Functions with the same defining rule can have different extrema, depe ing on the domain.

Example 2 EXPLORING ABSOLUTE EXTREMA

The absolute extrema of the following functions on their domains can be seen in Figure 4.2.

	Function Rule	Domain D	Absolute Extrema on D
(a)	$y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x =$
(b)	$y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x =$ Absolute minimum of 0 at $x =$
(c)	$y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x =$ No absolute minimum.
(d)	$y = x^2$	$(0, 2)$	No absolute extrema.

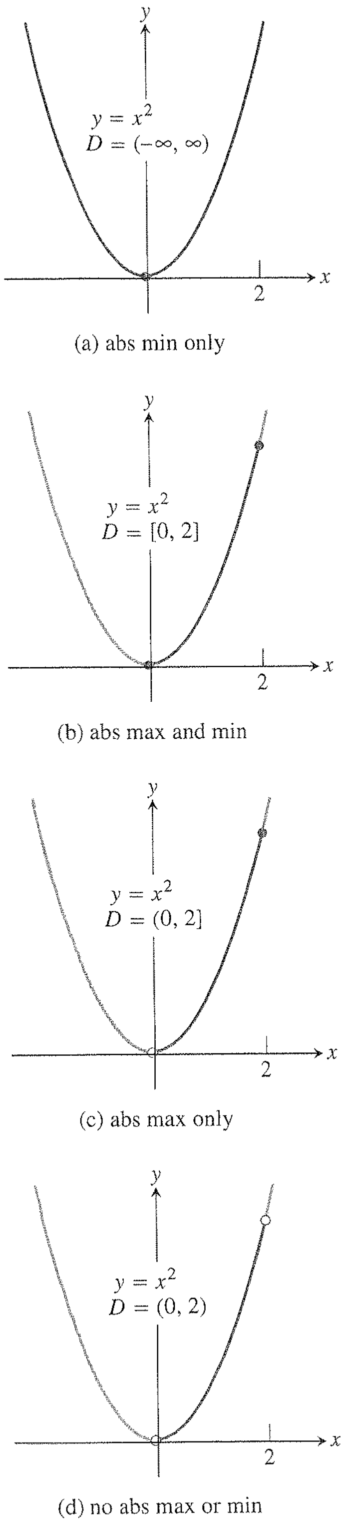


Figure 4.2 (Example 2)

Example 2 shows that a function may fail to have a maximum or m mum value. This cannot happen with a continuous function on a finite clo interval.

Theorem 1 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f has both a maximum value and a minimum value on the interval. (Figure 4.3)

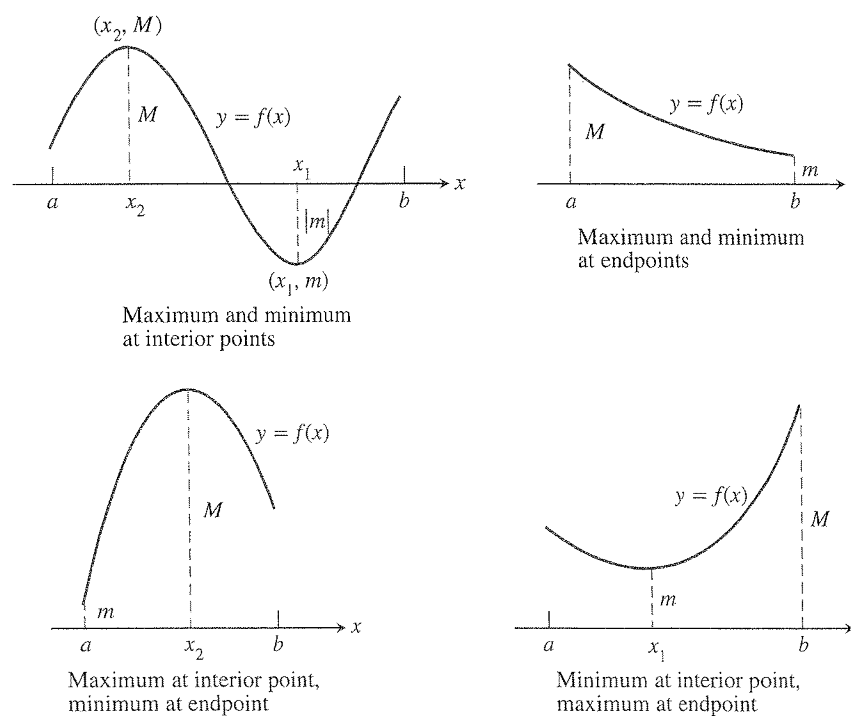


Figure 4.3 Some possibilities for a continuous function's maximum and minimu on a closed interval $[a, b]$.

Chapter 4 Overview

In the past, when virtually all graphing was done by hand—ofte ously—derivatives were the key tool used to sketch the graph of a Now we can graph a function quickly, and usually correctly, using a However, confirmation of much of what we see and conclude tru grapher view must still come from calculus.

This chapter shows how to draw conclusions from derivatives a extreme values of a function and about the general shape of a functior We will also see how a tangent line captures the shape of a curve near of tangency, how to deduce rates of change we cannot measure from change we already know, and how to find a function when we know first derivative and its value at a single point. The key to recovering f from derivatives is the Mean Value Theorem, a theorem whose corolla vide the gateway to *integral calculus*, which we begin in Chapter 5.

4.1 Extreme Values of Functions

Absolute (Global) Extreme Values • Local (Relative) Extreme Values • Finding Extreme Values

Absolute (Global) Extreme Values

One of the most useful things we can learn from a function's deri whether the function assumes any maximum or minimum values on interval and where these values are located if it does. Once we know find a function's extreme values, we will be able to answer such ques "What is the most effective size for a dose of medicine?" and "What is expensive way to pipe oil from an offshore well to a refinery down the We will see how to answer questions like these in Section 4.4.

Definition Absolute Extreme Values

Let f be a function with domain D. Then $f(c)$ is the

(a) absolute maximum value on D if and only if $f(x) \leq f(c)$ for all x in D.

(b) absolute minimum value on D if and only if $f(x) \geq f(c)$ for all x in D.

Absolute (or **global**) maximum and minimum values are also **absolute extrema** (plural of the Latin *extremum*). We often omit th "absolute" or "global" and just say maximum and minimum.

Example 1 shows that extreme values can occur at interior points points of intervals.

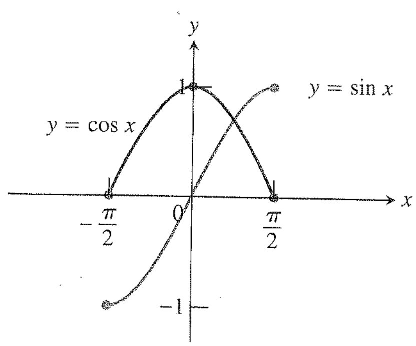


Figure 4.1 (Example 1)

Example 1 EXPLORING EXTREME VALUES

On $[-\pi/2, \pi/2]$, $f(x) = \cos x$ takes on a maximum value of 1 (onc a minimum value of 0 (twice). The function $g(x) = \sin x$ takes on a mum value of 1 and a minimum value of -1 (Figure 4.1).

Because of Theorem 2, we usually need to look at only a few points find a function's extrema. These consist of the interior domain points wh $f' = 0$ or f' does not exist (the domain points covered by the theorem) and domain endpoints (the domain points not covered by the theorem). At all ot domain points, $f' > 0$ or $f' < 0$.

The following definition helps us summarize these findings.

Definition Critical Point

A point in the interior of the domain of a function f at which $f' = 0$ or f' does not exist is a **critical point** of f.

Thus, in summary, extreme values occur only at critical points endpoints.

Example 3 FINDING ABSOLUTE EXTREMA

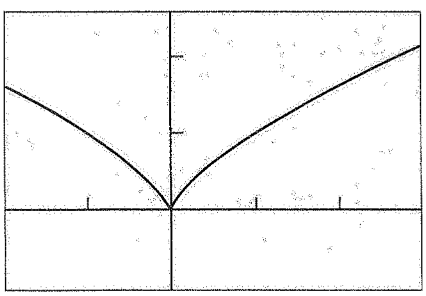
Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

Solution

Solve Graphically

Figure 4.5 suggests that f has an absolute maximum value of about 2 at $x = 3$ and an absolute minimum value of 0 at $x = 0$.

$y = x^{2/3}$



[–2, 3] by [–1, 2.5]

Figure 4.5 (Example 3)

Confirm Analytically

We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

has no zeros but is undefined at $x = 0$. The values of f at this one critical point and at the endpoints are

Critical point value: $f(0) = 0$;

Endpoint values: $f(-2) = (-2)^{2/3} = \sqrt[3]{4}$;

$f(3) = (3)^{2/3} = \sqrt[3]{9}$.

We can see from this list that the function's absolute maximum value is $\sqrt[3]{9} \approx 2.08$, and occurs at the right endpoint $x = 3$. The absolute minimum value is 0, and occurs at the interior point $x = 0$.

In Example 4, we investigate the function whose graph was drawn Example 3 of Section 1.2.

Example 4 FINDING EXTREME VALUES

Find the extreme values of $f(x) = \dfrac{1}{\sqrt{4 - x^2}}$.

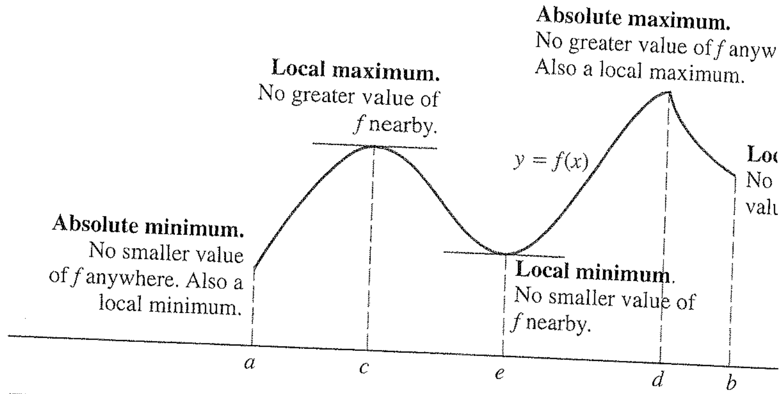


Figure 4.4 Classifying extreme values.

Local (Relative) Extreme Values

Figure 4.4 shows a graph with five points where a function has extre on its domain $[a, b]$. The function's absolute minimum occurs at a ev at e the function's value is smaller than at any other point *nearby.* ' rises to the left and falls to the right around c, making $f(c)$ a maximu The function attains its absolute maximum at d.

Definition Local Extreme Values

Let c be an interior point of the domain of the function f. Then $f($

(a) local maximum value at c if and only if $f(x) \leq f(c)$ for all some open interval containing c.

(b) local minimum value at c if and only if $f(x) \geq f(c)$ for all x some open interval containing c.

Local extrema are also called **relative extrema.** We can extend nitions of local extrema to endpoints of intervals. A function f has maximum or local minimum *at an endpoint* c if the appropriate in holds for all x in some half-open domain interval containing c.

An absolute extremum is also a local extremum, because being an value overall makes it an extreme value in its immediate neighborhood. *a list of local extrema will automatically include absolute extrema are any.*

Finding Extreme Values

The interior domain points where the function in Figure 4.4 has local e values are points where either f' is zero or f' does not exist. This is ge the case, as we see from the following theorem.

Theorem 2 Local Extreme Values

If a function f has a local maximum value or a local minimum valu at an interior point c of its domain, and if f' exists at c, then

$$f'(c) = 0.$$

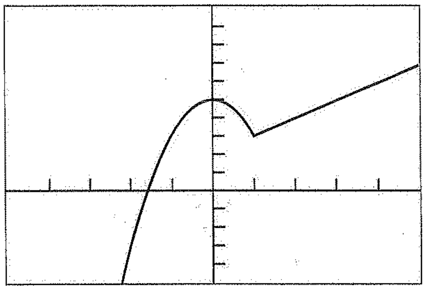
Example 5 FINDING EXTREME VALUES

Find the extreme values of

$$f(x) = \begin{cases} 5 - 2x^2, & x \le 1 \\ x + 2, & x > 1. \end{cases}$$

Solution

Solve Graphically



[–5, 5] by [–5, 10]

Figure 4.8 The function in Example 5.

The graph in Figure 4.8 suggests that $f'(0) = 0$ and that $f'(1)$ does no exist. There appears to be a local maximum value of 5 at $x = 0$ and a local minimum value of 3 at $x = 1$.

Confirm Analytically

For $x \neq 1$, the derivative is

$$f'(x) = \begin{cases} \dfrac{d}{dx}(5 - 2x^2) = -4x, & x < 1 \\ \dfrac{d}{dx}(x + 2) = 1, & x > 1. \end{cases}$$

The only point where $f' = 0$ is $x = 0$. What happens at $x = 1$?

At $x = 1$, the right- and left-hand derivatives are respectively

$$\lim_{h \to 0^+} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^+} \frac{(1 + h) + 2 - 3}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1,$$

$$\lim_{h \to 0^-} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^-} \frac{5 - 2(1 + h)^2 - 3}{h}$$

$$= \lim_{h \to 0^-} \frac{-2h(2 + h)}{h} = -4.$$

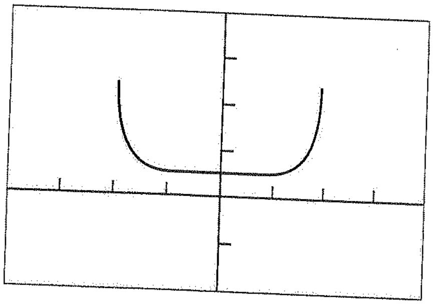
Since these one-sided derivatives differ, f has no derivative at $x = 1$, ε 1 is a second critical point of f.

The domain $(-\infty, \infty)$ has no endpoints, so the only values of f that mi; be local extrema are those at the critical points:

$$f(0) = 5 \quad \text{and} \quad f(1) = 3.$$

From the formula for f, we see that the values of f immediately to eithe side of $x = 0$ are less than 5, so 5 is a local maximum. Similarly, the values of f immediately to either side of $x = 1$ are greater than 3, so 3 a local minimum.

Most graphing calculators have built-in methods to find the coordinat points where extreme values occur. We must, of course, be sure that w correct graphs to find these values. The calculus that you learn in this ch should make you feel more confident about working with graphs.



[–4, 4] by [–2, 4]

Figure 4.6 The graph of

$$f(x) = \frac{1}{\sqrt{4 - x^2}}.$$

(Example 4)

Solution

Solve Graphically

Figure 4.6 suggests that f has an absolute minimum of about 0.5 at . There also appear to be local maxima at $x = -2$ and $x = 2$. Howeve defined at these points and there do not appear to be maxima anywh

Confirm Analytically

The function f is defined only for $4 - x^2 > 0$, so its domain is the interval $(-2, 2)$. The domain has no endpoints, so all the extreme v must occur at critical points. We rewrite the formula for f to find f'

$$f(x) = \frac{1}{\sqrt{4 - x^2}} = (4 - x^2)^{-1/2}.$$

Thus,

$$f'(x) = -\frac{1}{2}(4 - x^2)^{-3/2}(-2x) = \frac{x}{(4 - x^2)^{3/2}}.$$

The only critical point in the domain $(-2, 2)$ is $x = 0$. The value

$$f(0) = \frac{1}{\sqrt{4 - 0^2}} = \frac{1}{2}$$

is therefore the sole candidate for an extreme value.

