 4 times, so there are 4 real zeros.



$$x^4 - 17x^2 + 16 = 0$$

$$(x^2)^2 - 17(x^2) + 16 = 0 \quad \textit{Quadratic form}$$

$$(x^2 - 16)(x^2 - 1) = 0$$

$$(x - 4)(x + 4)(x - 1)(x + 1) = 0$$

(continued on the next page)

Use the zero product property.

$x - 4 = 0$ or $x + 4 = 0$ or $x - 1 = 0$ or $x + 1 = 0$

$x = 4$ $\quad x = -4$ $\quad x = 1$ $\quad x = -1$

The roots are −4, 4, −1, and 1, which are verified on the graph.

You can solve cubic equations with the quadratic formula if a quadratic factor can be found.

Example 2 **Solve $x^3 + 64 = 0$.**

$$x^3 + 64 = 0$$

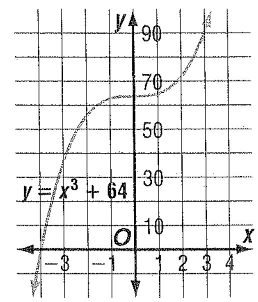
$$(x + 4)(x^2 - 4x + 16) = 0 \quad \textit{Factor.}$$

Use the zero product property.

$x + 4 = 0$ or $x^2 - 4x + 16 = 0$

$$x = -4 \qquad x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(16)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{-48}}{2} \text{ or } 2 \pm 2i\sqrt{3}$$



The roots are −4 and $2 \pm 2i\sqrt{3}$.

The only real root is −4.

The graph of the related function crosses the x-axis only once at −4.

You have studied the rule $(a^m)^n = a^{mn}$ in Chapter 5. This property of exponents is often used to solve equations that have terms with rational exponents.

Example 3

Real World APPLICATION

Finance

Isabel earned \$1000 from her summer job, and on August 1 she decided to put it in the bank to save it for a cruise she wants to take during the next spring break, which starts on April 1. The cruise costs \$1046, but Isabel figures that if her money earns some interest she may have enough money by April. As she shops around for interest rates at various banks, what interest rate should she be looking for so that her \$1000 on August 1 will grow to \$1046 by April 1? Use the interest formula $A = P(1 + r)^t$.

Let x represent $1 + r$ and substitute the known values into the formula: $A = \$1046$, $P = \$1000$, and $t = 8$ months or $\frac{2}{3}$ year.

$$A = P(1 + r)^t$$

$$1046 = 1000x^{\frac{2}{3}} \quad \textit{Substitute.}$$

$$1.046 = x^{\frac{2}{3}} \quad \textit{Divide each side by 1000.}$$

$$(1.046)^3 = \left(x^{\frac{2}{3}}\right)^3 \quad \textit{Cube each side.}$$

$$1.14 = x^2$$

$$\pm\sqrt{1.14} \text{ or } \pm 1.07 = x \quad \textit{Take the square root of each side. Why } \pm\textit{?}$$

Since $x = 1 + r$, then $r = 0.07$ or $r = -2.07$. Since interest rates cannot be negative, the interest rate is 0.07 or 7%.

Some equations involving rational exponents can be written in quadratic form.

Example 4 **Solve $x^{\frac{1}{2}} - 8x^{\frac{1}{4}} + 15 = 0$.**

$$x^{\frac{1}{2}} - 8x^{\frac{1}{4}} + 15 = 0$$

$$\left(x^{\frac{1}{4}}\right)^2 - 8\left(x^{\frac{1}{4}}\right) + 15 = 0 \quad \textit{Quadratic form}$$

$$\left(x^{\frac{1}{4}} - 5\right)\left(x^{\frac{1}{4}} - 3\right) = 0 \quad \textit{Factor.}$$

Use the zero product property.

$$x^{\frac{1}{4}} - 5 = 0 \quad \text{or} \quad x^{\frac{1}{4}} - 3 = 0$$

$$x^{\frac{1}{4}} = 5 \qquad x^{\frac{1}{4}} = 3$$

$$\left(x^{\frac{1}{4}}\right)^4 = 5^4 \qquad \left(x^{\frac{1}{4}}\right)^4 = 3^4$$

$$x = 625 \qquad x = 81$$

Check:

$$x^{\frac{1}{2}} - 8x^{\frac{1}{4}} + 15 = 0$$

$$625^{\frac{1}{2}} - 8(625)^{\frac{1}{4}} + 15 \stackrel{?}{=} 0$$

$$25 - 40 + 15 \stackrel{?}{=} 0$$

$$0 = 0 \ \checkmark$$

$$81^{\frac{1}{2}} - 8(81)^{\frac{1}{4}} + 15 = 0$$

$$9 - 24 + 15 \stackrel{?}{=} 0$$

$$0 = 0 \ \checkmark$$

The real roots are 81 and 625.

Example 5 **Solve $x - 2\sqrt{x} - 3 = 0$.**

$$x - 2\sqrt{x} - 3 = 0$$

$$\left(\sqrt{x}\right)^2 - 2(\sqrt{x}) - 3 = 0 \quad \textit{Quadratic form}$$

$$\sqrt{x} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \textit{Use the quadratic formula.}$$

$$\sqrt{x} = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-3)}}{2(1)} \quad \textit{a = 1, b = −2, and c = −3}$$

$$\sqrt{x} = \frac{2 \pm \sqrt{16}}{2}$$

$$\sqrt{x} = 3 \quad \text{or} \quad \sqrt{x} = -1$$

$$x = 9$$

There is no real number x such that $\sqrt{x} = -1$. The only real solution is 9.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **Explain** why the graph in Example 2 crosses the x-axis only once when three roots are given.
2. **Explain** the steps you would take to solve $\sqrt{x^4 + 48} = 4x$.
3. **Write** three examples of equations that are not quadratic but can be written in quadratic form. Then write them in quadratic form.

Guided Practice

Factor each polynomial. Identify the quadratic factor if one exists.

4. $x^4 - 3x^3 + 6x^2$
5. $2x^3 + 7x^2 - 8x$
6. $4m^3 + 9m - 16m^2$
7. $y^3 - y^5 - 100y$
8. $x^7 + x^{\frac{7}{2}} + x^5$
9. $x^3 - 729$

Write each equation in quadratic form if possible. If not, explain why not.

10. $3r + 7\sqrt{r} = 11$
11. $a^8 + 10a^4 - 16 = 0$

Solve each equation.

12. $x - 16x^{\frac{1}{2}} = -64$

13. $m^4 + 7m^3 + 12m^2 = 0$

14. $3m^{\frac{3}{2}} - 81 = 0$

15. $y^3 = 26.6y - 3.2y^2$

16. **Geometry** The width of a rectangular prism is w centimeters. The height is 2 centimeters less than the width. The length is 4 centimeters more than the width. If the volume of the prism is 8 times the measure of the length, find the dimensions of the prism.

EXERCISES

Practice

Write each equation in quadratic form if possible. If not, explain why not.

17. $x^8 + 10x^4 = -13.2$

18. $11x^4 + 3x = -8$

19. $84n^4 - 62n^2 = 0$

20. $7q + 8\sqrt{q} = 13$

21. $5y^4 + 7y = 8$

22. $11n^4 = -44n^2$

Solve each equation.

23. $x^3 - 3x^2 - 10x = 0$

24. $n^3 + 12n^2 + 32n = 0$

25. $b^3 = 1331$

26. $m^4 - 7m^2 + 12 = 0$

27. $z - 8\sqrt{z} - 240 = 0$

28. $y^3 - 729 = 0$

29. $y^{\frac{2}{3}} - 9y^{\frac{1}{3}} + 20 = 0$

30. $r - 19r^{\frac{1}{2}} + 60 = 0$

31. $6.25m^3 - 12.25m = 0$

32. $y^{\frac{1}{3}} = 7.5$

33. $m^5 + 1.5m^4 = 15.04m^3$

34. $p^{\frac{2}{3}} + 11p^{\frac{1}{3}} + 28 = 0$

35. Write an equation for a polynomial that has roots -3, 0, and 2.

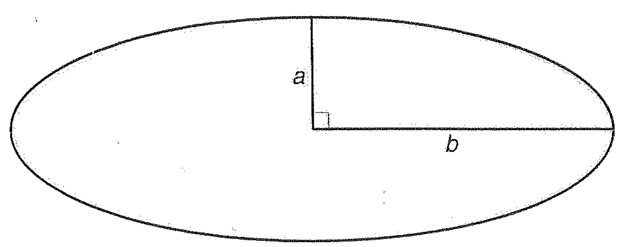
Critical Thinking

36. Write an explanation about how you would solve the equation $|a - 3|^2 - 9|a - 3| = -8$. Then solve the equation.

Applications and Problem Solving

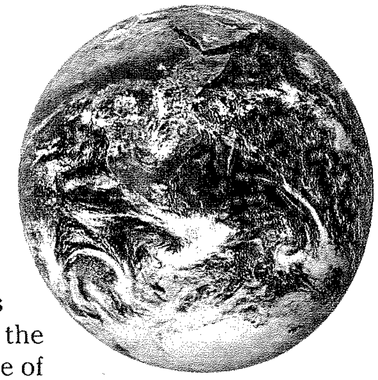
Real World

37. **Geometry** The formula for the area of an ellipse is $A = \pi ab$. Find the measure of a and b to the nearest hundredth of an inch if an ellipse has an area of 8.85 square inches and the measure of b is 2.3 inches greater than a.