To determine whether 1/2 is an extreme value of f, we examine the f

$$f(x) = \frac{1}{\sqrt{4 - x^2}}.$$

As x moves away from 0 on either side, the denominator gets smalle values of f increase, and the graph rises. We have a minimum value a $x = 0$, and the minimum is absolute.

The function has no maxima, either local or absolute. This does not v Theorem 1 (The Extreme Value Theorem) because here f is defined o *open* interval. To invoke Theorem 1's guarantee of extreme points, the interval must be closed.

While a function's extrema can occur only at critical points and end not every critical point or endpoint signals the presence of an extreme Figure 4.7 illustrates this for interior points. Exercise 53 describes a fu that fails to assume an extreme value at an endpoint of its domain.

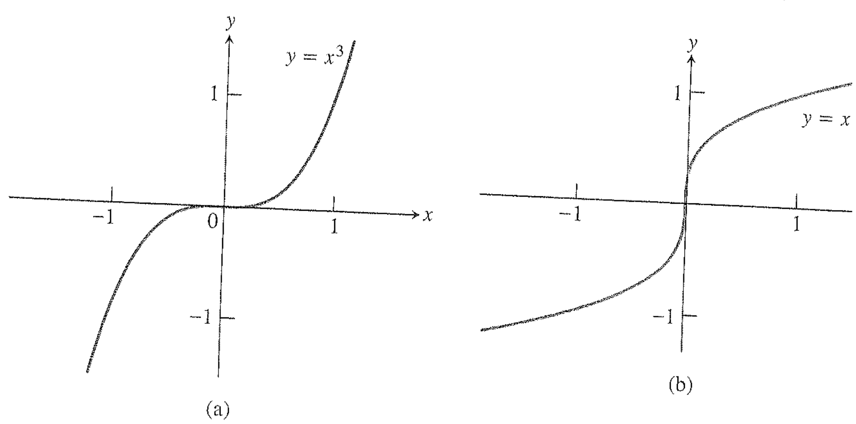
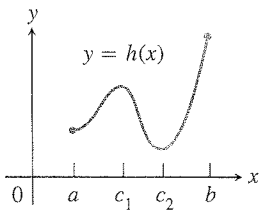


Figure 4.7 Critical points without extreme values. (a) $y' = 3x^2$ is 0 at $x = 0$, bu $y = x^3$ has no extremum there. (b) $y' = (1/3)x^{-2/3}$ is undefined at $x = 0$, but $y =$ has no extremum there.

Section 4.1 Exercises

In Exercises 1–6, identify each x-value at which any absolute extreme value occurs. Explain how your answer is consistent with the Extreme Value Theorem.

1.



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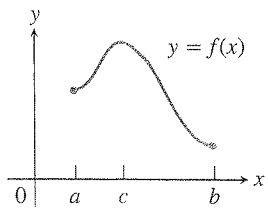

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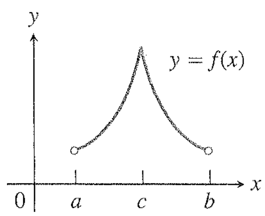

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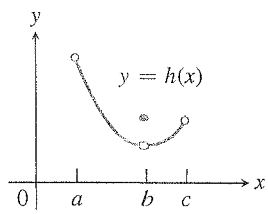

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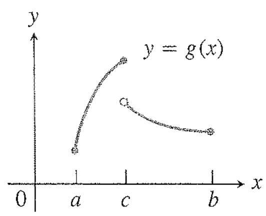

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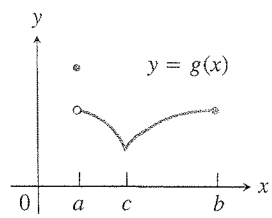
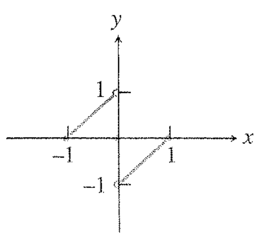


In Exercises 7–10, find the extreme values and where they occur.

7.



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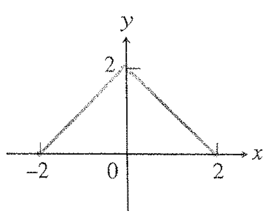

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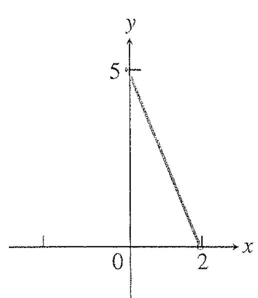

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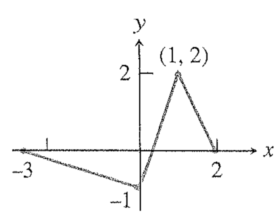


In Exercises 11–18, use analytic methods to find the extreme values of the function on the interval and where they occur.

11. $f(x) = \dfrac{1}{x} + \ln x, \quad 0.5 \leq x \leq 4$

12. $g(x) = e^{-x}, \quad -1 \leq x \leq 1$

13. $h(x) = \ln(x + 1), \quad 0 \leq x \leq 3$

14. $k(x) = e^{-x^2}, \quad -\infty < x < \infty$

15. $f(x) = \sin\left(x + \dfrac{\pi}{4}\right), \quad 0 \leq x \leq \dfrac{7\pi}{4}$

16. $g(x) = \sec x, \quad -\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$

17. $f(x) = x^{2/5}, \quad -3 \leq x < 1$

18. $f(x) = x^{3/5}, \quad -2 < x \leq 3$

In Exercises 19–30, find the extreme values of the function and where they occur.

19. $y = 2x^2 - 8x + 9$

20. $y = x^3 - 2x + 4$

21. $y = x^3 + x^2 - 8x + 5$

22. $y = x^3 - 3x^2 + 3x -$

23. $y = \sqrt{x^2 - 1}$

24. $y = \dfrac{1}{x^2 - 1}$

25. $y = \dfrac{1}{\sqrt{1 - x^2}}$

26. $y = \dfrac{1}{\sqrt[3]{1 - x^2}}$

27. $y = \sqrt{3 + 2x - x^2}$

28. $y = \dfrac{3}{2}x^4 + 4x^3 - 9x^2 + 10$

29. $y = \dfrac{x}{x^2 + 1}$

30. $y = \dfrac{x + 1}{x^2 + 2x + 2}$

In Exercises 31–34, *work in groups of two or three* to find tl extreme values of the function on the interval and where they occur.

31. $f(x) = |x - 2| + |x + 3|, \quad -5 \leq x \leq 5$

32. $g(x) = |x - 1| - |x - 5|, \quad -2 \leq x \leq 7$

33. $h(x) = |x + 2| - |x - 3|, \quad -\infty < x < \infty$

34. $k(x) = |x + 1| + |x - 3|, \quad -\infty < x < \infty$

Explorations

In Exercises 35 and 36, give reasons for your answers.

35. **Writing to Learn** Let $f(x) = (x - 2)^{2/3}$.

(a) Does $f'(2)$ exist?

(b) Show that the only local extreme value of f occurs a $x = 2$.

(c) Does the result in (b) contradict the Extreme Value Theorem?

(d) Repeat (a) and (b) for $f(x) = (x - a)^{2/3}$, replacing 2

36. **Writing to Learn** Let $f(x) = |x^3 - 9x|$.

(a) Does $f'(0)$ exist?

(b) Does $f'(3)$ exist?

(c) Does $f'(-3)$ exist?

(d) Determine all extrema of f.

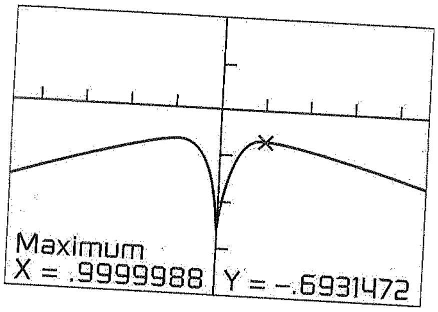
Example 6 USING GRAPHICAL METHODS

Find the extreme values of $f(x) = \ln \left| \frac{x}{1+x^2} \right|$.

Solution

Solve Graphically

The domain of f is the set of all nonzero real numbers. Figure 4.9 that f is an even function with a maximum value at two points. T nates found in this window suggest an extreme value of about −(approximately $x = 1$. Because f is even, there is another extrem same value at approximately $x = -1$. The figure also suggests a value at $x = 0$, but f is not defined there.



[–4.5, 4.5] by [–4, 2]

Figure 4.9 The function in Example 6.

Confirm Analytically

The derivative

$$f'(x) = \frac{1-x^2}{x(1+x^2)}$$

is defined at every point of the function's domain. The critical poin $f'(x) = 0$ are $x = 1$ and $x = -1$. The corresponding values of f $\ln(1/2) = -\ln 2 \approx -0.69$.

Exploration 1 **Finding Extreme Values**

Let $f(x) = \left| \frac{x}{x^2+1} \right|, \; -2 \le x \le 2$.

1. Determine graphically the extreme values of f and where they occur. Find f' at these values of x.
2. Graph f and f' (or NDER $(f(x), x, x)$) in the same viewing window. Comment on the relationship between the graphs.
3. Find a formula for $f'(x)$.

Quick Review 4.1

In Exercises 1–8, find the first derivative of the function.

1. $f(x) = \sqrt{4-x}$
2. $f(x) = x^{3/4}$
3. $f(x) = \frac{2}{\sqrt{9-x^2}}$
4. $f(x) = \frac{1}{\sqrt[3]{x^2-1}}$
5. $g(x) = \ln(x^2+1)$
6. $g(x) = \cos(\ln x)$
7. $h(x) = e^{2x}$
8. $h(x) = e^{\ln x}$

In Exercises 9 and 10, find the limit for

$$f(x) = \frac{2}{\sqrt{9-x^2}}.$$

9. $\lim_{x\to 3^-} f(x)$
10. $\lim_{x\to -3^+} f(x)$

In Exercises 11 and 12, let

$$f(x) = \begin{cases} x^3 - 2x, & x \le 2 \\ x + 2, & x > 2. \end{cases}$$

11. Find **(a)** $f'(1)$, **(b)** $f'(3)$, **(c)** $f'(2)$.
12. **(a)** Find the domain of f'.
    (b) Write a formula for $f'(x)$.

4.2 Mean Value Theorem

Mean Value Theorem • Physical Interpretation • Increasing an
Decreasing Functions • Other Consequences

Mean Value Theorem

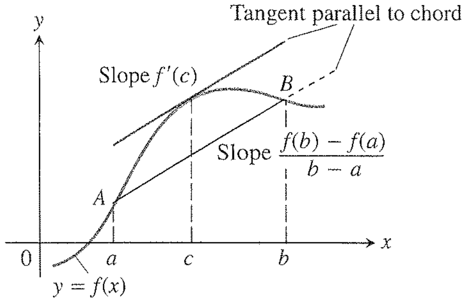


Figure 4.10 Figure for the Mean Value Theorem.

The Mean Value Theorem connects the average rate of change of a fun over an interval with the instantaneous rate of change of the function at a within the interval. Its powerful corollaries lie at the heart of some of the important applications of the calculus.

The theorem says that somewhere between points A and B on a diff tiable curve, there is at least one tangent line parallel to chord AB (Figure 4

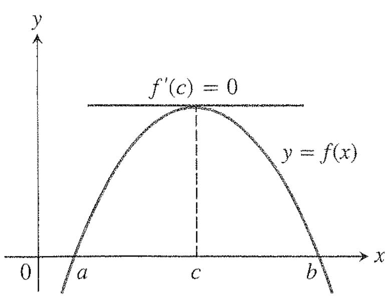
Theorem 3 Mean Value Theorem for Derivatives

If $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b), then there is at least one point c in (a, b) at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Rolle's Theorem

The first version of the Mean Value Theorem was proved by French mathematician Michel Rolle (1652–1719). His version had $f(a) = f(b) = 0$ and was proved only for polynomials, using algebra and geometry.



Rolle distrusted calculus and spent most of his life denouncing it. It is ironic that he is known today only for an unintended contribution to a field he tried to suppress.

The hypotheses of Theorem 3 cannot be relaxed. If they fail at ever point, the graph may fail to have a tangent parallel to the chord. For inst the function $f(x) = |x|$ is continuous on $[-1, 1]$ and differentiable at e point of the interior $(-1, 1)$ except $x = 0$. The graph has no tangent paral chord AB (Figure 4.11a). The function $g(x) = \text{int}(x)$ is differentiable at e point of $(1, 2)$ and continuous at every point of $[1, 2]$ except $x = 2$. Agai graph has no tangent parallel to chord AB (Figure 4.11b).

The Mean Value Theorem is an *existence theorem.* It tells us the num exists without telling how to find it. We can sometimes satisfy our curi about the value of c but the real importance of the theorem lies in the su ing conclusions we can draw from it.

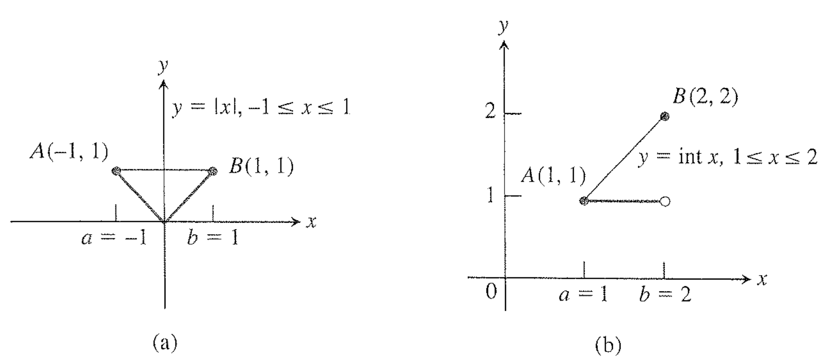


Figure 4.11 No tangent parallel to chord AB.

In Exercises 37–44, identify the critical point and determine the local extreme values.

37. $y = x^{2/3}(x + 2)$

38. $y = x^{2/3}(x^2 - 4)$

39. $y = x\sqrt{4 - x^2}$

40. $y = x^2\sqrt{3 - x}$

41. $y = \begin{cases} 4 - 2x, & x \le 1 \\ x + 1, & x > 1 \end{cases}$

42. $y = \begin{cases} 3 - x, & x < 0 \\ 3 + 2x - x^2, & x \ge 0 \end{cases}$

43. $y = \begin{cases} -x^2 - 2x + 4, & x \le 1 \\ -x^2 + 6x - 4, & x > 1 \end{cases}$

44. $y = \begin{cases} -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$

In Exercises 45–48, match the table with a graph of $f(x)$.

45.

x	$f'(x)$
a	0
b	0
c	5

46.

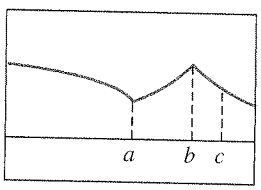
x	$f'(x)$
a	0
b	0
c	-5

47.

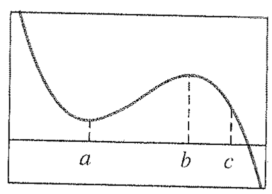
x	$f'(x)$
a	does not exist
b	0
c	-2

48.

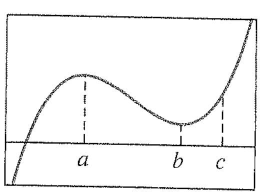
x	$f'(x)$
a	does not exist
b	does not exist
c	-1.7



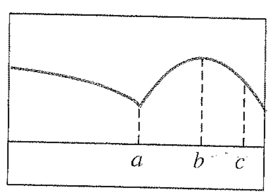
(a)



(b)



(c)



(d)

49. Writing to Learn The function

$$V(x) = x(10 - 2x)(16 - 2x), \quad 0 < x < 5,$$

models the volume of a box.

(a) Find the extreme values of V.

(b) Interpret any values found in (a) in terms of volume of the box.

50. Writing to Learn The function

$$P(x) = 2x + \frac{200}{x}, \quad 0 < x < \infty,$$

models the perimeter of a rectangle of dimensions x b $100/x$.

(a) Find any extreme values of P.

(b) Give an interpretation in terms of perimeter of the rectangle for any values found in (a).

Extending the Ideas

51. *Cubic Functions* Consider the cubic function

$$f(x) = ax^3 + bx^2 + cx + d.$$

(a) Show that f can have 0, 1, or 2 critical points. Giv examples and graphs to support your argument.

(b) How many local extreme values can f have?

52. *Proving Theorem 2* Assume that the function f has maximum value at the interior point c of its domain.

(a) Show that there is an open interval containing c su $f(x) - f(c) \le 0$ for all x in the open interval.

(b) Writing to Learn Now explain why we may say

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \le 0.$$

(c) Writing to Learn Now explain why we may say

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \ge 0.$$

(d) Writing to Learn Explain how (b) and (c) allow conclude $f'(c) = 0$.

(e) Writing to Learn Give a similar argument if f ha local minimum value at an interior point.

In Exercise 53, *work in groups of two or three.*

53. *Functions with No Extreme Values at Endpoints*

(a) Graph the function

$$f(x) = \begin{cases} \sin \frac{1}{x}, & x > 0 \\ 0, & x = 0. \end{cases}$$

Explain why $f(0) = 0$ is not a local extreme value of f

(b) Construct a function of your own that fails to have a extreme value at a domain endpoint.

Definitions Increasing Function, Decreasing Function

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. f **increases** on I if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
2. f **decreases** on I if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

Monotonic Functions

A function that is always increasing on an interval or always decreasing on an interval is said to be **monotonic** there.

The Mean Value Theorem allows us to identify exactly where graphs and fall. Functions with positive derivatives are increasing functions; funct with negative derivatives are decreasing functions.

Corollary 1 Increasing and Decreasing Functions

Let f be continuous on $[a, b]$ and differentiable on (a, b).

1. If $f' > 0$ at each point of (a, b), then f increases on $[a, b]$.
2. If $f' < 0$ at each point of (a, b), then f decreases on $[a, b]$.

Proof Let x_1 and x_2 be any two points in $[a, b]$ with $x_1 < x_2$. The Mean \ Theorem applied to f on $[x_1, x_2]$ gives

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some c between x_1 and x_2. The sign of the right-hand side of this equ is the same as the sign of $f'(c)$ because $x_2 - x_1$ is positive. Therefore,

(a) $f(x_1) < f(x_2)$ if $f' > 0$ on (a, b) (f is increasing), or

(b) $f(x_1) > f(x_2)$ if $f' < 0$ on (a, b) (f is decreasing).

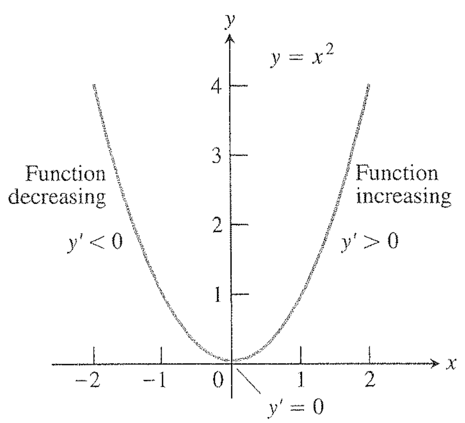


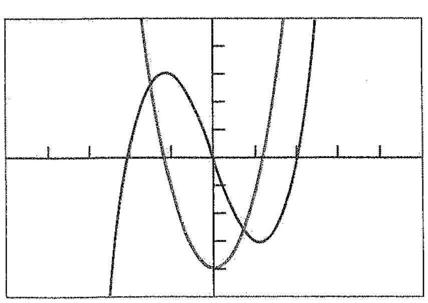
Figure 4.15 (Example 4)

Example 4 DETERMINING WHERE GRAPHS RISE OR FALL

The function $y = x^2$ (Figure 4.15) is

(a) decreasing on $(-\infty, 0]$ because $y' = 2x < 0$ on $(-\infty, 0)$.

(b) increasing on $[0, \infty)$ because $y' = 2x > 0$ on $(0, \infty)$.



[-5, 5] by [-5, 5]

Figure 4.16 By comparing the graphs of $f(x) = x^3 - 4x$ and $f'(x) = 3x^2 - 4$ we can relate the increasing and decreasing behavior of f to the sign of f'. (Example 5)

Example 5 DETERMINING WHERE GRAPHS RISE OR FALL

Where is the function $f(x) = x^3 - 4x$ increasing and where is it decreas

Solution

Solve Graphically

The graph of f in Figure 4.16 suggests that f is increasing from $-\infty$ to x-coordinate of the local maximum, decreasing between the two local extrema, and increasing again from the x-coordinate of the local minim to ∞. This information is supported by the superimposed graph of $f'(x) = 3x^2 - 4$.

Confirm Analytically

The function is increasing where $f'(x) > 0$.

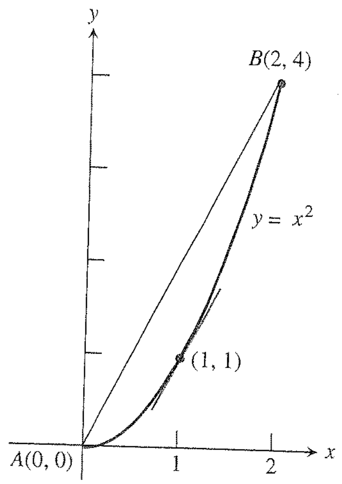


Figure 4.12 (Example 1)

Example 1 EXPLORING THE MEAN VALUE THEOREM

Show that the function $f(x) = x^2$ satisfies the hypotheses of the Mı Value Theorem on the interval $[0, 2]$. Then find a solution c to the eı

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

on this interval.

Solution The function $f(x) = x^2$ is continuous on $[0, 2]$ and differ tiable on $(0, 2)$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value The guarantees a point c in the interval $(0, 2)$ for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$2c = \frac{f(2) - f(0)}{2 - 0} = 2 \qquad f'(x) = 2x$$

$$c = 1.$$

Interpret

The tangent line to $f(x) = x^2$ at $x = 1$ has slope 2 and is parallel to chord joining $A(0, 0)$ and $B(2, 4)$ (Figure 4.12).

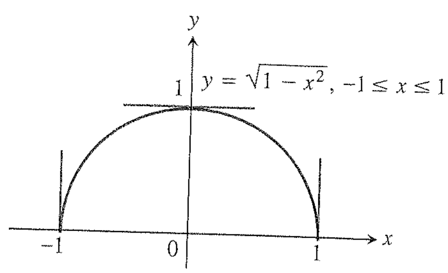


Figure 4.13 (Example 2)

Example 2 APPLYING THE MEAN VALUE THEOREM

Let $f(x) = \sqrt{1 - x^2}$, $A = (-1, f(-1))$, and $B = (1, f(1))$. Find a tangent to f in the interval $(-1, 1)$ that is parallel to the secant AB.

Solution The function f (Figure 4.13) is continuous on the interval [- and

$$f'(x) = \frac{-x}{\sqrt{1 - x^2}}$$

is defined on the interval $(-1, 1)$. The function is not differentiable at $x = -1$ and $x = 1$, but it does not need to be for the theorem to apply. Since $f(-1) = f(1) = 0$, the tangent we are looking for is horizontal. W find that $f' = 0$ at $x = 0$, where the graph has the horizontal tangent y

Physical Interpretation

If we think of the difference quotient $(f(b) - f(a))/(b - a)$ as the ave change in f over $[a, b]$ and $f'(c)$ as an instantaneous change, then the M Value Theorem says that the instantaneous change at some interior point ı equal the average change over the entire interval.

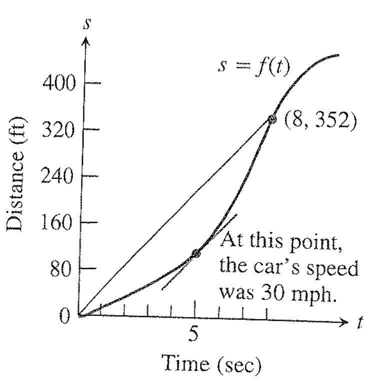


Figure 4.14 (Example 3)

Example 3 INTERPRETING THE MEAN VALUE THEOREM

If a car accelerating from zero takes 8 sec to go 352 ft, its average veloc for the 8-sec interval is $352/8 = 44$ ft/sec, or 30 mph. At some point du ing the acceleration, the theorem says, the speedometer must read exactl 30 mph (Figure 4.14).

Increasing and Decreasing Functions

Our first use of the Mean Value Theorem will be its application to increas and decreasing functions.

Proof Let $h = f - g$. Then for each point x in I,

$$h'(x) = f'(x) - g'(x) = 0.$$

It follows from Corollary 2 that there is a constant C such that $h(x) = ($ all x in I. Thus, $h(x) = f(x) - g(x) = C$, or $f(x) = g(x) + C$.

We know that the derivative of $f(x) = x^2$ is $2x$ on the interval $(-\infty$ So, any other function $g(x)$ with derivative $2x$ on $(-\infty, \infty)$ must have th mula $g(x) = x^2 + C$ for some constant C.

Example 6 APPLYING COROLLARY 3

Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passe through the point (0, 2).

Solution Since f has the same derivative as $g(x) = -\cos x$, we know $f(x) = -\cos x + C$ for some constant C. To identify C, we use the coı tion that the graph must pass through (0, 2). This is equivalent to saying

$$\begin{aligned} f(0) &= 2 \\ -\cos(0) + C &= 2 \qquad f(x) = -\cos x + C \\ -1 + C &= 2 \\ C &= 3. \end{aligned}$$

The formula for f is $f(x) = -\cos x + 3$.

In Example 6 we were given a derivative and asked to find a functior that derivative. This type of function is so important that it has a name.

Definition Antiderivative

A function $F(x)$ is an **antiderivative** of a function $f(x)$ if $F'(x) = f($ for all x in the domain of f. The process of finding an antiderivative is **antidifferentiation.**

We know that if f has one antiderivative F then it has infinitely many derivatives, each differing from F by a constant. Corollary 3 says these a there are. In Example 6, we found the particular antiderivative of $\sin x$ v graph passed through the point (0, 2).

Example 7 FINDING VELOCITY AND POSITION

Find the velocity and position functions of a freely falling body for eac the following sets of conditions:

(a) The acceleration is 9.8 m/sec^2 and the body falls from rest.

(b) The acceleration is 9.8 m/sec^2 and the body is propelled downwar with an initial velocity of 1 m/sec.

$$3x^2 - 4 > 0$$
$$x^2 > \frac{4}{3}$$
$$x < -\sqrt{\frac{4}{3}} \quad \text{or} \quad x > \sqrt{\frac{4}{3}}$$

The function is decreasing where $f'(x) < 0$.

$$3x^2 - 4 < 0$$
$$x^2 < \frac{4}{3}$$
$$-\sqrt{\frac{4}{3}} < x < \sqrt{\frac{4}{3}}$$

Interpret

To describe the intervals we use the approximation $\sqrt{4/3} \approx 1.15$. The function increases on $(-\infty, -1.15]$ and $[1.15, \infty)$. It decreases $[-1.15, 1.15]$.

In Example 5, the exact interval on which the function f d is $[-\sqrt{4/3}, \sqrt{4/3}]$. We reported it as $[-1.15, 1.15]$. As we have be all along, we continue to display results to a number of decimal pl seems appropriate to the problem.

Other Consequences

We know that constant functions have the zero function as their deriva can now use the Mean Value Theorem to show conversely that the on tions with the zero function as derivative are constant functions.

Corollary 2 Functions with $f' = 0$ are Constant

If $f'(x) = 0$ at each point of an interval I, then there is a constant for which $f(x) = C$ for all x in I.

Proof Our plan is to show that $f(x_1) = f(x_2)$ for any two points x_1 in I. We can assume the points are numbered so that $x_1 < x_2$. Since f is entiable at every point of $[x_1, x_2]$, it is continuous at every point as well. f satisfies the hypotheses of the Mean Value Theorem on $[x_1, x_2]$. The there is a point c between x_1 and x_2 for which

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Because $f'(c) = 0$, it follows that $f(x_1) = f(x_2)$.

We can use Corollary 2 to show that if two functions have the same ative, they differ by a constant.

Corollary 3 Functions with the Same Derivative Differ by a Constant

If $f'(x) = g'(x)$ at each point of an interval I, then there is a constan C such that $f(x) = g(x) + C$ for all x in I.

Section 4.2 Exercises

In Exercises 1–8, use analytic methods to find **(a)** the local extrema, **(b)** the intervals on which the function is increasing, and **(c)** the intervals on which the function is decreasing.

1. $f(x) = 5x - x^2$

2. $g(x) = x^2 - x - 12$

3. $h(x) = \dfrac{2}{x}$

4. $k(x) = \dfrac{1}{x^2}$

5. $f(x) = e^{2x}$

6. $f(x) = e^{-0.5x}$

7. $y = 4 - \sqrt{x + 2}$

8. $y = x^4 - 10x^2 + 9$

In Exercises 9–14, find **(a)** the local extrema, **(b)** the intervals on which the function is increasing, and **(c)** the intervals on which the function is decreasing.