38. **Geometry** A piece of wire is cut into two pieces. One piece is bent into the shape of a square, and the other into the shape of an equilateral triangle. The side of the square and a side of the equilateral triangle have the same length, and the measure of that length, in inches, is an integer. The original piece of wire was less than 50 inches long before it was cut.
    a. Find all possible integral measurements for the length of the side of the square and the triangle.
    b. What is the shortest possible length for the original piece of wire?
    c. What is the longest possible length for the original piece of wire?

39. **Aerospace** The force of gravity decreases with the square of the distance from the center of Earth. So, as an object moves further from Earth, its weight decreases. The radius of Earth is approximately 3960 miles. The formula relating weight and distance is $(3960 + r)^2 = \frac{3960^2 \cdot W_E}{W_S}$, where W_E represents the weight of a body on Earth, W_S represents the weight of a body a certain distance from the center of Earth, and r represents the distance of an object above Earth's surface.



a. An astronaut weighs 140 pounds on Earth and 120 pounds in space. How far is he above Earth's surface?

b. An astronaut weighs 125 pounds on Earth. What is her weight in space if she is 99 miles above the surface of Earth?

Mixed Review

40. **Manufacturing** The volume of a milk carton is 200 cubic inches. The base of the carton is square, and the height is 3 inches more than the length of the base. What are the dimensions of the carton? (Lesson 8–5)

41. Find the value of c such that the points at (7, 2) and (3, c) are 5 units apart. (Lesson 7–1)

42. **Agriculture** The function $f(x) = -x^2 + 8x$, where x is the number of apple trees planted in a given area and $f(x)$ is the number of pounds of apples produced per day, can be used to determine how many apples are to be planted in a certain area. (Lesson 6–7)

a. Graph $f(x) = -x^2 + 8x$.

b. If Wessel Farm wants to produce at least 12 pounds of apples per day, how many trees should they plant in the area? Write as an inequality.

c. According to the graph of $f(x)$, production is low if few or many trees are planted and production is high if a medium number of trees is planted. Give some possible reasons why this might be true in real life.

internet CONNECTION

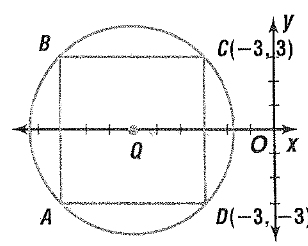
Data Update For more information on fruit production, visit: **www.algebra2.glencoe.com**

43. **ACT Practice** In the figure, what is the equation of circle Q that is circumscribed around the square $ABCD$?

A $(x - 6)^2 - y^2 = 6$ B $(x - 6)^2 + y^2 = 6$

C $(x + 6)^2 + y^2 = 6$ D $(x + 6)^2 + y^2 = 18$

E $(x - 6)^2 + y^2 = 18$



44. Find M if $\begin{bmatrix} -9 & 12 \\ 4 & -7 \end{bmatrix} \cdot M = \begin{bmatrix} -9 & 12 \\ 4 & -7 \end{bmatrix}$. (Lesson 4–5)

45. Use Cramer's rule to solve the system of equations. (Lesson 3–3)

$$\frac{x}{2} - \frac{2y}{3} = 2\frac{1}{3}$$
$$3x + 4y = -50$$

46. **Geography** The following numbers are the percent of people in the South American countries who live in urban areas. (Lesson 1–3)

84, 87, 51, 76, 65, 54, 46, 70, 86, 83, 35

a. Make a stem-and-leaf plot of this data.

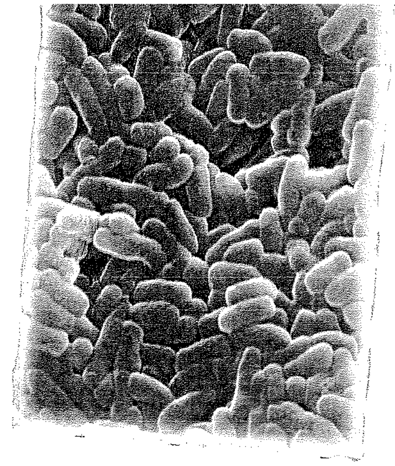
b. How many countries are less than 60% urban?

c. Argentina is the most urbanized country in South America. What percent of it is urban?

d. Guyana is the least urbanized country in South America. What percent of it is urban?

For Extra Practice, see page 896.

8–7 Composition of Functions



What YOU'LL LEARN

- To find the composition of functions.

Why IT'S IMPORTANT

You can use composition of functions to solve problems involving biology and foreign currency.

CONNECTION

Biology

Temperature is measured in different units in different countries. An American scientist and a German scientist are working on incubating a bacterium in their respective countries. They are sharing their findings with each other through the Internet. The last message from the German scientist says that her bacterium died at a temperature of 312° K, which she discovered was not warm enough. The American scientist's temperature for incubation is 98.2° F. Should the American scientist be worried? *This problem will be solved in Example 5.*

Let $K(x)$ be the function for converting Celsius temperatures to Kelvin, and let $C(x)$ be the function for converting Fahrenheit temperatures to Celsius.

$K(x) = x + 273$ *Converting Celsius to Kelvin*

$C(x) = \frac{5}{9}(x - 32)$ *Converting Fahrenheit to Celsius*

Using **composition of functions** is one way to solve the problem.

Composition of Functions

Suppose f and g are functions such that the range of g is a subset of the domain of f. Then the composite function $f \circ g$ can be described by the equation $[f \circ g](x) = f[g(x)]$.

$[f \circ g](x)$ and $f[g(x)]$ are both read "f of g of x."

Given two functions f and g, you can find $f \circ g$ and $g \circ f$ if the range of each function is a subset of the domain of the other function.

Example 1 **If $f = \{(1, 2), (3, 4), (5, 4)\}$ and $g = \{(2, 5), (4, 3)\}$, find $f \circ g$ and $g \circ f$.**

$f[g(2)] = f(5)$ or 4 $\quad g(2) = 5$

$f[g(4)] = f(3)$ or 4 $\quad g(4) = 3$

$f \circ g = \{(2, 4), (4, 4)\}$

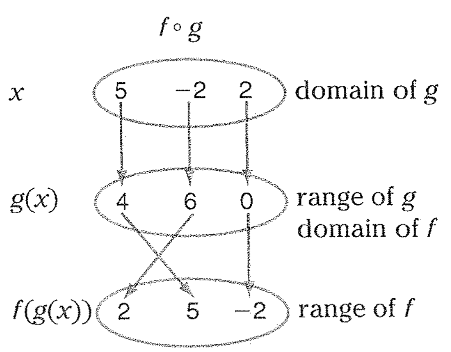
$g[f(1)] = g(2)$ or 5 $\quad f(1) = 2$

$g[f(3)] = g(4)$ or 3 $\quad f(3) = 4$

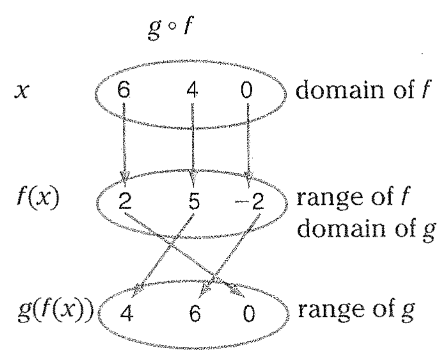
$g[f(5)] = g(4)$ or 3 $\quad f(5) = 4$

$g \circ f = \{(1, 5), (3, 3), (5, 3)\}$

The composition of functions can be shown by mappings. Suppose $f = \{(6, 2), (4, 5), (0, -2)\}$ and $g = \{(5, 4), (-2, 6), (2, 0)\}$. The composition of these functions is shown below.



$f \circ g = \{(5, 5), (-2, 2), (2, -2)\}$



$g \circ f = \{(6, 0), (4, 4), (0, 6)\}$

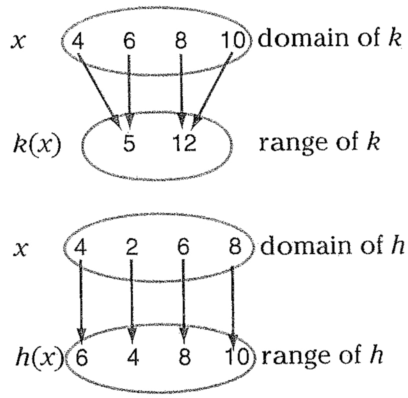
LOOK BACK

Refer to Lesson 2-1 for information on mappings.

The composition of two functions may not exist. Look back at the definition of the composition of functions. The composition of functions f and g, $f \circ g$, is defined when the range of g is a subset of the domain of f. If this condition is not met, the composition is not defined.