9. $f(x) = x\sqrt{4 - x}$

10. $g(x) = x^{1/3}(x + 8)$

11. $h(x) = \dfrac{-x}{x^2 + 4}$

12. $k(x) = \dfrac{x}{x^2 - 4}$

13. $f(x) = x^3 - 2x - 2\cos x$

14. $g(x) = 2x + \cos x$

In Exercises 15–18, **(a)** show that the function f satisfies the hypotheses of the Mean Value Theorem on the given interval $[a, b]$. **(b)** Find each value of c in (a, b) that satisfies the equation

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

15. $f(x) = x^2 + 2x - 1, \quad [0, 1]$

16. $f(x) = x^{2/3}, \quad [0, 1]$

17. $f(x) = \sin^{-1} x, \quad [-1, 1]$

18. $f(x) = \ln(x - 1), \quad [2, 4]$

In Exercises 19 and 20, the interval $a \le x \le b$ is given. Let $A = (a, f(a))$ and $B = (b, f(b))$. Write an equation for

(a) the secant line AB.

(b) a tangent line to f in the interval (a, b) that is parallel to AB.

19. $f(x) = x + \dfrac{1}{x}, \quad 0.5 \le x \le 2$

20. $f(x) = \sqrt{x - 1}, \quad 1 \le x \le 3$

In Exercises 21–24, **(a)** show that the function f does not satisfy the hypotheses of the Mean Value Theorem on the given interval $[a, b]$. **(b)** Graph f together with the line through the points $A(a, f(a))$ and $B(b, f(b))$. **(c)** Find any values of c in (a, b) that satisfy the equation

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

21. $f(x) = x^{1/3}, \quad [-1, 1]$

22. $f(x) = |x - 1|, \quad [0, 3]$

23. $f(x) = 1 - |x|, \quad [-1, 1]$

24. $f(x) = \begin{cases} \cos x, & -\pi \le x < 0, \\ 1 + \sin x, & 0 \le x \le \pi, \end{cases} \quad$ on $[-\pi, \pi]$

In Exercises 25–30, find all possible functions f with the gi derivative.

25. $f'(x) = x$

26. $f'(x) = 2$

27. $f'(x) = 3x^2 - 2x + 1$

28. $f'(x) = \sin x$

29. $f'(x) = e^x$

30. $f'(x) = \dfrac{1}{x - 1}, \quad x$:

In Exercises 31–34, find the function with the given derivat whose graph passes through the point P.

31. $f'(x) = -\dfrac{1}{x^2}, \quad x > 0, \quad P(2, 1)$

32. $f'(x) = \dfrac{1}{4x^{3/4}}, \quad P(1, -2)$

33. $f'(x) = \dfrac{1}{x + 2}, \quad x > -2, \quad P(-1, 3)$

34. $f'(x) = 2x + 1 - \cos x, \quad P(0, 3)$

In Exercises 35–38, *work in groups of two or three* and ske a graph of a differentiable function $y = f(x)$ that has the gi properties.

35. **(a)** local minimum at $(1, 1)$, local maximum at $(3, 3)$

(b) local minima at $(1, 1)$ and $(3, 3)$

(c) local maxima at $(1, 1)$ and $(3, 3)$

36. $f(2) = 3, \quad f'(2) = 0,$ and

(a) $f'(x) > 0$ for $x < 2$, $f'(x) < 0$ for $x > 2$.

(b) $f'(x) < 0$ for $x < 2$, $f'(x) > 0$ for $x > 2$.

(c) $f'(x) < 0$ for $x \ne 2$.

(d) $f'(x) > 0$ for $x \ne 2$.

37. $f'(-1) = f'(1) = 0, \quad f'(x) > 0$ on $(-1, 1)$, $f'(x) < 0$ for $x < -1$, $f'(x) > 0$ for $x > 1$.

38. A local minimum value that is greater than one of its lc maximum values

39. *Speeding* A trucker handed in a ticket at a toll booth showing that in 2 h she had covered 159 mi on a toll road with speed limit 65 mph. The trucker was cited fo speeding. Why?

40. *Temperature Change* It took 20 sec for the temperat rise from 0°F to 212°F when a thermometer was taken a freezer and placed in boiling water. Explain why at sc moment in that interval the mercury was rising at exact 10.1°F/sec.

41. *Triremes* Classical accounts tell us that a 170-oar trire (ancient Greek or Roman warship) once covered 184 se miles in 24 h. Explain why at some point during this fe trireme's speed exceeded 7.5 knots (sea miles per hour

42. *Running a Marathon* A marathoner ran the 26.2-mi York City Marathon in 2.2 h. Show that at least twice, marathoner was running at exactly 11 mph.

Solution

(a) *Falling from rest.* We measure distance fallen in meters and tir seconds, and assume that the body is released from rest at time $t =$

Velocity: We know that the velocity $v(t)$ is an antiderivative of the c function 9.8. We also know that $g(t) = 9.8t$ is an antiderivative of Corollary 3,

$$v(t) = 9.8t + C$$

for some constant C. Since the body falls from rest, $v(0) = 0$. Thus

$$9.8(0) + C = 0 \quad \text{and} \quad C = 0.$$

The body's velocity function is $v(t) = 9.8t$.

Position: We know that the position $s(t)$ is an antiderivative of $9.8t$. W know that $h(t) = 4.9t^2$ is an antiderivative of $9.8t$. By Corollary 3,

$$s(t) = 4.9t^2 + C$$

for some constant C. Since $s(0) = 0$,

$$4.9(0)^2 + C = 0 \quad \text{and} \quad C = 0.$$

The body's position function is $s(t) = 4.9t^2$.

(b) *Propelled downward.* We measure distance fallen in meters and tir seconds, and assume that the body is propelled downward with velocit 1 m/sec at time $t = 0$.

Velocity: The velocity function still has the form $9.8t + C$, but insteai being zero, the initial velocity (velocity at $t = 0$) is now 1 m/sec. Thus

$$9.8(0) + C = 1 \quad \text{and} \quad C = 1.$$

The body's velocity function is $v(t) = 9.8t + 1$.

Position: We know that the position $s(t)$ is an antiderivative of $9.8t +$ We also know that $k(t) = 4.9t^2 + t$ is an antiderivative of $9.8t + 1$. By Corollary 3,

$$s(t) = 4.9t^2 + t + C$$

for some constant C. Since $s(0) = 0$,

$$4.9(0)^2 + 0 + C = 0 \quad \text{and} \quad C = 0.$$

The body's position function is $s(t) = 4.9t^2 + t$.

Quick Review 4.2

In Exercises 1 and 2, find exact solutions to the inequality.

1. $2x^2 - 6 < 0$

2. $3x^2 - 6 > 0$

In Exercises 3–5, let $f(x) = \sqrt{8 - 2x^2}$.

3. Find the domain of f.

4. Where is f continuous?

5. Where is f differentiable?

In Exercises 6–8, let $f(x) = \dfrac{x}{x^2 - 1}$.

6. Find the domain of f.

7. Where is f continuous?

8. Where is f differentiable?

In Exercises 9 and 10, find C so that the graph of the function passes through the specified point.

9. $f(x) = -2x + C, \quad (-2, 7)$

10. $g(x) = x^2 + 2x + C, \quad (1, -1)$

Extending the Ideas

53. *Geometric Mean* The **geometric mean** of two positive numbers a and b is $\sqrt{ab}$. Show that for $f(x) = 1/x$ on any interval $[a, b]$ of positive numbers, the value of c in the conclusion of the Mean Value Theorem is $c = \sqrt{ab}$.

54. *Arithmetic Mean* The **arithmetic mean** of two numbers a and b is $(a + b)/2$. Show that for $f(x) = x^2$ on any interval $[a, b]$, the value of c in the conclusion of the Mean Value Theorem is $c = (a + b)/2$.

55. *Upper Bounds* Show that for any numbers a and b, $|\sin b - \sin a| \leq |b - a|$.

56. *Sign of f′* Assume that f is differentiable on $a \leq x \leq b$ and that $f(b) < f(a)$. Show that f' is negative at some point between a and b.

57. *Monotonic Functions* Show that monotonic increasing and decreasing functions are one-to-one.

4.3 Connecting f' and f'' with the Graph of f

First Derivative Test for Local Extrema • Concavity • Points of Inflection • Second Derivative Test for Local Extrema • Learning about Functions from Derivatives

First Derivative Test for Local Extrema

As we see once again in Figure 4.17, a function f may have local extrema at some critical points while failing to have local extrema at others. The key is the sign of f' in a critical point's immediate vicinity. As x moves from left to right, the values of f increase where $f' > 0$ and decrease where $f' < 0$.

At the points where f has a minimum value, we see that $f' < 0$ on the interval immediately to the left and $f' > 0$ on the interval immediately to the right. (If the point is an endpoint, there is only the interval on the appropriate side to consider.) This means that the curve is falling (values decreasing) on the left of the minimum value and rising (values increasing) on its right. Similarly, at the points where f has a maximum value, $f' > 0$ on the interval immediately to the left and $f' < 0$ on the interval immediately to the right. This means that the curve is rising (values increasing) on the left of the maximum value and falling (values decreasing) on its right.

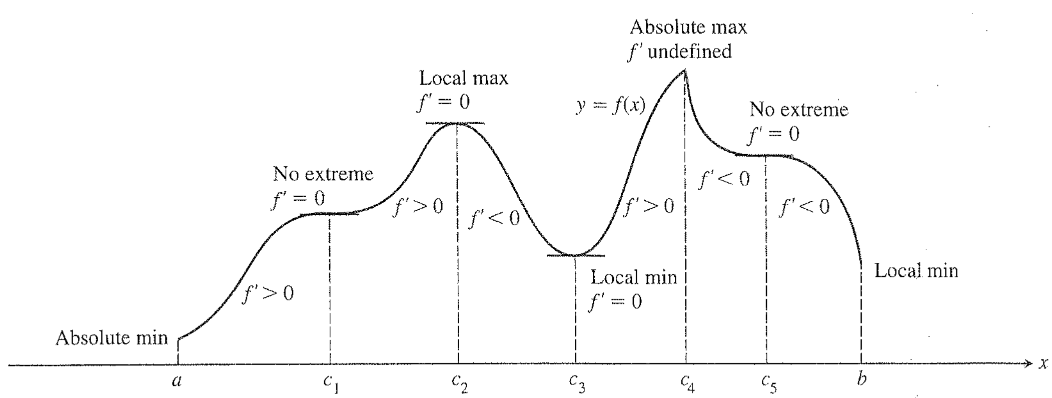


Figure 4.17 A function's first derivative tells how the graph rises and falls.

43. *Free Fall* On the moon, the acceleration due to gravity is $1.6\ \text{m/sec}^2$.

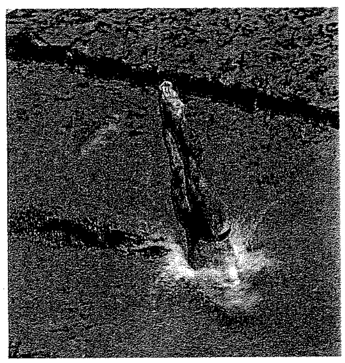
(a) If a rock is dropped into a crevasse, how fast will it be going just before it hits bottom 30 sec later?

(b) How far below the point of release is the bottom of the crevasse?

(c) If instead of being released from rest, the rock is thrown into the crevasse from the same point with a downward velocity of 4 m/sec, when will it hit the bottom and how fast will it be going when it does?

44. *Diving* **(a)** With what velocity will you hit the water if you step off from a 10-m diving platform?

(b) With what velocity will you hit the water if you dive off the platform with an upward velocity of 2 m/sec?



45. Writing to Learn The function

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

is zero at $x = 0$ and at $x = 1$. Its derivative is equal to 1 at every point between 0 and 1, so f' is never zero between 0 and 1, and the graph of f has no tangent parallel to the chord from (0, 0) to (1, 0). Explain why this does not contradict the Mean Value Theorem.

46. Writing to Learn Explain why there is a zero of $y = \cos x$ between every two zeros of $y = \sin x$.

47. *Unique Solution* Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Also assume that $f(a)$ and $f(b)$ have opposite signs and $f' \neq 0$ between a and b. Show that $f(x) = 0$ exactly once between a and b.

In Exercises 48 and 49, show that the equation has exactly one solution in the interval. (*Hint:* See Exercise 47.)

48. $x^4 + 3x + 1 = 0, \quad -2 \le x \le -1$

49. $x + \ln (x + 1) = 0, \quad 0 \le x \le 3$

50. *Parallel Tangents* Assume that f and g are differentiable on $[a, b]$ and that $f(a) = g(a)$ and $f(b) = g(b)$. Show that there is at least one point between a and b where the tangents to the graphs of f and g are parallel or the same line. Illustrate with a sketch.

Explorations

51. *Analyzing Derivative Data* Assume that f is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$. The table gives some values of $f'(x)$.

x	$f'(x)$	x	$f'(x)$
−2	7	0.25	−4.81
−1.75	4.19	0.5	−4.25
−1.5	1.75	0.75	−3.31
−1.25	−0.31	1	−2
−1	−2	1.25	−0.31
−0.75	−3.31	1.5	1.75
−0.5	−4.25	1.75	4.19
−0.25	−4.81	2	7
0	−5		

(a) Estimate where f is increasing, decreasing, and has local extrema.

(b) Find a quadratic regression equation for the data in the table and superimpose its graph on a scatter plot of the data.

(c) Use the model in (b) for f' and find a formula for f that satisfies $f(0) = 0$.

52. *Analyzing Motion Data* Priya's distance D in meters from a motion detector is given by the data in Table 4.1.

Table 4.1 Motion Detector Data

t (sec)	D (m)	t (sec)	D (m)
0.0	3.36	4.5	3.59
0.5	2.61	5.0	4.15
1.0	1.86	5.5	3.99
1.5	1.27	6.0	3.37
2.0	0.91	6.5	2.58
2.5	1.14	7.0	1.93
3.0	1.69	7.5	1.25
3.5	2.37	8.0	0.67
4.0	3.01		

(a) Estimate when Priya is moving toward the motion detector; away from the motion detector.

(b) Writing to Learn Give an interpretation of any local extreme values in terms of this problem situation.

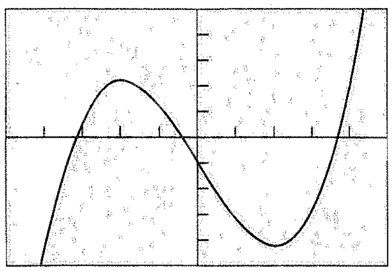
(c) Find a cubic regression equation $D = f(t)$ for the data in Table 4.1 and superimpose its graph on a scatter plot of the data.

(d) Use the model in (c) for f to find a formula for f'. Use this formula to estimate the answers to (a). ■

Here is how we apply the first derivative test to find the local extrema of a function. The critical points of a function f partition the x-axis into intervals on which f' is either positive or negative. We determine the sign of f' in each interval by evaluating f' for one value of x in the interval. Then we apply Theorem 4 as shown in Examples 1 and 2.

Example 1 USING THE FIRST DERIVATIVE TEST

Find the critical points of $f(x) = x^3 - 12x - 5$. Find the function's local and absolute extreme values. Identify the intervals on which f is increasing and decreasing.



[-5, 5] by [-25, 25]

Figure 4.18 The graph of $f(x) = x^3 - 12x - 5$. (Example 1)

Solution Figure 4.18 suggests that f has two critical points.

Solve Analytically

Since f is continuous and differentiable for all real numbers, the critical points occur only at the zeros of f'.

$$f'(x) = 3x^2 - 12 = 0$$
$$x^2 = 4$$
$$x = -2, 2$$

The zeros of f' partition the x-axis into intervals as follows.

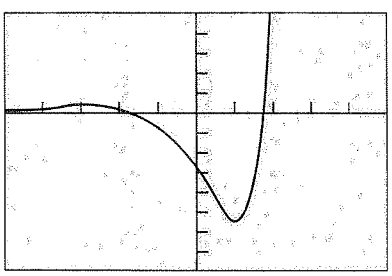
Intervals	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
Sign of f'	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing

We can see from the table that there is a local maximum at $x = -2$ and a local minimum at $x = 2$. The local maximum value is $f(-2) = 11$, and the local minimum value is $f(2) = -21$. There are no absolute extrema. The function increases on the intervals $(-\infty, -2]$ and $[2, \infty)$, and decreases on the interval $[-2, 2]$.

If a function increases, then decreases, and finally increases again like the function in Example 1, you might conjecture that it is a polynomial function. Example 2 shows that this is not necessarily true.

Example 2 USING THE FIRST DERIVATIVE TEST

Find the critical points of $f(x) = (x^2 - 3)e^x$. Find the function's local and absolute extreme values. Identify the intervals on which f is increasing and decreasing.



[-5, 5] by [-8, 5]

Figure 4.19 The graph of $f(x) = (x^2 - 3)e^x$. (Example 2)

Solution This time it is a little harder to see one of the extrema (Figure 4.19).

Solve Analytically

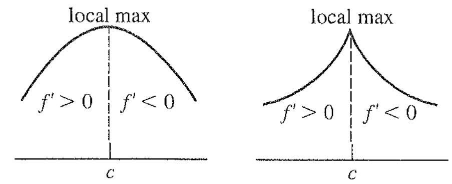
The function f is continuous and differentiable for all real numbers, so the critical points occur only at the zeros of f'.

Theorem 4 First Derivative Test for Local Extrema

The following test applies to a continuous function $f(x)$.

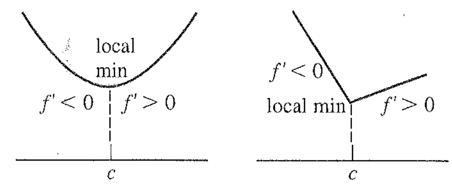
At a critical point c:

1. If f' changes sign from positive to negative at c ($f' > 0$ for $x < c$ and $f' < 0$ for $x > c$), then f has a local maximum value at c.



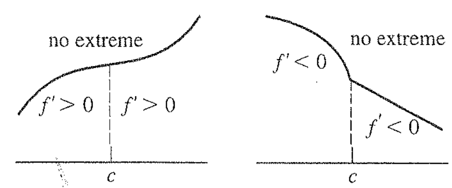
(a) $f'(c) = 0$ (b) $f'(c)$ undefined

2. If f' changes sign from negative to positive at c ($f' < 0$ for $x < c$ and $f' > 0$ for $x > c$), then f has a local minimum value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

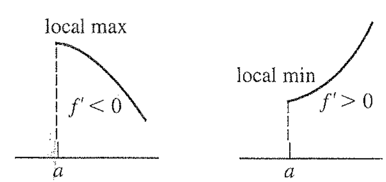
3. If f' does not change sign at c (f' has the same sign on both sides of c), then f has no local extreme value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

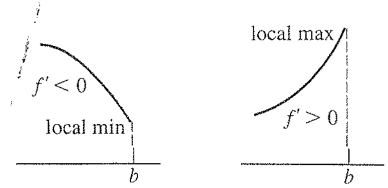
At a left endpoint a:

If $f' < 0$ ($f' > 0$) for $x > a$, then f has a local maximum (minimum) value at a.



At a right endpoint b:

If $f' < 0$ ($f' > 0$) for $x < b$, then f has a local minimum (maximum) value at b.



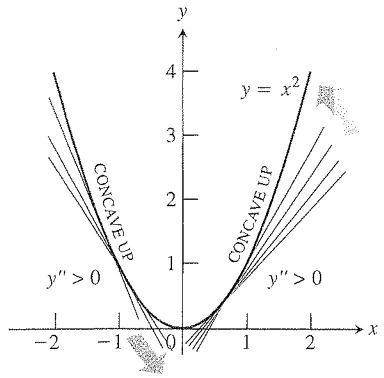
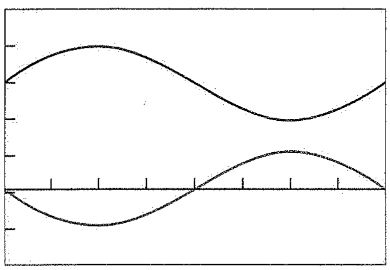


Figure 4.21 The graph of $y = x^2$ is concave up on any interval. (Example 3)

$y_1 = 3 + \sin x$, $y_2 = -\sin x$



[0, 2π] by [−2, 5]

Figure 4.22 Using the graph of y'' to determine the concavity of y. (Example 4)

Example 3 APPLYING THE CONCAVITY TEST

The curve $y = x^2$ (Figure 4.21) is concave up on $(-\infty, \infty)$ because its second derivative $y'' = 2$ is always positive.

Example 4 DETERMINING CONCAVITY

Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

Solution The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$, where $y'' = -\sin x$ is negative. It is concave up on $(\pi, 2\pi)$, where $y'' = -\sin x$ is positive (Figure 4.22).

Points of Inflection

The curve $y = 3 + \sin x$ in Example 4 changes concavity at the point $(\pi, 3)$. We call $(\pi, 3)$ a *point of inflection* of the curve.

Definition **Point of Inflection**

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection.**

A point on a curve where y'' is positive on one side and negative on the other is a point of inflection. At such a point, y'' is either zero (because derivatives have the intermediate value property) or undefined. If y is a twice-differentiable function, $y'' = 0$ at a point of inflection and y' has a local maximum or minimum.

To study the motion of a body moving along a line, we often graph the body's position as a function of time. One reason for doing so is to reveal where the body's acceleration, given by the second derivative, changes sign. On the graph, these are the points of inflection.

Example 5 STUDYING MOTION ALONG A LINE

A particle is moving along a horizontal line with position function

$$s(t) = 2t^3 - 14t^2 + 22t - 5, \quad t \geq 0.$$

Find the velocity and acceleration, and describe the motion of the particle.

Solution

Solve Analytically

The velocity is

$$v(t) = s'(t) = 6t^2 - 28t + 22 = 2(t - 1)(3t - 11),$$

and the acceleration is

$$a(t) = v'(t) = s''(t) = 12t - 28 = 4(3t - 7).$$

When the function $s(t)$ is increasing, the particle is moving to the right; when $s(t)$ is decreasing, the particle is moving to the left. Figure 4.23 shows the graphs of the position, velocity, and acceleration of the particle.

Using the Product Rule we find

$$f'(x) = (x^2 - 3) \cdot \frac{d}{dx}e^x + \frac{d}{dx}(x^2 - 3) \cdot e^x$$
$$= (x^2 - 3) \cdot e^x + (2x) \cdot e^x$$
$$= (x^2 + 2x - 3)e^x.$$

Since e^x is never zero, the first derivative is zero if and only if

$$x^2 + 2x - 3 = 0$$
$$(x + 3)(x - 1) = 0.$$

The zeros $x = -3$ and $x = 1$ partition the x-axis into intervals as follows.

Intervals	$x < -3$	$-3 < x < 1$	$1 < x$
Sign of f'	+	−	+
Behavior of f	increasing	decreasing	increasing

We can see from the table that there is a local maximum (about 0.299) at $x = -3$, and a local minimum (about -5.437) at $x = 1$. The local minimum value is also an absolute minimum because $f(x) > 0$ for $|x| > \sqrt{3}$. There is no absolute maximum. The function increases on $(-\infty, -3]$ and $[1, \infty)$, and decreases on $[-3, 1]$.

Concavity

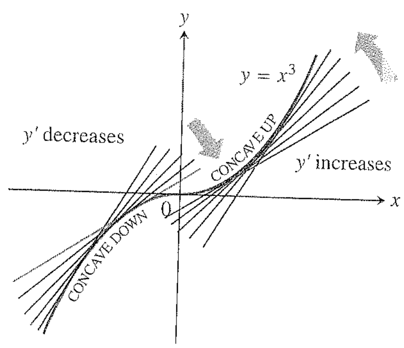


Figure 4.20 The graph of $y = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

As you can see in Figure 4.20, the function $y = x^3$ rises as x increases, but the portions defined on the intervals $(-\infty, 0)$ and $(0, \infty)$ *turn* in different ways. Looking at tangents as we scan from left to right, we see that the slope y' of the curve decreases on the interval $(-\infty, 0)$ and then increases on the interval $(0, \infty)$. The curve $y = x^3$ is *concave down* on $(-\infty, 0)$ and *concave up* on $(0, \infty)$. The curve lies below the tangents where it is concave down, and above the tangents where it is concave up.

Definition Concavity

The graph of a differentiable function $y = f(x)$ is

(a) concave up on an open interval I if y' is increasing on I.

(b) concave down on an open interval I if y' is decreasing on I.

If a function $y = f(x)$ has a second derivative, then we can conclude that y' increases if $y'' > 0$ and y' decreases if $y'' < 0$.

Concavity Test

The graph of a twice-differentiable function $y = f(x)$ is

(a) concave up on any interval where $y'' > 0$.

(b) concave down on any interval where $y'' < 0$.

Table 4.2 NFL Average Salaries

Year	Average Salary (dollars)
1983	141,000
1984	206,000
1985	217,000
1986	220,000
1987	220,000
1988	250,000
1989	319,000
1990	365,000
1991	425,000
1992	492,000
1993	683,000
1994	636,000
1995	714,000
1996	795,000

Source: NFLPA as reported by Gordon Forbes in *USA Today*, May 7, 1997.

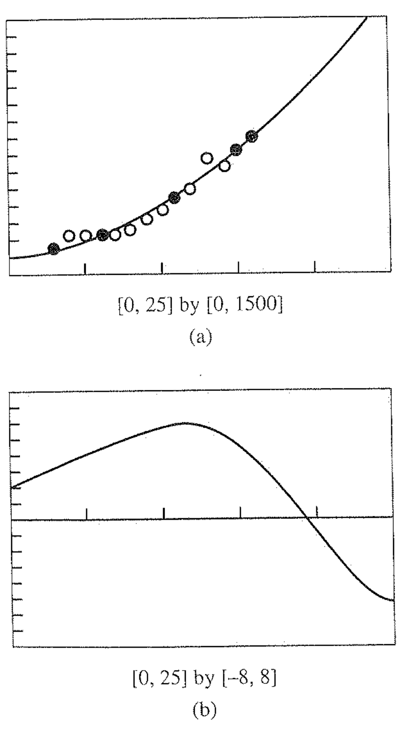


Figure 4.26 The graph of (a)

$$y = \frac{2171}{1 + 24.72e^{-0.168x}}$$

and (b) y''. (Example 6)

Example 6 USING LOGISTIC REGRESSION

Table 4.2 shows the average salaries for National Football League (NFL) players.

(a) Find the logistic regression equation for the data.

(b) Use the regression equation to predict when the rate of salary increases (f') will start to decrease, and the average salary at that time.

(c) Is there a ceiling to the salaries? If so, what is it?

Solution

(a) We let $x = 0$ represent 1980, $x = 1$ represent 1981, and so forth. We enter the salaries in thousands, using 141 for the salary in 1983, and so on. The logistic regression equation is approximately

$$y = \frac{2171}{1 + 24.72e^{-0.168x}}.$$

Its graph is superimposed on a scatter plot of the data in Figure 4.26a.

(b) We need to find the point of inflection, so we need to solve the equation

$$y'' = 0.$$

Using the graph of y'' in Figure 4.26b we find that $y'' = 0$ when $x \approx 19.097$. The corresponding salary is $y(19.097) \approx 1086$. So, the rate of increase should start to decrease in 1999, and the average player salary at that time should be approximately $1,086,000.

(c) Notice that

$$\lim_{x\to\infty} \frac{2171}{1 + 24.72e^{-0.168x}} = 2171,$$

so the average salaries have a ceiling of about $2,171,000.

Second Derivative Test for Local Extrema

Instead of looking for sign changes in y' at critical points, we can sometimes use the following test to determine the presence of local extrema.

Theorem 5 Second Derivative Test for Local Extrema

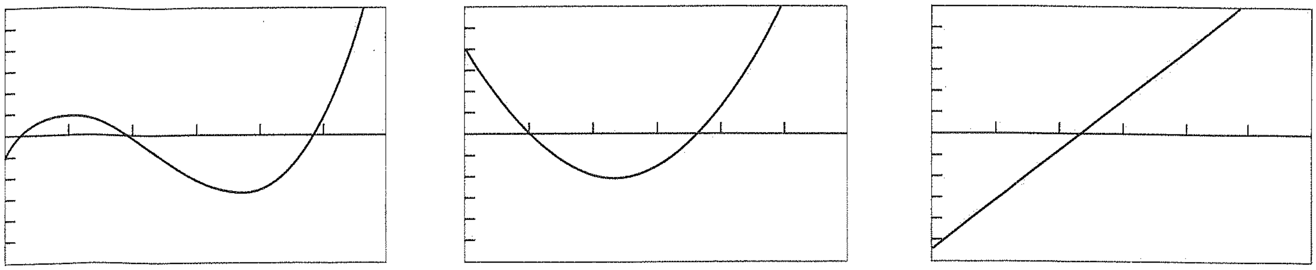
1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

This test requires us to know f'' *only at c itself* and not in an interval about c. This makes the test easy to apply. That's the good news. The bad news is that the test fails if $f''(c) = 0$ or if $f''(c)$ fails to exist. When this happens, go back to the first derivative test for local extreme values.

In Example 7, we apply the second derivative test to the function in Example 1.

Example 7 USING THE SECOND DERIVATIVE TEST

Find the extreme values of $f(x) = x^3 - 12x - 5$.



[0, 6] by [–30, 30]
(a)

[0, 6] by [–30, 30]
(b)

[0, 6] by [–30, 30]
(c)

Figure 4.23 The graph of (a) $s(t) = 2t^3 - 14t^2 + 22t - 5,\ t \ge 0$, (b) $s'(t) = 6t^2 - 28t + 22$, and (c) $s''(t) = 12t - 28$. (Example 5)

Notice that the first derivative ($v = s'$) is zero when $t = 1$ and $t = 11/3$.