Example 2 **If $h = \{(4, 6), (2, 4), (6, 8), (8, 10)\}$ and $k = \{(4, 5), (6, 5), (8, 12), (10, 12)\}$, find $h \circ k$ and $k \circ h$, if they exist.**

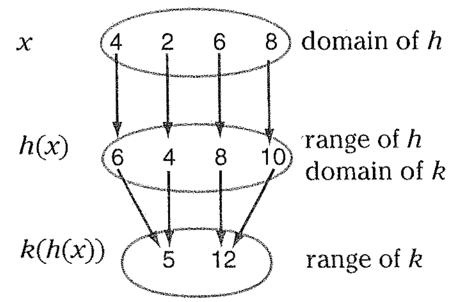
$h \circ k$



$h \circ k$ does not exist.

The range of k is not a subset of the domain of h. *Why not?*

$k \circ h$



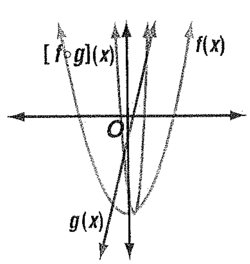
$k \circ h = \{(4, 5), (2, 5), (6, 12), (8, 12)\}$

Sometimes, when two functions are composed, the graph of the composition resembles the graph of one of the original functions.

Example 3 **If $f(x) = x^2 - 4$ and $g(x) = 4x - 1$, find $[f \circ g](x)$.**

$$
\begin{aligned}
[f \circ g](x) &= f[g(x)] \\
&= f(4x - 1) && \textit{Substitute } 4x - 1 \textit{ for } g(x). \\
&= (4x - 1)^2 - 4 && \textit{Evaluate } f \textit{ when } x \textit{ is } (4x - 1). \\
&= 16x^2 - 8x + 1 - 4 \\
&= 16x^2 - 8x - 3 && \textit{Simplify.}
\end{aligned}
$$

The graphs of each function are shown below.



$f(x)$ is quadratic.
$g(x)$ is linear.
$[f \circ g](x)$ is quadratic.

Example 4 **If $f(x) = x + 5$ and $g(x) = x^2 - 2$, find $[f \circ g](3)$ and $[g \circ f](3)$.**

$[f \circ g](3) = f[g(3)]$

$= f(3^2 - 2)$ *Substitute $3^2 - 2$ for $g(3)$.*

$= f(7)$ *Simplify.*

$= 7 + 5$ or 12 *Evaluate f when x is 7.*

$[g \circ f](3) = g[f(3)]$

$= g(3 + 5)$ *Substitute $3 + 5$ for $f(3)$.*

$= g(8)$ *Simplify.*

$= 8^2 - 2$ or 62 *Evaluate g when x is 8.*

Example 5

CONNECTION

Biology

Refer to the application at the beginning of the lesson. Should the American scientist be worried that her incubation temperature is not warm enough?

In order to compare the two temperatures, we need to convert the American scientist's temperature from Fahrenheit to Kelvin. To do this, convert the temperature from Fahrenheit to Celsius and then from Celsius to Kelvin. This can be written as $[K \circ C](x)$ or $K[C(x)]$.

$[K \circ C](x) = K[C(x)]$

$= K\left[\frac{5}{9}(x - 32)\right]$ *Substitute $\frac{5}{9}(x - 32)$ for $C(x)$.*

$= \left[\frac{5}{9}(x - 32)\right] + 273$

$[K \circ C](98.2) = \left[\frac{5}{9}(98.2 - 32)\right] + 273$ *Replace x with 98.2.*

≈ 309.8 *Simplify*

The American scientist is incubating her bacterium at 309.8° K. This is 2.2° K cooler than the German scientist's incubation temperature in which the bacterium died, so she should be worried.

Iteration is a special type of composition, the composition of a function with itself.

Example 6 **Find $f(4)$ if $f(0) = 2$ and $f(n) = f(n - 1) + n$.**

$f(0) = 2$

$f(1) = f(1 - 1) + 1 = f(0) + 1 = 3$ *Use $f(0)$ to find $f(1)$.*

$f(2) = f(2 - 1) + 2 = f(1) + 2 = 5$ *Use $f(1)$ to find $f(2)$.*

$f(3) = f(3 - 1) + 3 = f(2) + 3 = 8$ *Use $f(2)$ to find $f(3)$.*

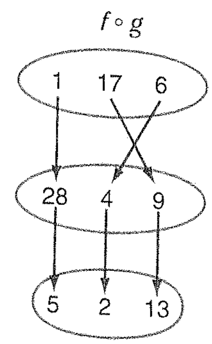
$f(4) = f(4 - 1) + 4 = f(3) + 4 = 12$ *Use $f(3)$ to find $f(4)$.*

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **Write** out how you would read $[g \circ h](x)$.
2. In Example 2, what values would the domain of h need for $h \circ k$ to exist?
3. Look at the mapping of $[f \circ g](x)$ at the right.
   a. State the domain and the range of f and g.
   b. Write the functions f and g as a set of ordered pairs.
   c. Does $[g \circ f](x)$ exist? Explain.



4. Refer to Example 3. What type of function is $[g \circ f](x)$?
5. **Show** that if $f(x) = x^2$ and $g(x) = x - 4$, then $[f \circ g](x) \neq [g \circ f](x)$.
6. **Write** two functions $f(x)$ and $g(x)$ such that $f[g(x)] = x^2 - 6$.
7. **Explain** how you could find $f(2)$ and $f(3)$ if $f(x)$ is defined by $f(x + 1) = \frac{1}{4}f(x)$ and $f(1) = 24$.

Math Journal

8. Draw your family tree. Explain how your family tree could be a composition of functions.

Guided Practice

Find $[f \circ g](2)$ and $[g \circ f](2)$.

9. $f(x) = x + 6$
   $g(x) = x - 3$
10. $f(x) = x^2 + 3$
    $g(x) = x + 1$

Find $g[h(x)]$ and $h[g(x)]$.

11. $g(x) = 4x$
    $h(x) = 2x - 1$
12. $g(x) = x + 2$
    $h(x) = x^2$

If $f(x) = x^2$, $g(x) = 3x$, and $h(x) = x + 2$, find each value.

13. $f[g(1)]$
14. $[f \circ h](4)$
15. $h[f(x)]$

Find the first four iterations of each function, given the initial value.

16. $f(0) = 1, f(n) = f(n - 1) + 3$
17. $f(1) = 3, f(n) = 2f(n - 1)$
18. **Bonus** A sales representative for a furniture manufacturer is paid an annual salary plus a bonus of 3% of her sales *over* \$275,000. Let $f(x) = x - 275{,}000$ and let $h(x) = 0.03x$.
    a. If x is greater than 275,000, is her bonus represented by $f[h(x)]$ or $h[f(x)]$? Explain.
    b. Find her bonus if her sales for the year are \$400,000.

EXERCISES

Practice

Find $[f \circ g](3)$ and $[g \circ f](3)$.

19. $f(x) = x$
    $g(x) = -x$

20. $f(x) = x^2$
    $g(x) = x^3$

21. $f(x) = x + 1$
    $g(x) = x^2 + 6$

22. $f = \{(1, -7), (2, 3), (3, 0)\}$
    $g = \{(0, 11), (3, 1)\}$

23. $f = \{(-1, 9), (3, 6)\}$
    $g = \{(-5, 3), (6, 12), (3, -1)\}$

24. $f(x) = 7x - 5$
    $g(x) = x^2 - 3x + 7$

Find $g[h(x)]$ and $h[g(x)]$.

25. $g(x) = x + 7$
    $h(x) = x + 4$

26. $g(x) = 5x$
    $h(x) = 2x$

27. $g(x) = x - 2$
    $h(x) = x^2$

28. $g(x) = -2x$
    $h(x) = -3x + 1$

29. $g(x) = x + 1$
    $h(x) = x^3$

30. $g(x) = |x|$
    $h(x) = x - 3$

If $f(x) = x^2$, $g(x) = 4x$, and $h(x) = x - 1$, find each value.

31. $h[g(2)]$
32. $[f \circ g](4)$
33. $[h \circ f](3)$
34. $[f \circ h](-3)$
35. $h[g(-2)]$
36. $h[f(-4)]$
37. $g[f(x)]$
38. $[f \circ g](x)$
39. $[f \circ (g \circ h)](x)$

Find the first four iterations of each function, given the initial value.

40. $f(0) = 4, f(n) = f(n - 1) - 5$
41. $f(1) = 2, f(n) = 3f(n - 1)$
42. $f(1) = 3, f(n) = f(n - 1) + 4n$
43. $f(0) = 0.2, f(n) = n(f(n - 1))$
44. $f(0) = -2, f(n) = 3f(n - 1) - 2n$
45. $f(0) = 6, f(n) = 4f(n - 1) + n^2$

Express $f \circ g$ and $g \circ f$, if they exist, as sets of ordered pairs.