Intervals	$0 < t < 1$	$1 < t < 11/3$	$11/3 < t$
Sign of $v = s'$	+	–	+
Behavior of s	increasing	decreasing	increasing
Particle motion	right	left	right

The particle is moving to the right in the time intervals [0, 1) and $(11/3, \infty)$, and moving to the left in (1, 11/3).

The acceleration $a(t) = s''(t) = 12t - 28 = 4(3t - 7)$ is zero when $t = 7/3$.

Intervals	$0 < t < 7/3$	$7/3 < t$
Sign of $a = s''$	–	+
Graph of s	concave down	concave up

The accelerating force is directed toward the left during the time interval [0, 7/3), is momentarily zero at $t = 7/3$, and is directed toward the right thereafter.

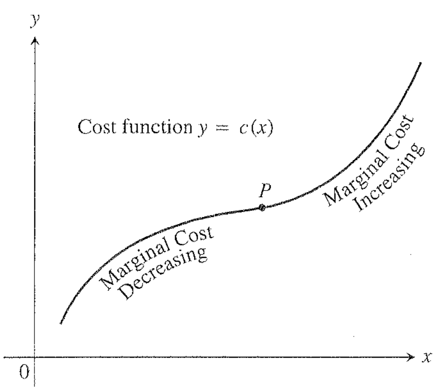


Figure 4.24 On a typical cost curve, a point of inflection separates an interval of decreasing marginal cost from an interval of increasing marginal cost.

Inflection points have applications in some areas of economics. If $y = c(x)$ (Figure 4.24) is the total cost of producing x units of something, the point of inflection at P is then the point at which the marginal cost (the cost of producing one more unit) changes from decreasing to increasing; i.e., it is where the marginal cost reaches a minimum.

The growth of an individual company, of a population, in sales of a new product, or of salaries often follows a *logistic* or *life cycle curve* like the one shown in Figure 4.25. For example, sales of a new product will generally grow slowly at first, then experience a period of rapid growth. Eventually, sales growth slows down again. The function f in Figure 4.25 is increasing. Its rate of increase, f', is at first increasing ($f'' > 0$) up to the point of inflection, and then its rate of increase, f', is decreasing ($f'' < 0$). This is, in a sense, the opposite of what happens in Figure 4.20.

Some graphers have the logistic curve as a built-in regression model. We use this feature in Example 6.

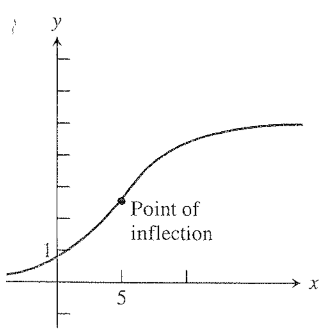


Figure 4.25 A logistic curve

$$y = \frac{c}{1 + ae^{-bx}}.$$

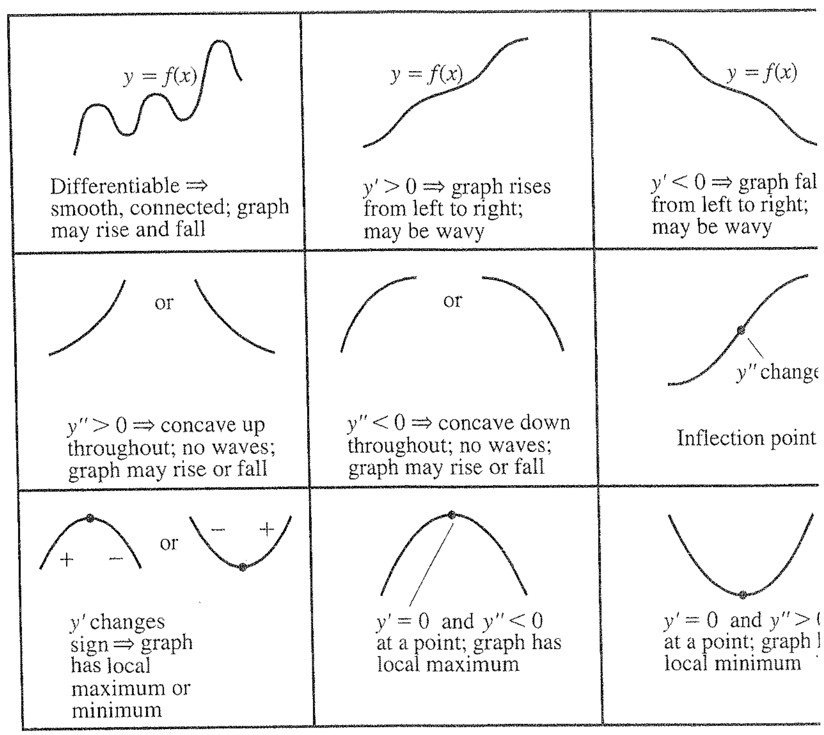
Exploration 1 **Finding f from f'**

Let $f'(x) = 4x^3 - 12x^2$.

1. Find three different functions with derivative equal to $f'(x)$. How are the graphs of the three functions related?
2. Compare their behavior with the behavior found in Example 8.

Learning about Functions from Derivatives

We have seen in Example 8 and Exploration 1 that we are able to re almost everything we need to know about a differentiable function $y = f$ examining y'. We can find where the graph rises and falls and where any extrema are assumed. We can differentiate y' to learn how the graph be it passes over the intervals of rise and fall. We can determine the shape function's graph. The only information we cannot get from the derivative to place the graph in the xy-plane. As we discovered in Section 4.2, th additional information we need to position the graph is the value of f point.



Solution We have

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$$
$$f''(x) = 6x.$$

Testing the critical points $x = \pm 2$ (there are no endpoints), we find

$f''(-2) = -12 < 0 \Rightarrow f$ has a local maximum at $x = -2$ an

$f''(2) = 12 > 0 \Rightarrow f$ has a local minimum at $x = 2$.

Example 8 USING f' AND f'' TO GRAPH f

Let $f'(x) = 4x^3 - 12x^2$.

(a) Identify where the extrema of f occur.

(b) Find the intervals on which f is increasing and the intervals on whic is decreasing.

(c) Find where the graph of f is concave up and where it is concave dow

(d) Sketch a possible graph for f.

Solution f is continuous since f' exists. The domain of f' is $(-\infty, \infty)$, the domain of f is also $(-\infty, \infty)$. Thus, the critical points of f occur only the zeros of f'. Since

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3),$$

the first derivative is zero at $x = 0$ and $x = 3$.

Intervals	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

Note

The *second derivative test* does not apply at $x = 0$ because $f''(0) = 0$. We need the first derivative test to see that there is no local extremum at $x = 0$.

(a) Using the first derivative test and the table above we see that there is n extremum at $x = 0$ and a local minimum at $x = 3$.

(b) Using the table above we see that f is decreasing in $(-\infty, 0]$ and $[0, 3]$, and increasing in $[3, \infty)$.

(c) $f''(x) = 12x^2 - 24x = 12x(x - 2)$ is zero at $x = 0$ and $x = 2$.

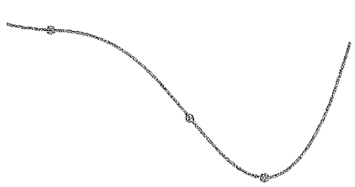
Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f''	$+$	$-$	$+$
Behavior of f	concave up	concave down	concave up

We see that f is concave up on the intervals $(-\infty, 0)$ and $(2, \infty)$, and concave down on $(0, 2)$.

(d) Summarizing the information in the two tables above we obtain

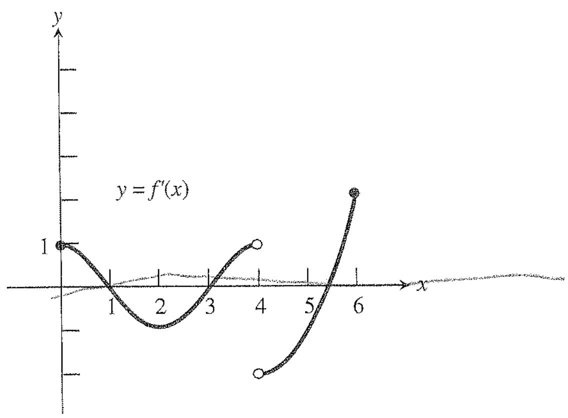
$x < 0$	$0 < x < 2$	$2 < x < 3$	$x < 3$
decreasing	decreasing	decreasing	increasing
concave up	concave down	concave up	concave up

Figure 4.27 shows one possibility for the graph of f.

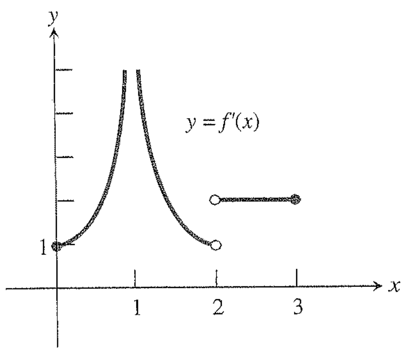


gure 4.27 The graph for f has no tremum but has points of inflection where = 0 and $x = 2$, and a local minimum ere $x = 3$. (Example 8)

5. The domain of f' is $[0, 4) \cup (4, 6]$.



6. The domain of f' is $[0, 1) \cup (1, 2) \cup (2, 3]$.



In Exercises 7–12, use analytic methods to find the intervals on which the function is

(a) increasing, **(b)** decreasing,

(c) concave up, **(d)** concave down.

Then find any

(e) local extreme values, **(f)** inflection points.

Support your answers graphically.

7. $y = x^2 - x - 1$

8. $y = -2x^3 + 6x^2 - 3$

9. $y = 2x^4 - 4x^2 + 1$

10. $y = xe^{1/x}$

11. $y = x\sqrt{8 - x^2}$

12. $y = \begin{cases} 3 - x^2, & x < 0 \\ x^2 + 1, & x \geq 0 \end{cases}$

In Exercises 13–28, find the intervals on which the function is

(a) increasing, **(b)** decreasing,

(c) concave up, **(d)** concave down.

Then find any

(e) local extreme values, **(f)** inflection points.

13. $y = 4x^3 + 21x^2 + 36x - 20$

14. $y = -x^4 + 4x^3 - 4x + 1$

15. $y = 2x^{1/5} + 3$

16. $y = 5 - x^{1/3}$

17. $y = \dfrac{5e^x}{e^x + 3e^{0.8x}}$

18. $y = \dfrac{8e^{-x}}{2e^{-x} + 5e^{-1.5x}}$

19. $y = \begin{cases} 2x, & x < 1 \\ 2 - x^2, & x \geq 1 \end{cases}$

20. $y = e^x, \quad 0 \leq x \leq 2\pi$

21. $y = xe^{1/x^2}$

22. $y = x^2\sqrt{9 - x^2}$

23. $y = \tan^{-1} x$

24. $y = x^{3/4}(5 - x)$

25. $y = x^{1/3}(x - 4)$

26. $y = x^{1/4}(x + 3)$

27. $y = \dfrac{x^3 - 2x^2 + x - 1}{x - 2}$

28. $y = \dfrac{x}{x^2 + 1}$

In Exercises 29 and 30, use the derivative of the function $y = f(x)$ to find the points at which f has a

(a) local maximum, **(b)** local minimum,

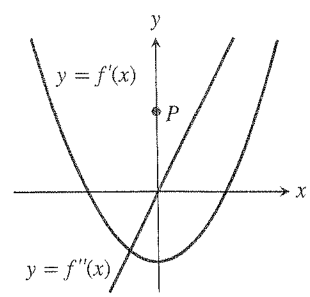
(c) point of inflection.

29. $y' = (x - 1)^2(x - 2)$

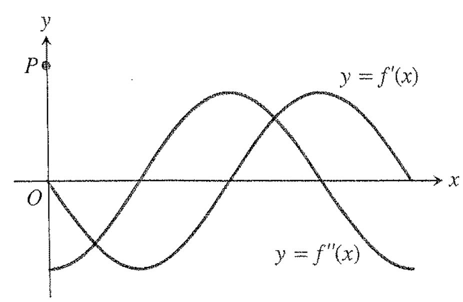
30. $y' = (x - 1)^2(x - 2)(x - 4)$

Exercises 31 and 32 show the graphs of the first and secon derivatives of a function $y = f(x)$. *Work in groups of two three.* Copy the figure and add a sketch of a possible graph of f that passes through the point P.

31.



32.



In Exercises 33 and 34, *work in groups of two or three.*

(a) Find the absolute extrema of f and where they occ

(b) Find any points of inflection.

(c) Sketch a possible graph of f.

33. f is continuous on $[0, 3]$ and satisfies the following.

x	0	1	2	3
f	0	2	0	-2
f'	3	0	does not exist	-3
f''	0	-1	does not exist	0

x	$0 < x < 1$	$1 < x < 2$	$2 < x < 3$
f	$+$	$+$	$-$
f'	$+$	$-$	$-$
f''	$-$	$-$	$-$

Exploration 2 **Finding f from f' and f''**

A function f is continuous on its domain $[-2, 4]$, $f(-2) = 5$, $f(4) = 1$, and f' and f'' have the following properties.

x	$-2 < x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$2 < x$
f'	+	does not exist	−	0	−
f''	+	does not exist	+	0	−

1. Find where all absolute extrema of f occur.
2. Find where the points of inflection of f occur.
3. Sketch a possible graph of f.

Quick Review 4.3

In Exercises 1 and 2, factor the expression and use sign charts to solve the inequality.

1. $x^2 - 9 < 0$

2. $x^3 - 4x > 0$

In Exercises 3–6, find the domains of f and f'.

3. $f(x) = xe^x$

4. $f(x) = x^{3/5}$

5. $f(x) = \dfrac{x}{x-2}$

6. $f(x) = x^{2/5}$

In Exercises 7–10, find the horizontal asymptotes of the function's graph.

7. $y = (4 - x^2)e^x$

8. $y = (x^2 - x)e^{-x}$

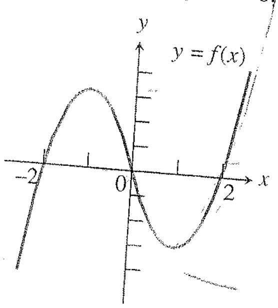
9. $y = \dfrac{200}{1 + 10e^{-0.5x}}$

10. $y = \dfrac{750}{2 + 5e^{-0.1x}}$

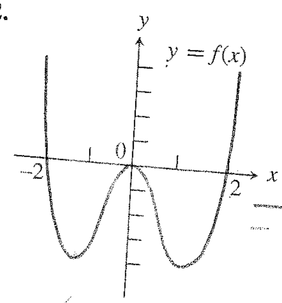
Section 4.3 Exercises

In Exercises 1 and 2, use the graph of the function f to estimate where **(a)** f' and **(b)** f'' are 0, positive, and negative.

1.