46. $f = \{(1, 1), (0, -3)\}$
    $g = \{(1, 0), (-3, 1), (2, 1)\}$

47. $f = \{(3, 8), (4, 0), (6, 3), (7, -1)\}$
    $g = \{(0, 4), (8, 6), (3, 6), (-1, 8)\}$

Critical Thinking

48. Name two functions f and g such that $f[g(x)] = g[f(x)]$.
49. If $f(0) = 4$ and $f(x + 1) = 3f(x) - 2$, find $f(4)$.

Applications and Problem Solving

Real World

50. **Foreign Currency** The British Isles, located northwest of the European mainland, consist of two large islands, Great Britain and Ireland, and many smaller islands. Carolyn Martinez took a trip to the British Isles during the summer of 1995 and needed to exchange American dollars for British pounds. After she arrived in Ireland, she needed to exchange her British pounds for Irish punts. Look at the functions below.

$B(x) = 0.9733x$ *Converting British pounds to Irish punts*

$A(x) = 0.6252x$ *Converting American dollars to British pounds*

a. Find the equation of the composition function $B[A(x)]$. Explain what this composition represents.
b. How many Irish punts will she get for $500?

51. **Discounts** Jeanette bought a new electric wok that was originally priced at \$38. The department store advertised a rebate of \$5 as well as a discount of 25% off all small appliances.
   a. Express the price of the wok after the rebate and the price after the discount using function notation. Let x represent the price of the wok, $r(x)$ represent the price after the rebate, and $p(x)$ represent the price after the discount.
   b. Find $r[p(x)]$ and explain what this value represents.
   c. Find $p[r(x)]$ and explain what this value represents.

52. **Finance** Nashota pays \$30 each month on a credit card that charges 1.4% interest monthly. She has a balance of \$450. The balance at the beginning of the nth month is given by the following function that is defined recursively.

   $f(1) = 450$

   $f(n) = f(n-1) + 0.014f(n-1) - 30$

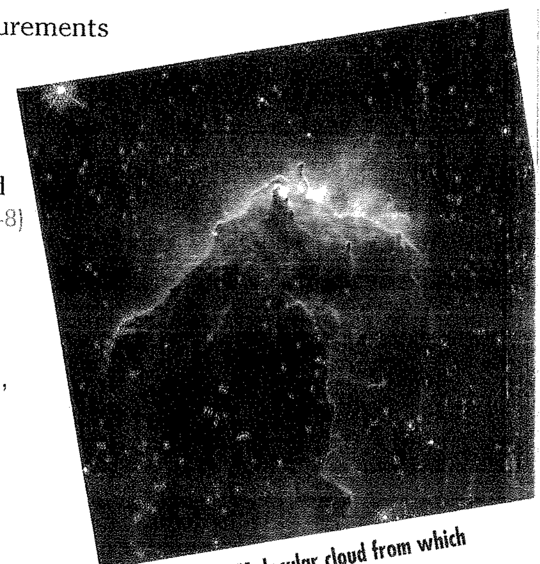
   Find the balances at the beginning of the first five months.

Mixed Review

53. Write $x^6 + 3x^3 - 10 = 0$ in quadratic form. (Lesson 8–6)

54. Write $y^2 = 6x$ in the form $x = a(y-k)^2 + h$. (Lesson 7–2)

55. **Statistics** An astronomer made ten measurements in minutes of degrees (') of the angular distance between two stars. The measurements were 11.20', 11.17', 10.92', 11.06', 11.19', 10.97', 11.09', 11.05', 11.22', and 11.03'. Find the mean and the standard deviation of the measurements. (Lesson 6–8)



Molecular cloud from which new stars emerge

56. Find the product $(m+7)^2$. (Lesson 5–2)

57. **SAT Practice** If $n + 5 > 0$ and $2 - 3n > -1$, then n could equal each of the following EXCEPT
   A -5 B -4 C -2
   D 0 E $\frac{1}{2}$

58. **Art** Cecilia would like to place a picture of her triangle-shaped painting in the center of a page in her portfolio. She would like the longest side on the bottom. The lengths of the sides of the picture are 4 inches, 3 inches, and 3 inches. Suppose the center of her page is represented by the origin. Find a system of inequalities that describes the points her picture would occupy on the page, so that the top and bottom of the picture are at an equal distance above and below the origin, and the left and right corners are at an equal distance from the origin. (Lesson 3–5)

59. What type of special function is $f(x) = x$? (Lesson 2–6)
   a. constant b. identity c. absolute value d. step

60. Find a value of b for which the graph of $y = bx - 7$ is perpendicular to the graph of $x - 2y = 18$. (Lesson 2–3)

61. Simplify $-4(3a + 2b) - 3(-7a - 6b)$. (Lesson 1–2)

For Extra Practice, see page 896.

MODELING MATHEMATICS

An Extension of Lesson 8–7

8–7B Exploring Iteration

Materials: grid paper

Each result of the iteration process is called an **iterate.** To iterate a function $f(x)$, begin with a starting value x_0, find $f(x_0)$, and call the result x_1. Then find $f(x_1)$, and call the result x_2. Find $f(x_2)$ and call the result x_3, and so on.

Activity 1 **Find the first three iterates, x_1, x_2, and x_3, of the function $f(x) = \frac{1}{2}x + 5$ for an initial value of $x_0 = 2$.**

Step 1 To obtain the first iterate, find the value of the function for $x_0 = 2$.

$f(x_0) = f(2)$

$= \frac{1}{2}(2) + 5$ or 6 So, $x_1 = 6$.

Step 2 To obtain the second iterate x_2, substitute the function value for the first iterate, x_1, for x.

$f(x_1) = f(6)$

$= \frac{1}{2}(6) + 5$ or 8 So, $x_2 = 8$.

Step 3 Now find the third iterate, x_3, by substituting x_2 for x.

$f(x_2) = f(8)$

$= \frac{1}{2}(8) + 5$ or 9 So, $x_3 = 9$.

Therefore, the first three iterates for the function $f(x) = \frac{1}{2}x + 5$ for an initial value of $x_0 = 2$ are 6, 8, 9.

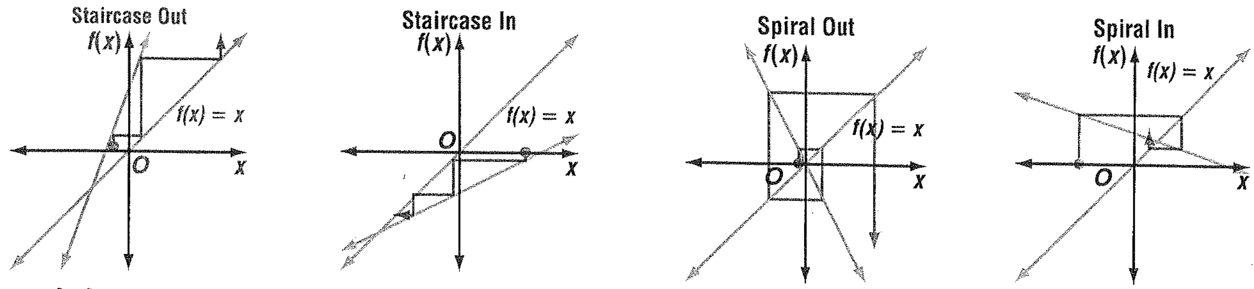
Graphing the iterations of a function can help us understand the process of iteration better. Follow these steps.

- Graph a function $g(x)$ and the line $f(x) = x$ on the coordinate plane.
- Choose an initial value, x_0, and locate the point $(x_0, 0)$.
- Draw a vertical line from $(x_0, 0)$ to the graph of $g(x)$. This will be the segment from the point $(x_0, 0)$ to $(x_0, g(x_0))$.
- Now draw a horizontal segment from this point to the graph of the line $f(x) = x$. This will be the segment from $(x_0, g(x_0))$ to $(g(x_0), g(x_0))$.
- Repeat the process for many iterations.

This process is called **graphical iteration.**

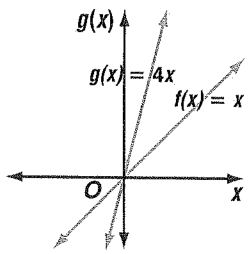
You can think of the line $f(x) = x$ as a mirror that reflects each function value to become the input for the next iteration of the function. The points at which the graph of the function $g(x)$ intersects the graph of the line $f(x) = x$ are called **fixed points.** If you try to iterate the initial value that corresponds to the x-coordinate of a fixed point, the iterates will all be the same.

Four basic paths are possible when a linear function is iterated.

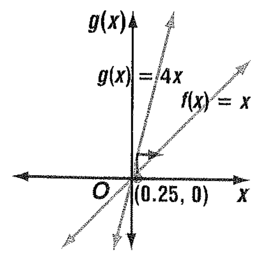


Activity 2 **Perform graphical iteration on the function $g(x) = 4x$ for the first three iterates if the initial value is $x_0 = 0.25$. Which of the four types of paths does the iteration take?**

Step 1 To do the graphical iteration, first graph the functions $f(x) = x$ and $g(x) = 4x$.

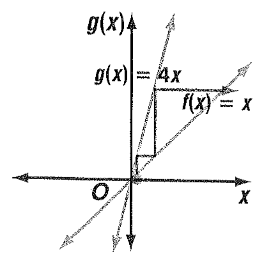


Step 2 Start at the point (0.25, 0) and draw a vertical line to the graph of $g(x) = 4x$. From that point, draw a horizontal line to the graph of $f(x) = x$.



Step 3 Repeat the process from the point on $f(x) = x$. Then repeat again.

The path of the iterations staircases out.



Model **Find the first three iterates of each function using the given initial value. If necessary, round your answers to the nearest hundredth.**

1. $g(x) = 5x;\ x_0 = 0.2$
2. $g(x) = -2x + 1;\ x_0 = -0.5$
3. $g(x) = 3 - 0.4x;\ x_0 = -4$
4. $g(x) = 3x - 0.5x^2;\ x_0 = 1$

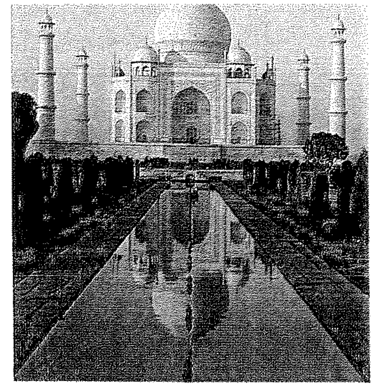
Draw **Graph each function and the function $f(x) = x$ on the same set of axes. Then draw the graphical iteration for $x_0 = 1$. State the slope of the linear function and tell what type of path the graphical iteration forms.**

5. $g(x) = 4x + 12$
6. $g(x) = \frac{3}{5}x + 2$
7. $g(x) = -2x - 3$
8. $g(x) = 5x - 7$
9. $g(x) = \frac{1}{4}x + 1$
10. $g(x) = -\frac{1}{3}x + 4$

Write

11. Write a paragraph explaining the relationship between the slope of a linear function and the type of path that the graphical iteration forms.
12. What type of path do you think is formed when you perform the graphical iteration on the function $f(x) = 5x - x^2$? How does it compare to the iteration of linear functions?

8-8 Inverse Functions and Relations



The Tāj Mahal, Āgra, India

What YOU'LL LEARN

- To determine the inverse of a function or relation,
- to graph functions and their inverses, and
- to work backward to solve problems.

Why IT'S IMPORTANT

You can use inverses to solve problems involving shopping and world cultures.

Real World APPLICATION

World Cultures

The ancient Hindus loved to do number puzzles. Aryabhata, a mathematician who lived in India during the sixth century A.D., was especially drawn to these puzzles. Look at the number puzzle below.

Choose a number between 1 and 10.
Multiply that number by 4.
Add 6 to the resulting number.
Divide by 2.
Then subtract 5.

Aryabhata could have correctly told you your original number. How? Suppose your original number was 8. The chart below shows the steps of the puzzle.

Verbal Instructions	Number	Functional Representation
Choose a number between 1 and 10.	8	$f(x) = x$
Multiply that number by 4.	8×4 or 32	$g(x) = 4[f(x)] = 4x$
Add 6.	$32 + 6$ or 38	$h(x) = g(x) + 6 = 4x + 6$
Divide by 2.	$38 \div 2$ or 19	$j(x) = h(x) \div 2 = \frac{4x+6}{2}$
Then subtract 5.	$19 - 5$ or 14	$k(x) = j(x) - 5 = \frac{4x+6}{2} - 5$

GLOBAL CONNECTIONS

Aryabhata I (476–550), a Hindu mathematician and astronomer, was the most important early scholar in Indian mathematics. In his work *Aryabhatiya,* he gave a value of 3.1416 for π, used the decimal and place-value system, and supplied a variety of rules for algebra.

To tell you the original number, Aryabhata would **work backward** and do the inverse, or opposite, of the steps as he went along. Subtraction is the inverse operation of addition, and division is the inverse operation of multiplication.

The chart below shows the inverse of the puzzle shown above.

Verbal Instructions	Number	Functional Representation
Tell me the number that you ended with.	14	$p(x) = x$
Add 5 to the number.	$14 + 5$ or 19	$r(x) = p(x) + 5 = x + 5$
Multiply by 2.	19×2 or 38	$t(x) = r(x) \times 2 = 2x + 10$
Subtract 6.	$38 - 6$ or 32	$v(x) = t(x) - 6 = 2x + 4$
Divide by 4.	$32 \div 4$ or 8	$w(x) = \frac{v(x)}{4} = \frac{1}{2}x + 1$

The functions $k(x) = \frac{4x+6}{2} - 5$ and $w(x) = \frac{1}{2}x + 1$ are **inverse functions.**

Definition of Inverse Functions

Two functions f and g are inverse functions if and only if both of their compositions are the identity function. That is,

$$[f \circ g](x) = x \text{ and } [g \circ f](x) = x.$$

You can determine if two functions are inverse functions by finding both of their compositions. If both compositions equal the identity function $h(x) = x$, then the functions are inverse functions.

You can also determine if two functions are inverse functions by graphing. The graphs of a function and its inverse are mirror images, or reflections, of each other with respect to the graph of the identity function $h(x) = x$. Its graph is the line of symmetry.

Example 1 **Determine whether $f(x) = 3x - 4$ and $g(x) = \frac{x+4}{3}$ are inverse functions.**

Method 1: Finding Compositions

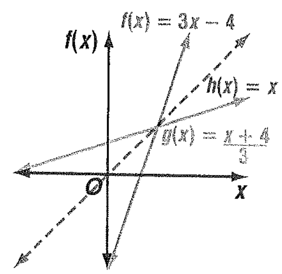
Find $[f \circ g](x)$ and $[g \circ f](x)$ to determine whether these functions are inverse functions.

$$\begin{aligned}[f \circ g](x) &= f[g(x)] \\ &= f\left(\frac{x+4}{3}\right) \\ &= 3\left(\frac{x+4}{3}\right) - 4 \\ &= x\end{aligned} \qquad \begin{aligned}[g \circ f](x) &= g[f(x)] \\ &= g(3x-4) \\ &= \frac{(3x-4)+4}{3} \\ &= x\end{aligned}$$

Since both $[f \circ g](x)$ and $[g \circ f](x)$ equal x, then $f(x)$ and $g(x)$ are inverse functions. That is, f is the inverse of g and g is the inverse of f.

Method 2: Graphing

Graph both functions.



Suppose the plane containing the graphs could be folded along the line $h(x) = x$. Then the graphs would coincide. This verifies that the functions are inverses.

We can write f is the inverse of g and g is the inverse of f using the notation $f = g^{-1}$ and $g = f^{-1}$. The symbol g^{-1} is read "g inverse" or "the inverse of g." By the definition of inverse functions, we can write the following.

The −1 is not an exponent.

$$[f \circ f^{-1}](x) = x \text{ and } [f^{-1} \circ f](x) = x$$

The ordered pairs of inverse functions are related. Use the functions in Example 1 and evaluate $f(5)$. Then find $f^{-1}[f(5)]$.

$f(5) = 3(5) - 4$ or 11

The ordered pair (5, 11) belongs to f.

$$\begin{aligned}f^{-1}[f(5)] &= f^{-1}(11) \\ &= \frac{(11)+4}{3} \text{ or } 5\end{aligned}$$

The ordered pair (11, 5) belongs to f^{-1}.

So, the inverse of a function can be found by exchanging the domain and range of the function.

Property of Inverse Functions	Suppose f and f^{-1} are inverse functions. Then $f(a) = b$ if and only if $f^{-1}(b) = a$.

To find the inverse of a function f, you can interchange the variables in the equation $y = f(x)$.

Example 2 **Find the inverse of $f(x) = 3x + 6$. Then verify that f and f^{-1} are inverse functions.**

Method 1: Algebra

$y = 3x + 6$ *Rewrite $f(x) = 3x + 6$ as $y = 3x + 6$.*

$x = 3y + 6$ *Interchange x and y.*

$y = \frac{x - 6}{3}$ *The inverse is also a function.*

The inverse of $f(x) = 3x + 6$ is $f^{-1}(x) = \frac{x - 6}{3}$.

Check: To verify that f and f^{-1} are inverses, show that the compositions of f and f^{-1} are identity functions.

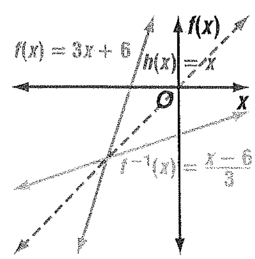
$$[f \circ f^{-1}](x) = f[f^{-1}(x)] = f\left(\frac{x-6}{3}\right) = 3\left(\frac{x-6}{3}\right) + 6 = x$$

$$[f^{-1} \circ f](x) = f^{-1}[f(x)] = f^{-1}(3x + 6) = \frac{(3x + 6) - 6}{3} = x$$

The functions are inverses, since both $[f \circ f^{-1}](x)$ and $[f^{-1} \circ f](x)$ equal x.

Method 2: Graphing

Now graph both functions.