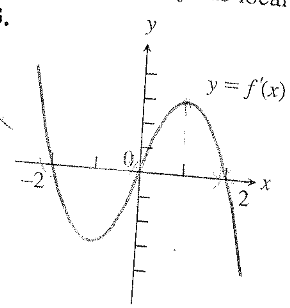


2.

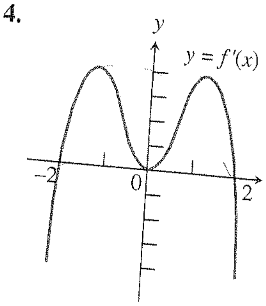


In Exercises 3–6, use the graph of f' to estimate the intervals on which the function f is **(a)** increasing or **(b)** decreasing. **(c)** Estimate where f has local extreme values.

3.



4.



Exploration

50. *Graphs of Cubics* There is almost no leeway in the locations of the inflection point and the extrema of $f(x) = ax^3 + bx^2 + cx + d,\ a \neq 0,$ because the one inflection point occurs at $x = -b/(3a)$ and the extrema, if any, must be located symmetrically about this value of x. Check this out by examining **(a)** the cubic in Exercise 13 and **(b)** the cubic in Exercise 8. Then **(c)** prove the general case. ■

Extending the Ideas

In Exercises 51 and 52, feel free to use a CAS (computer al system), if you have one, to solve the problem.

51. *Logistic Functions* Let $f(x) = c/(1 + ae^{-bx})$ with a $abc \neq 0$.

    (a) Show that f is increasing on the interval $(-\infty, \infty)$ i $abc > 0$, and decreasing if $abc < 0$.

    (b) Show that the point of inflection of f occurs at $x = (\ln|a|)/b$.

52. *Quartic Polynomial Functions* Let $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ with $a \neq 0$.

    (a) Show that the graph of f has 0 or 2 points of inflect

    (b) Write a condition that must be satisfied by the coefficients if the graph of f has 0 or 2 points of inflect

4.4 Modeling and Optimization

Examples from Business and Industry • Examples from Mathematics • Examples from Economics • Modeling Discret Phenomena with Differentiable Functions

Examples from Business and Industry

To *optimize* something means to maximize or minimize some aspect What is the size of the most profitable production run? What is the least e: sive shape for an oil can? What is the stiffest rectangular beam we can cut a 12-inch log? We usually answer such questions by finding the great smallest value of some function that we have used to model the situation

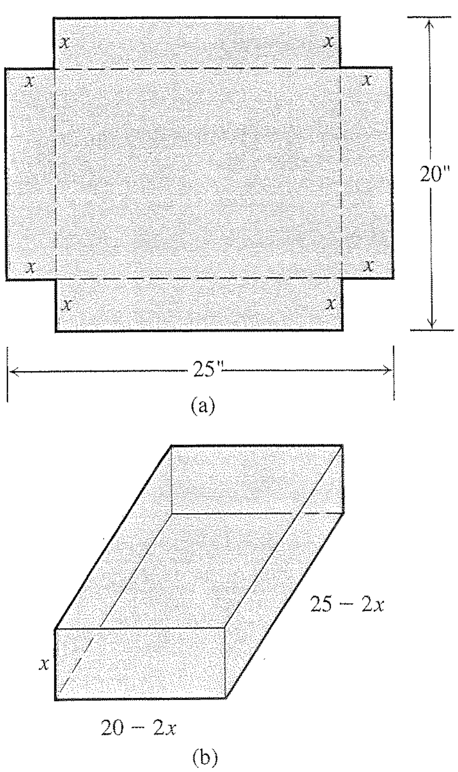


Figure 4.28 An open box made by cutting the corners from a piece of tin. (Example 1)

Example 1 FABRICATING A BOX

An open-top box is to be made by cutting congruent squares of side leng from the corners of a 20- by 25-inch sheet of tin and bending up the side (Figure 4.28). How large should the squares be to make the box hold as much as possible? What is the resulting maximum volume?

Solution

Model

The height of the box is x, and the other two dimensions are $(20 - 2x$ $(25 - 2x)$. Thus, the volume of the box is

$$V(x) = x(20 - 2x)(25 - 2x).$$

Solve Graphically

Because $2x$ cannot exceed 20, we have $0 \leq x \leq 10$. Figure 4.29 sugg that the maximum value of V is about 820.53 and occurs at $x \approx 3.68$.

Confirm Analytically

Expanding, we obtain $V(x) = 4x^3 - 90x^2 + 500x$. The first derivativ of V is

$$V'(x) = 12x^2 - 180x + 500.$$

34. f is an even function, continuous on $[-3, 3]$, and satisfies the following.

x	0	1	2
f	2	0	-1
f'	does not exist	0	does not exist
f''	does not exist	0	does not exist

x	$0 < x < 1$	$1 < x < 2$	$2 < x < 3$
f	+	−	−
f'	−	−	−
f''	+	−	+

(d) What can you conclude about $f(3)$ and $f(-3)$?

In Exercises 35 and 36, *work in groups of two or three.* Sketch a possible graph of a continuous function f that has the given properties.

35. Domain $[0, 6]$, graph of f' given in Exercise 5, and $f(0) = 2$.

36. Domain $[0, 3]$, graph of f' given in Exercise 6, and $f(0) = -3$.

In Exercises 37–40, a particle is moving along a line with position function $s(t)$. Find the **(a)** velocity and **(b)** acceleration, and **(c)** describe the motion of the particle for $t \geq 0$.

37. $s(t) = t^2 - 4t + 3$

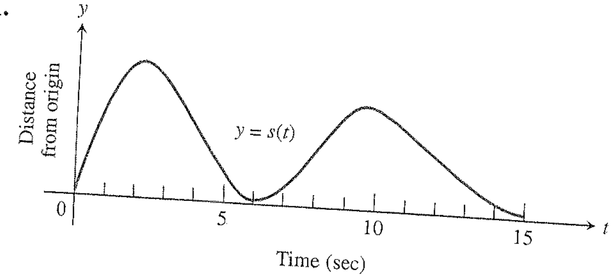
38. $s(t) = 6 - 2t - t^2$

39. $s(t) = t^3 - 3t + 3$

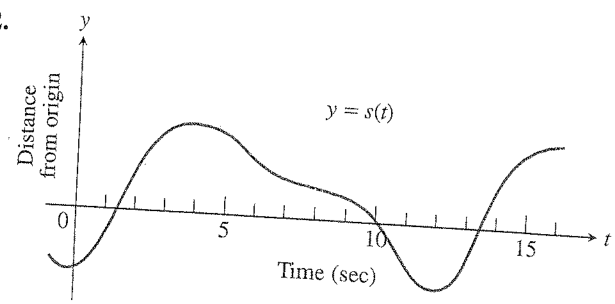
40. $s(t) = 3t^2 - 2t^3$

In Exercises 41 and 42, the graph of the position function $y = s(t)$ of a particle moving along a line is given. At approximately what times is the particle's **(a)** velocity equal to zero? **(b)** acceleration equal to zero?

41.



42.



43. Writing to Learn If $f(x)$ is a differentiable function and $f'(c) = 0$ at an interior point c of f's domain, must f have a local maximum or minimum at $x = c$? Explain.

44. Writing to Learn If $f(x)$ is a twice-differential and $f''(c) = 0$ at an interior point c of f's domai have an inflection point at $x = c$? Explain.

45. *Connecting f and f'* Sketch a smooth curve $y =$ through the origin with the properties that $f'(x) <$ $x < 0$ and $f'(x) > 0$ for $x > 0$.

46. *Connecting f and f"* Sketch a smooth curve $y =$ through the origin with the properties that $f''(x) <$ $x < 0$ and $f''(x) > 0$ for $x > 0$.

47. *Connecting f, f', and f"* Sketch a continuous cu $y = f(x)$ with the following properties. Label coor where possible.

$f(-2) = 8$ $\qquad f'(x) > 0$ for $|x|$

$f(0) = 4$ $\qquad f'(x) < 0$ for $|x|$

$f(2) = 0$ $\qquad f''(x) < 0$ for $x <$

$f'(2) = f'(-2) = 0$ $\qquad f''(x) > 0$ for $x >$

48. *Using Behavior to Sketch* Sketch a continuous cu $y = f(x)$ with the following properties. Label coord where possible.

x	y	Curve
$x < 2$		falling, concave up
2	1	horizontal tangent
$2 < x < 4$		rising, concave up
4	4	inflection point
$4 < x < 6$		rising, concave down
6	7	horizontal tangent
$x > 6$		falling, concave down

49. Table 4.3 gives the sales of prepaid phone cards for se years (the values for 1997 and 1998 were estimated). I $x = 0$ represent 1992, $x = 1$ represent 1993, and so i

(a) Find the logistic regression equation and superimpo graph on a scatter plot of the data.

(b) Use the regression equation to predict when the rate increase in sales will start to decrease, and to predict th sales at that time.

(c) At what amount does the regression equation predict sales will stabilize?

Table 4.3 Prepaid Phone Card Sales

Year	Sales (millions of dollars)
1992	12
1993	114
1994	348
1995	792
1996	1110
1997	1520
1998	1900

Source: Atlantic ACM as reported by Grant Jerding in *USA Today,* April 16, 1997.

Notice from the graph that for small r (a tall thin container, like a piece pipe), the term $2000/r$ dominates and A is large. For large r (a short v container, like a pizza pan), the term $2\pi r^2$ dominates and A again is la

Since A is differentiable on $r > 0$, an interval with no endpoints, it ca have a minimum value only where its first derivative is zero.

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2}$$

$$0 = 4\pi r - \frac{2000}{r^2} \qquad \text{Set } dA/dr = 0.$$

$$4\pi r^3 = 2000 \qquad \text{Multiply by } r^2.$$

$$r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42 \qquad \text{Solve for } r.$$

Something happens at $r = \sqrt[3]{500/\pi}$, but what?

If the domain of A were a closed interval, we could find out by evaluat A at this critical point and the endpoints and comparing the results. Bu the domain is an open interval, so we must learn what is happening at $r = \sqrt[3]{500/\pi}$ by referring to the shape of A's graph. The second deriv

$$\frac{d^2A}{dr^2} = 4\pi + \frac{4000}{r^3}$$

is positive throughout the domain of A. The graph is therefore concave and the value of A at $r = \sqrt[3]{500/\pi}$ an absolute minimum.

The corresponding value of h (after a little algebra) is

$$h = \frac{1000}{\pi r^2} = 2\sqrt[3]{\frac{500}{\pi}} = 2r.$$

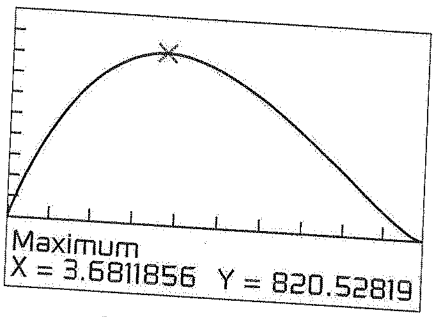
Interpret

The one-liter can that uses the least material has height equal to the di: meter, with $r \approx 5.42$ cm and $h \approx 10.84$ cm.

Strategy for Solving Max-Min Problems

1. **Understand the Problem** Read the problem carefully. Identify tl information you need to solve the problem.
2. **Develop a Mathematical Model of the Problem** Draw pictures and label the parts that are important to the problem. Introduce a variable to represent the quantity to be maximized or minimized. Using that variable, write a function whose extreme value gives tl information sought.
3. **Graph the Function** Find the domain of the function. Determine what values of the variable make sense in the problem.
4. **Identify the Critical Points and Endpoints** Find where the deri ative is zero or fails to exist.
5. **Solve the Mathematical Model** If unsure of the result, support c confirm your solution with another method.
6. **Interpret the Solution** Translate your mathematical result into tl problem setting and decide whether the result makes sense.

$y = x(20 - 2x)(25 - 2x)$



[0, 10] by [−300, 1000]

Figure 4.29 We chose the −300 in $-300 \le y \le 1000$ so that the coordinates of the local maximum at the bottom of the screen would not interfere with the graph. (Example 1)

The two solutions of the quadratic equation $V'(x) = 0$ are

$$c_1 = \frac{180 - \sqrt{180^2 - 48(500)}}{24} \approx 3.68 \text{ and}$$

$$c_2 = \frac{180 + \sqrt{180^2 - 48(500)}}{24} \approx 11.32.$$

Only c_1 is in the domain [0, 10] of V. The values of V at this one c point and the two endpoints are

Critical point value: $V(c_1) \approx 820.53$

Endpoint values: $V(0) = 0, \quad V(10) = 0.$

Interpret

Cutout squares that are about 3.68 in. on a side give the maximum about 820.53 in^3.

Example 2 DESIGNING A CAN

You have been asked to design a one-liter oil can shaped like a righ lar cylinder (Figure 4.30). What dimensions will use the least mater

Solution

Volume of can: If r and h are measured in centimeters, then the volu the can in cubic centimeters is

$$\pi r^2 h = 1000.$$

1 liter = 1000 cm^3

Surface area of can:

$$A = \underbrace{2\pi r^2}_{\text{circular ends}} + \underbrace{2\pi rh}_{\text{cylinder wall}}$$

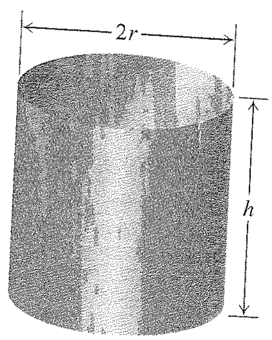


Figure 4.30 This one-liter can uses the least material when $h = 2r$. (Example 2)

How can we interpret the phrase "least material"? One possibility is t ignore the thickness of the material and the waste in manufacturing. T we ask for dimensions r and h that make the total surface area as sma possible while satisfying the constraint $\pi r^2 h = 1000$. (Exercise 17 describes one way to take waste into account.)

Model

To express the surface area as a function of one variable, we solve for of the variables in $\pi r^2 h = 1000$ and substitute that expression into th surface area formula. Solving for h is easier,

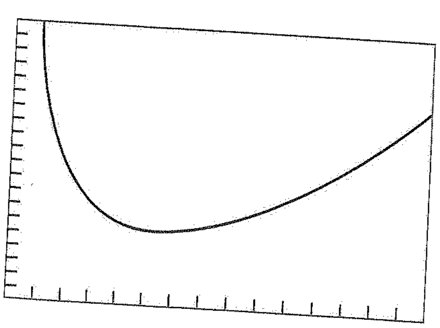
$$h = \frac{1000}{\pi r^2}.$$

Thus,

$$A = 2\pi r^2 + 2\pi rh$$

$$= 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{2000}{r}.$$

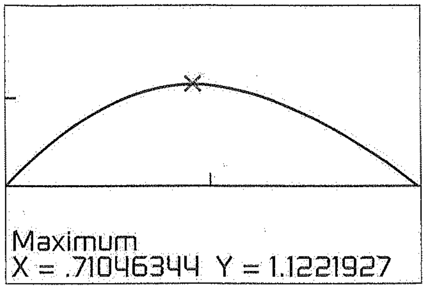


[0, 15] by [0, 2000]

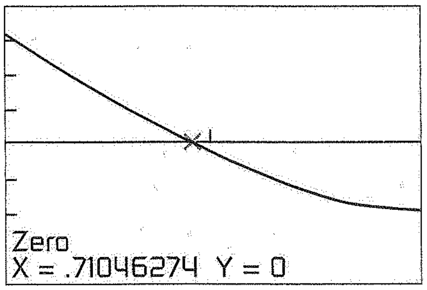
Figure 4.31 The graph of $A = 2\pi r^2 + 2000/r$, $r > 0$. (Example 2)

Solve Analytically

Our goal is to find a value of $r > 0$ that minimizes the value of A. Figure 4.31 suggests that such a value exists.