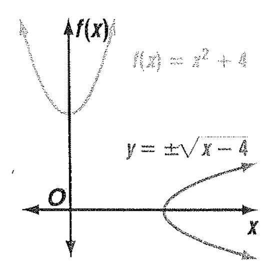


The graphs are reflections of each other over the line $h(x) = x$. This verifies that the functions are inverses.

In Example 2, f and f^{-1} were both functions. However, the inverse of a function is not always a function.

Example 3 **Find the inverse of $f(x) = x^2 + 4$. Determine whether the inverse is a function.**

Rewrite $f(x)$ as $y = x^2 + 4$.



$x = y^2 + 4$ *Interchange x and y.*

$\pm\sqrt{x - 4} = y$ *Solve for y.*

The inverse of $f(x) = x^2 + 4$ is $y = \pm\sqrt{x - 4}$. This inverse is not a function, since the graph does not pass the vertical line test for functions.

Check this result

To make the inverse of a quadratic function a function, only nonnegative values of the range are considered. Using only these values, the inverse is called a square root function. In Example 3, the square root function is $y = \sqrt{x - 4}$.

MODELING MATHEMATICS

Inverses of Functions

Materials: geomirror, grid paper, straightedge

Use a full sheet of grid paper. Draw axes in the center of the page, and label each mark on each axis as one unit.

Your Turn

a. Use a straightedge to graph $y = 2x - 8$ on the grid paper. Label the graph with its equation.

b. On the same set of axes, use a straightedge to graph $y = x$ as a dashed line.

c. Place the reflective mirror so that the drawing edge is on the line $y = x$ and carefully plot points that are part of the reflection of the original line with respect to the line of symmetry.

d. Draw a line through the points. This is the inverse of the original function.

e. What is the equation of the inverse?

f. Try this activity with the function $y = x^3 - 5$. Is the inverse also a function? Explain.

You may recall that a relation is a set of ordered pairs. The **inverse relation** is the set of ordered pairs obtained by reversing the coordinates of each original ordered pair.

Definition of Inverse Relations

Two relations are inverse relations if and only if whenever one relation contains the element (a, b), the other relation contains the element (b, a).

Example 4

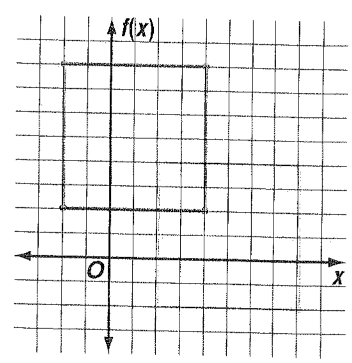
INTEGRATION Geometry

The vertices of square *ABCD* form the relation $\{(2, 4), (8, 4), (2, -2), (8, -2)\}$. Find the inverse of this relation and determine if the resulting ordered pairs are also the vertices of a square.

To find the inverse of this relation, reverse the coordinates of the ordered pairs.

The inverse of the relation is $\{(4, 2), (4, 8), (-2, 2), (-2, 8)\}$.

Plotting the points shows that the ordered pairs are also vertices of a square.



CHECK FOR UNDERSTANDING

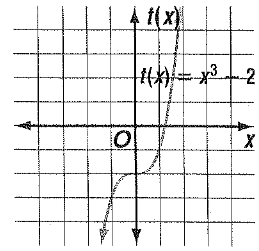
Communicating Mathematics

Study the lesson. Then complete the following.

1. **Explain** the difference between inverse functions and inverse relations. Are inverse functions also inverse relations? Explain.
2. **Describe** in your own words how to find the inverse of a function.
3. **Explain** why the inverse relation in Example 4 is not a function.
4. **Explain** how the graph of a function is related to the graph of its inverse.

5. **Sketch** the graph of the inverse of $t(x)$, shown at the right. Is the inverse a function? Explain.

6. Use a reflective mirror and grid paper to graph the inverse of the function $f(x) = \frac{1}{2}x^2 - 3$.

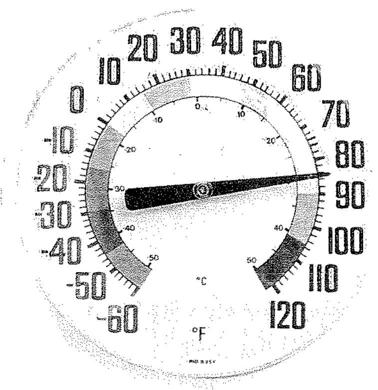


Guided Practice

Find the inverse of each relation and determine whether the inverse is a function.

7. $\{(3, 2), (4, 2)\}$

8. $\{(3, 8), (4, -2), (5, -3)\}$



Find the inverse of each function. Then graph the function and its inverse.

9. $y = 7x$

10. $y = x$

11. $f(x) = x - 6$

12. $y = -2x - 1$

13. Determine whether $f(x) = 6x + 2$ and $g(x) = \frac{x - 2}{6}$ are inverse functions.

14. **Temperature** Refer to the application at the beginning of Lesson 8-7. The formula for converting Fahrenheit to Celsius is $C(x) = \frac{5}{9}(x - 32)$. Find $C^{-1}(x)$, the inverse of $C(x)$, and explain what practical purpose the inverse serves.

EXERCISES

Practice

Find the inverse of each relation and determine whether the inverse is a function.

15. $\{(2, 4), (-3, 1), (2, 8)\}$

16. $\{(-1, -2), (-3, -2), (-1, -4), (0, 6)\}$

17. $\{(1, 3), (1, -1), (1, -3), (1, 1)\}$

18. $\{(6, 11), (-2, 7), (0, 3), (-5, 3)\}$

Find the inverse of each function. Then graph the function and its inverse.

19. $y = 6$

20. $y = 4x$

21. $f(x) = 4x + 4$

22. $g(x) = \frac{1}{2}x + 2$

23. $f(x) = -x$

24. $g(x) = x - 2$

25. $y = x^2 - 9$

26. $y = (x - 9)^2$

27. $f(x) = \frac{2x - 1}{3}$

28. $y = x^2 + 1$

29. $y = (x - 4)^2$

30. $y = (x + 2)^2 - 3$

Determine whether each pair of functions are inverse functions.

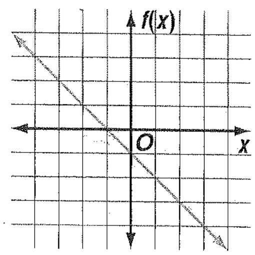
31. $f(x) = x + 7$
$g(x) = x - 7$

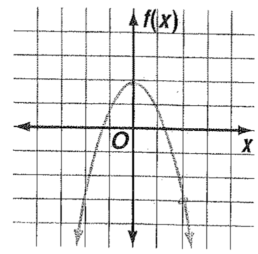
32. $g(x) = 2x - 3$
$h(x) = -2x + 3$

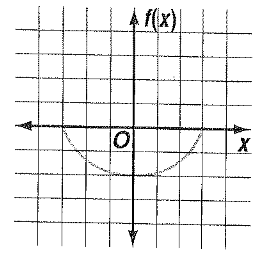
33. $f(x) = \frac{x - 2}{3}$
$g(x) = 3x - 2$

34. $f(x) = \frac{x - 1}{2}$
$g(x) = 2x + 1$

Sketch the graph of the inverse of each relation. Then determine if the inverse is a function.

35. 

36. 

37. 

Programming

38. The graphing calculator program at the right evaluates $f[g(x)]$ and $g[f(x)]$ if $f(x) = 2x - 1$ and $g(x) = x^2 + 1$. The calculator prompts you to enter a value for x and finds the values of $f[g(x)]$ and $g[f(x)]$. It also tells you if the functions are inverses. To use this program for other pairs of functions, change the second and third lines to reflect the expressions for $f[g(x)]$ and $g[f(x)]$.

```
Program: INVERSES
: Prompt X
: 2(X² + 1) − 1→A
: (2X − 1)² + 1→B
: Disp "F(G(X)) = ",A
: Disp "G(F(X)) = ",B
: If A = B
: Then
: Disp "INVERSES"
: Else
: Disp "NOT INVERSES"
```

Use this program and a chart to determine if each pair of functions are inverses of each other.

a. $f(x) = x + 1$
   $g(x) = x - 1$

b. $f(x) = \frac{1}{2}x^2 + 4$
   $g(x) = 2x + 8$

c. $g(x) = \frac{1}{5}(x + 7)$
   $h(x) = 5x - 7$

Critical Thinking

39. Find a function that is its own inverse. Can you find more than one?

Applications and Problem Solving

Real World

40. **Consumerism** LaKisha bought a stereo on sale from Electronics Unlimited. She used a $40 gift certificate to help pay for the stereo. If the stereo was on sale at 25% off and the final bill was $522.50 plus tax, what is its regular price?

41. **Work Backward** Jake asked Cynthia to choose a number between 1 and 20. He told her to add 7 to that number, then multiply by 4, then subtract 6, then divide by 2. Cynthia told Jake that her final number was 35. What was her original number?

42. **Sales** Sales associates at Electronics Unlimited earn $8 an hour plus a 4% commission on the merchandise they sell. Write a function to describe their weekly income and find how much merchandise they must sell in order to earn $500 in a 40-hour week.

Mixed Review

43. **Chemistry** While performing an experiment, Joy Chen found the temperature of a solution at different times. She needs to record the temperature in degrees Kelvin, but only has a thermometer with a Fahrenheit scale. Joy knows that a Kelvin temperature is 273 degrees greater than an equivalent Celsius temperature and that the formula $C = \frac{5}{9}(F - 32)$ converts a Fahrenheit temperature to Celsius. What will she record when the thermometer reads 59° F? (Lesson 8–7)

44. Write an equation for the parabola with focus at (2, 4) and directrix $y = 6$. Then draw the graph. (Lesson 7–2)

45. **ACT Practice** What is the quadratic equation that has roots of $2\frac{1}{2}$ and $\frac{2}{3}$?

A $5x^2 + 11x - 7 = 0$ B $5x^2 - 11x + 10 = 0$
C $6x^2 - 19x + 10 = 0$ D $6x^2 + 11x + 10 = 0$
E $6x^2 + 19x + 10 = 0$

46. Divide $(n^3 - n^2 + 4n + 6)$ by $(n + 1)$ using synthetic division. (Lesson 5–3)

47. Find $-2\begin{bmatrix} 0 & 3 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} 5 & 3 \\ -3 & 9 \end{bmatrix}$. (Lesson 4–2)

48. Graph $|x| + y \geq 3$. (Lesson 2–7)

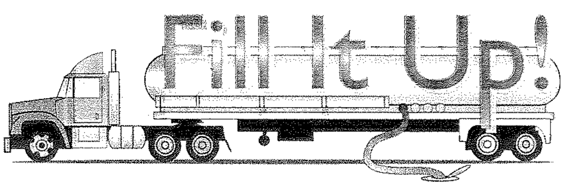
49. State the domain and range of the relation $\{(9, 0), (3, 1), (12, 7), (1, -4), (12, 8), (-11, -3), (0, -6)\}$. Is this relation a function? (Lesson 2–1)

For Extra Practice, see page 896.

50. **Manufacturing** A company manufactures auto parts. The diameter of a piston cannot vary more than 0.001 cm. Write an inequality to represent the diameter of a piston if its diameter is supposed to be 10 cm. (Lesson 1–7)

WORKING ON THE Investigation

Refer to the Investigation on pages 474–475.



Now that you have performed your own experiments on coffee cans of different sizes, you decide to get some data on the markings on a real gasoline tank from another colleague. The length of the tank is 20 feet and its radius is 10 feet. The table shows the distance in feet measured on the dipstick, the volume in cubic feet of gasoline in the tank, the percent of the tank that is full, and the percent of the dipstick that is wet.

meas. on stick (ft)	vol. of gas in tank (ft^3)	% of vol. in tank	% of stick covered	change in vol. %
0	0.0	0.0	0	
1	327.0	5.2	10	
2	894.6	14.2	20	
3	1585.3	25.2	30	
4	2347.0	37.4	40	
5	3141.6	50.0	50	
6	3936.2	62.6	60	
7	4697.8	74.8	70	
8	5388.6	85.8	80	
9	5956.2	94.8	90	
10	6283.2	100.0	100	

1 What are the differences and similarities in your charts and this chart? Are there any differences in the dipstick measurement increments and the volume measurement increments in your data and these data? Explain.
2 What is the volume of this tank in cubic feet?
3 Where on the dipstick would the markings be to show that the tank is one-half full, one-fourth full, and three-fourths full? Does this coincide with the locations at which you found the markings should be for your data?
4 Copy and complete the table.
5 Make a graph of the change in percentage of capacity for each increment. Describe the graph. How does this graph differ from those you made in your experiments? How is this graph similar to the graphs you made in your experiments?
6 Graph the inverse of this relation. Does the original graph represent a function? Does the graph of the inverse represent a function?

Add the results of your work to your Investigation Folder.

8–8B Square Root Functions and Relations

What YOU'LL LEARN

- To graph and analyze square root functions, and
- to graph square root inequalities.

Why IT'S IMPORTANT

You can use square root functions to solve problems involving weather and oceanography.

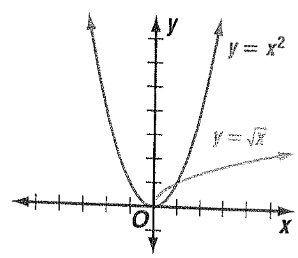
Real World APPLICATION

Weather

The function $d = \sqrt{\frac{3h}{2}}$ represents the greatest distance d in miles that a person h feet high can see on a clear day. Suppose Kenyatta is standing on the 102nd floor observation deck of the Empire State Building, 1250 feet high, on a clear day. What is the greatest distance that he can see? *You will solve this problem in Example 2.*

Because the function described above involves a square root, it is called a **square root function.** In order for a square root to be a real number, the radicand cannot be negative. When graphing a square root function, determine when the radicand would be negative and remember to exclude those values from the domain.

In Lesson 8-8, you learned that the inverse of a quadratic function is a square root function if only the nonnegative range is considered. Look at the graph at the right. Notice that for a square root function, negative values are excluded from the range.

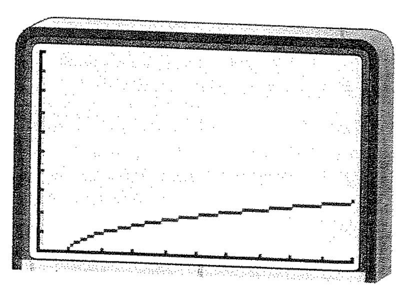


For many families of square root functions, the parent function is $y = \sqrt{x}$.

EXPLORATION GRAPHING CALCULATORS

To graph square root function like $y = \sqrt{x - 1}$, use the [$\sqrt{\ }$] key.

Enter: [Y=] [2nd] [$\sqrt{\ }$] [(] [X,T,θ,n] [−] 1 [)] [ZOOM] 6



Your Turn

a. Graph $y = \sqrt{x}$, $y = \sqrt{x} + 1$, and $y = \sqrt{x} - 2$ in the viewing window $[-2, 8]$ by $[-4, 6]$. State the domain and range of each function and describe the similarities and differences among the graphs.

b. Graph $y = \sqrt{x}$, $y = \sqrt{2x}$, and $y = \sqrt{8x}$ in the viewing window $[0, 10]$ by $[0, 10]$. State the domain and range of each function and describe the similarities and differences among the graphs.

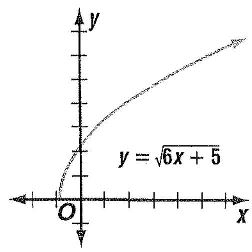
Example 1 **Graph $y = \sqrt{6x + 5}$. State the domain, range, and x- and y-intercepts.**

Since the radicand cannot be negative, identify the domain.

$$6x + 5 \geq 0$$
$$6x \geq -5$$
$$x \geq -\frac{5}{6}$$

Thus, the x-intercept is $-\frac{5}{6}$. Generate a table of values and graph the function.

x	y
$-\frac{5}{6}$	0
0	2.2
2	4.1
4	5.4
6	6.4
8	7.3
10	8.1



The domain is $x \geq -\frac{5}{6}$, and the range is $y \geq 0$. The y-intercept is $\sqrt{5}$ or about 2.24.

Example 2 Real World APPLICATION

Weather

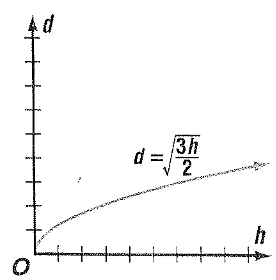
Refer to the application at the beginning of the lesson.

a. Graph the function $d = \sqrt{\frac{3h}{2}}$. State the domain and range.

b. Find the greatest distance that Kenyatta can see.

a. Make a table of values and graph the function.

h	d
0	0
2	$\sqrt{3}$ or 1.73
4	$\sqrt{6}$ or 2.45
6	$\sqrt{9}$ or 3.00
8	$\sqrt{12}$ or 3.46
10	$\sqrt{15}$ or 3.87



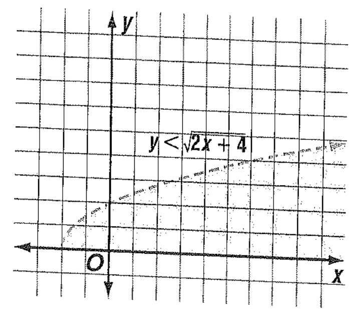
The domain and range of the function are both all nonnegative real numbers.

b. If the observation deck is 1250 feet high, then the greatest distance Kenyatta can see is $\sqrt{\frac{3(1250)}{2}}$ or about 43.3 miles. *Check this result against the graph.*

You can use what you know about square root functions to graph *square root inequalities*.

Example 3 **Graph $y < \sqrt{2x + 4}$.**

Generate a table of values and graph the relation. Since the boundary should not be included, the graph should be dashed.



Because the range includes only nonnegative real numbers, the graph is only in the first and second quadrants. Select a point on one of the half-planes and test its ordered pair. For example, use (1, 1). Since $1 < \sqrt{2(1) + 4}$, the half-plane containing (1, 1) should be shaded.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **Describe** the differences between the graphs of $y = \sqrt{x} - 6$ and $y = \sqrt{x - 6}$.
2. **Find** the minimum value in the domain for $y = \sqrt{2x + 3}$.
3. Refer to the application at the beginning of the lesson.
   a. Write a mathematical sentence that describes *all* of the distances Kenyatta would be able to see.
   b. Graph the relation that you came up with in part a.

Guided Practice

Graph each function. State the domain and range of each function.

4. $y = -\sqrt{x}$
5. $y = \sqrt{7x}$
6. $y = \sqrt{x - 1} + 5$
7. $y = \sqrt{5x + 1}$

Graph each inequality.

8. $y > \sqrt{x + 9} - 8$
9. $y \leq \sqrt{3x + 4} - 4$

10. **Oceanography** A *tsunami* is a large ocean wave generated by an undersea earthquake. The formula for a tsunami's speed s in meters per second is $s = 3.1\sqrt{d}$, where d is the depth of the ocean in meters.
    a. Describe the domain and range of the function.
    b. Determine the speed of a tsunami if the earthquake occurs in a part of the ocean that is 10,000 meters deep.
    c. Graph the function and use the graph to verify your answer to part b.

EXERCISES

Practice

Graph each function. State the domain and range of each function.

11. $y = \sqrt{3x}$
12. $y = -\sqrt{4x}$
13. $y = -2\sqrt{x}$
14. $y = 5\sqrt{x}$
15. $y = \sqrt{x + 2}$
16. $y = \sqrt{x - 7}$
17. $y = -\sqrt{2x + 1}$
18. $y = \sqrt{3x - 2}$
19. $y = \sqrt{x + 6} - 5$
20. $y = \sqrt{x - 5} + 3$
21. $y = \sqrt{7x - 1} + 2$
22. $y = 2\sqrt{3 - 4x} + 6$

Graph each inequality.

23. $y \leq -6\sqrt{x}$
24. $y < \sqrt{x + 8}$
25. $y > \sqrt{5x + 7}$
26. $y \geq \sqrt{2x - 7}$
27. $y \geq \sqrt{x - 3} + 4$
28. $y < \sqrt{6x - 2} + 3$

Critical Thinking

29. In Lesson 6-6A, you investigated the role that a, h, and k played in the graph of a quadratic function of the form $y = a(x - h)^2 + k$.

 a. Describe the roles a, h, and k play for the family of square root functions of the form $y = a\sqrt{x - h} + k$.

 b. Use what you found in part a to describe the graphs of the functions below without graphing them. Relate your answers to the graph of the parent function $y = \sqrt{x}$.

 $y = -2\sqrt{x}$ $y = \sqrt{x - 4}$ $y = \sqrt{x} + 3$ $y = 3\sqrt{x - 1} + 5$

Applications and Problem Solving

Real World

30. **Geometry** There are several sizes of ice cream cones available at Johnson's Real Ice Cream Shoppe, but all of them are 5 inches long.

 a. Write a square root function that expresses the radius r of the cones as a function of volume V. Use the formula $V = \frac{1}{3}\pi r^2 h$, where h is the height.

 b. Describe the domain and range of the function.

 c. Determine the volume of a cone that has a radius of 2 inches.

 d. Graph the function and use the graph to verify your answer to part c.

31. **Geometry** The volume V and the surface area A of a soap bubble are related by the formula $V = 0.094\sqrt{A^3}$.

 a. Describe the domain and range of the function.

 b. Determine the volume of a soap bubble that has a surface area of 12 cm^2.

 c. Graph the function and use the graph to verify your answer to part b.

 d. Does this graph look like the graphs of the other square root functions you have graphed in this lesson? Why or why not?

CHAPTER 8 HIGHLIGHTS

INTERNET CONNECTION

Chapter Review For additional lesson-by-lesson review, visit: **www.algebra2.glencoe.com**

VOCABULARY

After completing this chapter, you should be able to define each term, property, or phrase and give an example or two of each.

Algebra

complex conjugates theorem (p. 504)
composition of functions (p. 520)
depressed polynomial (p. 487)
Descartes' rule of signs (p. 505)
factor theorem (p. 487)
fundamental theorem of algebra (p. 503)
integral zero theorem (p. 509)
inverse function (p. 528)
inverse relation (p. 531)
leading coefficient (p. 480)
location principle (p. 494)
polynomial function (p. 479)
polynomial in one variable (p. 478)
quadratic form (p. 515)
rational zero theorem (p. 509)
relative maximum (p. 494)
relative minimum (p. 494)
remainder theorem (p. 485)
square root function (pp. 530, 535)
synthetic substitution (p. 486)

Discrete Mathematics

fixed points (p. 526)
graphical iteration (p. 526)
iterate (p. 526)
iteration (p. 522)

Problem Solving

work backward (p. 528)

UNDERSTANDING AND USING THE VOCABULARY

Choose the letter of the term that best matches each statement or phrase.

1. A point on the graph of a polynomial function that has no other nearby points with lesser y-coordinates is the __?__.
2. The __?__ is the factor in the term in a polynomial function with the highest degree.
3. The __?__ says that in any polynomial function, if an imaginary number is a zero of that function, then its conjugate is also a zero.
4. When a polynomial is divided by one of its binomial factors, the quotient is called a __?__.
5. A point on the graph of a polynomial function that has no other nearby points with a greater y-coordinate is the __?__.
6. $f \circ g(x) = f[g(x)]$ represents a __?__.
7. $(x^2)^2 - 17(x^2) + 16 = 0$ is written in __?__.
8. $f(x) = 6x - 2$ and $g(x) = \frac{x+2}{6}$ are __?__ since $[f \circ g](x) = x$ and $[g \circ f](x) = x$.

a. complex conjugates theorem
b. composition of functions
c. depressed polynomial
d. inverse functions
e. leading coefficient
f. quadratic form
g. relative maximum
h. relative minimum

SKILLS AND CONCEPTS

OBJECTIVES AND EXAMPLES

Upon completing this chapter, you should be able to:

- evaluate polynomial functions (Lesson 8–1)

Find $p(a + 1)$ if $p(x) = 5x - x^2 + 3x^3$.

$p(a + 1) = 5(a + 1) - (a + 1)^2 + 3(a + 1)^3$

$= 5a + 5 - (a^2 + 2a + 1) + 3(a + 1)(a^2 + 2a + 1)$

$= 5a + 5 - a^2 - 2a - 1 + 3(a^3 + 3a^2 + 3a + 1)$

$= 5a + 5 - a^2 - 2a - 1 + 3a^3 + 9a^2 + 9a + 3$

$= 3a^3 + 8a^2 + 12a + 7$

REVIEW EXERCISES

Use these exercises to review and prepare for the chapter test.

Find $p(-4)$ and $p(x + h)$ for each function.

9. $p(x) = x - 2$
10. $p(x) = -x + 4$
11. $p(x) = 6x + 3$
12. $p(x) = x^2 + 5$
13. $p(x) = x^2 - x$
14. $p(x) = 2x^3 - 1$

- find factors of polynomials by using the factor theorem and synthetic division (Lesson 8–2)

Show that $x + 2$ is a factor of $x^3 - 2x^2 - 5x + 6$. Then find any remaining factors.

$$\begin{array}{r|rrrr} -2 & 1 & -2 & -5 & 6 \\ & & -2 & 8 & -6 \\ \hline & 1 & -4 & 3 & 0 \end{array}$$

The remainder is 0, so $x + 2$ is a factor of $x^3 - 2x^2 - 5x + 6$, and $x^3 - 2x^2 - 5x + 6 = (x + 2)(x^2 - 4x + 3)$.

Since $x^2 - 4x + 3 = (x - 3)(x - 1)$, $x^3 - 2x^2 - 5x + 6 = (x + 2)(x - 3)(x - 1)$.

Use synthetic substitution to find $f(3)$ and $f(-2)$ for each function.

15. $f(x) = x^2 - 5$
16. $f(x) = x^2 - 4x + 4$
17. $f(x) = x^3 - 3x^2 + 4x + 8$
18. $f(x) = x^4 - 5x + 2$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

19. $x^3 + 5x^2 + 8x + 4$; $x + 1$
20. $x^3 + 4x^2 + 7x + 6$; $x + 2$
21. $x^3 - x^2 - 4x + 4$; $x + 2$
22. $x^4 - 6x^3 + 22x + 15$; $x + 1$

- approximate the real zeros of polynomial functions (Lesson 8–3)

The location principle states that if $y = f(x)$ represents a polynomial function, and a and b are two numbers such that $f(a) < 0$ and $f(b) > 0$, then the function has at least one zero between a and b.

Approximate the real zeros of each function to the nearest tenth. Then use the functional values to graph the function.

23. $h(x) = x^3 - 6x - 9$
24. $f(x) = x^4 + 7x + 1$
25. $p(x) = x^5 + x^4 - 2x^3 + 1$
26. $g(x) = x^3 - x^2 + 1$
27. $r(x) = 4x^3 + x^2 - 11x + 3$
28. $f(x) = x^3 + 4x^2 + x - 2$