$[0, \pi/2]$ by $[-1, 2]$
(a)



$[0, \pi/2]$ by $[-4, 4]$
(b)

Figure 4.34 The graph of (a) $A(x) = (\pi - 2x) \sin x$ and (b) A' in the interval $0 \le x \le \pi/2$. (Example 4)

Solve Analytically and Graphically

We can assume that $0 \le x \le \pi/2$. Notice that $A = 0$ at the endpoin $x = 0$ and $x = \pi/2$. Since A is differentiable, the only critical points occur at the zeros of the first derivative,

$$A'(x) = -2 \sin x + (\pi - 2x) \cos x.$$

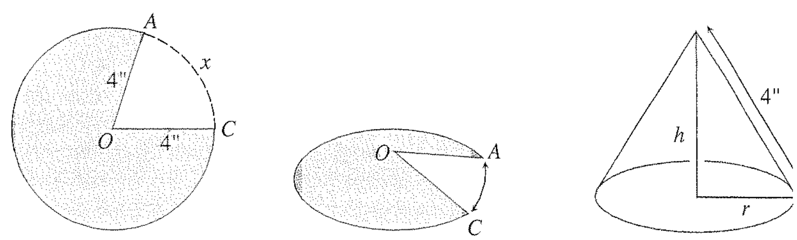
It is not possible to solve the equation $A'(x) = 0$ using algebraic meth We can use the graph of A (Figure 4.34a) to find the maximum value a where it occurs. Or, we can use the graph of A' (Figure 4.34b) to find v the derivative is zero, and then evaluate A at this value of x to find the r mum value. The two x-values appear to be the same, as they should.

Interpret

The rectangle has a maximum area of about 1.12 square units when $x \approx 0.71$. At this point, the rectangle is $\pi - 2x \approx 1.72$ units long by $\sin x \approx 0.65$ unit high.

Exploration 1 **Constructing Cones**

A cone of height h and radius r is constructed from a flat, circular disk of radius 4 in. by removing a sector AOC of arc length x in. an then connecting the edges OA and OC. What arc length x will produce the cone of maximum volume, and what is that volume?



NOT TO SCALE

1. Show that

$$r = \frac{8\pi - x}{2\pi}, \qquad h = \sqrt{16 - r^2}, \qquad \text{and}$$

$$V(x) = \frac{\pi}{3}\left(\frac{8\pi - x}{2\pi}\right)^2 \sqrt{16 - \left(\frac{8\pi - x}{2\pi}\right)^2}.$$

2. Show that the natural domain of V is $0 \le x \le 16\pi$. Graph V ov this domain.
3. Explain why the restriction $0 \le x \le 8\pi$ makes sense in the prol lem situation. Graph V over this domain.
4. Use graphical methods to find where the cone has its maximum volume, and what that volume is.
5. Confirm your findings in part 4 analytically. (*Hint:* Use $V(x) = (1/3)\pi r^2 h,\ h^2 + r^2 = 16,$ and the Chain Rule.)

Examples from Mathematics

Example 3 USING THE STRATEGY

Find two numbers whose sum is 20 and whose product is as large as possible.

Solution

Model

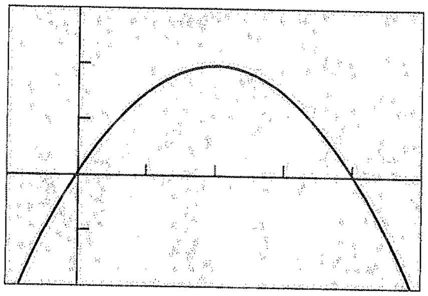
If one number is x, the other is $(20 - x)$, and their product is $f(x) = x(20 - x)$.

Solve Graphically

We can see from the graph of f in Figure 4.32 that there is a maximum. From what we know about parabolas, the maximum occurs at $x = 10$.

Interpret

The two numbers we seek are $x = 10$ and $20 - x = 10$.



[–5, 25] by [–100, 150]

Figure 4.32 The graph of $f(x) = x(20 - x)$ with domain $(-\infty, \infty)$ has an absolute maximum of 100 at $x = 10$. (Example 3)

Sometimes we find it helpful to use both analytic and graphical metl together, as in Example 4.

Example 4 INSCRIBING RECTANGLES

A rectangle is to be inscribed under one arch of the sine curve (Figure 4.3 What is the largest area the rectangle can have, and what dimensions give that area?

Solution

Model

Let $(x, \sin x)$ be the coordinates of point P in Figure 4.33. From what w know about the sine function the x-coordinate of point Q is $(\pi - x)$. Thu

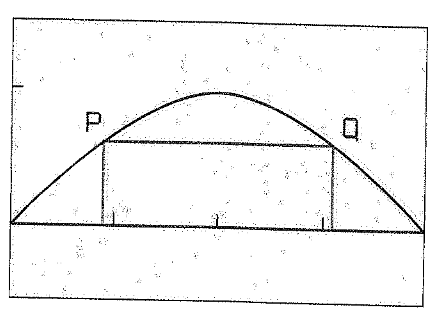
$$\pi - 2x = \text{length of rectangle}$$

and

$$\sin x = \text{height of rectangle.}$$

The area of the rectangle is

$$A(x) = (\pi - 2x) \sin x.$$



[0, π] by [–0.5, 1.5]

Figure 4.33 A rectangle inscribed under one arch of $y = \sin x$. (Example 4)

What guidance do we get from this observation? We know that a prc tion level at which $p'(x) = 0$ need not be a level of maximum profit. It mig a level of minimum profit, for example. But if we are making financial pr tions for our company, we should look for production levels at which mar cost seems to equal marginal revenue. If there is a most profitable produ level, it will be one of these.

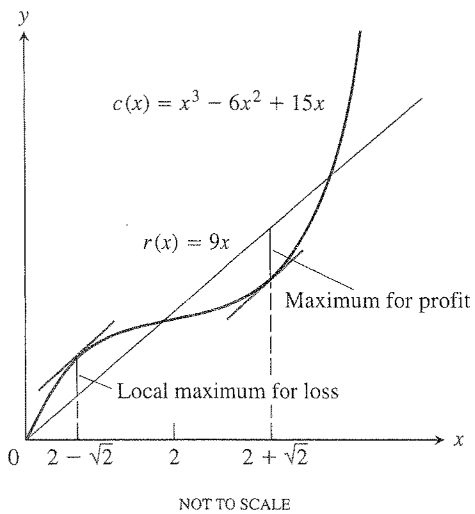


Figure 4.36 The cost and revenue curves for Example 5.

Example 5 MAXIMIZING PROFIT

Suppose that $r(x) = 9x$ and $c(x) = x^3 - 6x^2 + 15x$, where x represe thousands of units. Is there a production level that maximizes profit? If what is it?

Solution Notice that $r'(x) = 9$ and $c'(x) = 3x^2 - 12x + 15$.

$$3x^2 - 12x + 15 = 9 \quad \text{Set } c'(x) = r'(x).$$

$$3x^2 - 12x + 6 = 0$$

The two solutions of the quadratic equation are

$$x_1 = \frac{12 - \sqrt{72}}{6} = 2 - \sqrt{2} \approx 0.586 \quad \text{and}$$

$$x_2 = \frac{12 + \sqrt{72}}{6} = 2 + \sqrt{2} \approx 3.414.$$

The possible production levels for maximum profit are $x \approx 0.586$ thous units or $x \approx 3.414$ thousand units. The graphs in Figure 4.36 show that maximum profit occurs at about $x = 3.414$ and maximum loss occurs a about $x = 0.586$.

Another way to look for optimal production levels is to look for levels minimize the average cost of the units produced. Theorem 7 helps us find tl

> **Theorem 7 Minimizing Average Cost**
>
> The production level (if any) at which average cost is smallest is a level at which the average cost equals the marginal cost.

Proof We assume that $c(x)$ is differentiable.

$$c(x) = \text{cost of producing } x \text{ items}, \quad x > 0$$

$$\frac{c(x)}{x} = \text{average cost of producing } x \text{ items}$$

If the average cost can be minimized, it will be a production level at w

$$\frac{d}{dx}\left(\frac{c(x)}{x}\right) = 0$$

$$\frac{xc'(x) - c(x)}{x^2} = 0 \quad \text{Quotient Rule}$$

$$xc'(x) - c(x) = 0 \quad \text{Multiply by } x^2.$$

$$\underbrace{c'(x)}_{\text{marginal cost}} = \underbrace{\frac{c(x)}{x}}_{\text{average cost}}.$$

Examples from Economics

Here we want to point out two more places where calculus makes a cont tion to economic theory. The first has to do with maximizing profit. The ond has to do with minimizing average cost.

Suppose that

$$r(x) = \text{the revenue from selling } x \text{ items},$$
$$c(x) = \text{the cost of producing the } x \text{ items},$$
$$p(x) = r(x) - c(x) = \text{the profit from selling } x \text{ items}.$$

Marginal Analysis

Because differentiable functions are locally linear, we can use the marginals to approximate the extra revenue, cost, or profit resulting from selling or producing one more item. Using these approximations is referred to as *marginal analysis.*

The marginal revenue, marginal cost, and marginal profit at this produc level (x items) are

$$\frac{dr}{dx} = \text{marginal revenue},$$
$$\frac{dc}{dx} = \text{marginal cost},$$
$$\frac{dp}{dx} = \text{marginal profit}.$$

The first observation is about the relationship of p to these derivatives.

Theorem 6 Maximum Profit

Maximum profit (if any) occurs at a production level at which marginal revenue equals marginal cost.

Proof We assume that $r(x)$ and $c(x)$ are differentiable for all $x > 0$, s $p(x) = r(x) - c(x)$ has a maximum value, it occurs at a production leve which $p'(x) = 0$. Since $p'(x) = r'(x) - c'(x)$, $p'(x) = 0$ implies that

$$r'(x) - c'(x) = 0 \quad \text{or} \quad r'(x) = c'(x).$$

Figure 4.35 gives more information about this situation.

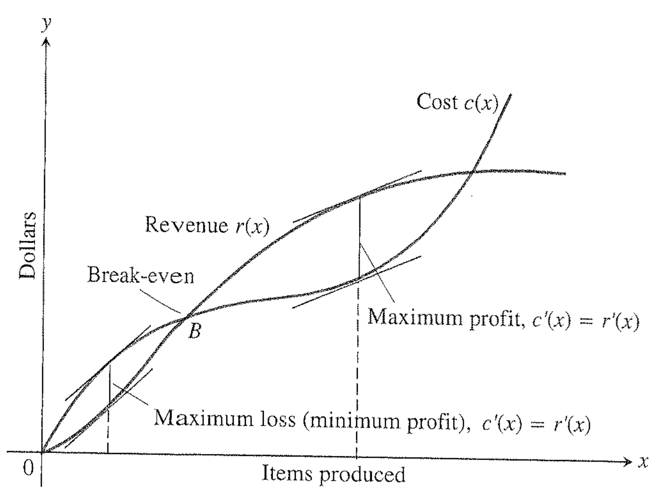


Figure 4.35 The graph of a typical cost function starts concave down and later turns concave up. It crosses the revenue curve at th break-even point B. To the left of B, the company operates at a loss. To the right, the company operates at a profit, the maximum pro occurring where $r'(x) = c'(x)$. Farther to the right, cost exceeds revenue (perhaps because of a combination of market saturation and rising labor and material costs) and production levels become unprofitable again.

Quick Review 4.4

1. Use the first derivative test to identify the local extrema of $y = x^3 - 6x^2 + 12x - 8$.
2. Use the second derivative test to identify the local extrema of $y = 2x^3 + 3x^2 - 12x - 3$.
3. Find the volume of a cone with radius 5 cm and height 8 cm.
4. Find the dimensions of a right circular cylinder with volume 1000 cm^3 and surface area 600 cm^2.

In Exercises 5–8, rewrite the expression as a trigonometr function of the angle α.

5. $\sin(-\alpha)$
6. $\cos(-\alpha)$
7. $\sin(\pi - \alpha)$
8. $\cos(\pi - \alpha)$

In Exercises 9 and 10, use substitution to find the exact s of the system of equations.

9. $\begin{cases} x^2 + y^2 = 4 \\ y = \sqrt{3}x \end{cases}$

10. $\begin{cases} \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1 \\ y = x + 3 \end{cases}$

Section 4.4 Exercises

In Exercises 1–10, solve the problem analytically. Support your answer graphically.

1. *Finding Numbers* The sum of two nonnegative numbers is 20. Find the numbers if

   (a) the sum of their squares is as large as possible; as small as possible.

   (b) one number plus the square root of the other is as large as possible; as small as possible.

2. *Maximizing Area* What is the largest possible area for a right triangle whose hypotenuse is 5 cm long, and what are its dimensions?

3. *Maximizing Perimeter* What is the smallest perimeter possible for a rectangle whose area is 16 in^2, and what are its dimensions?

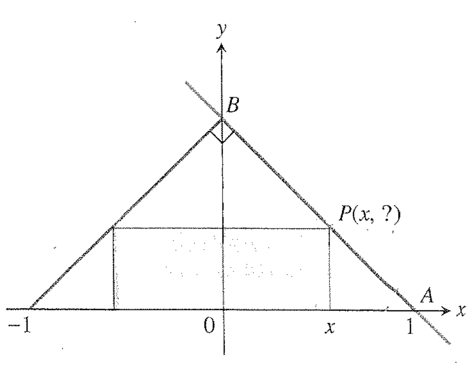
4. *Finding Area* Show that among all rectangles with an 8-m perimeter, the one with largest area is a square.

5. *Inscribing Rectangles* The figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.

   (a) Express the y-coordinate of P in terms of x. (*Hint:* Write an equation for the line AB.)

   (b) Express the area of the rectangle in terms of x.

   (c) What is the largest area the rectangle can have, and what are its dimensions?



6. *Largest Rectangle* A rectangle has its base on the and its upper two vertices on the parabola $y = 12 -$ What is the largest area the rectangle can have, and v its dimensions?

7. *Optimal Dimensions* You are planning to make an rectangular box from an 8- by 15-in. piece of cardbo cutting congruent squares from the corners and foldi the sides. What are the dimensions of the box of larg volume you can make this way, and what is its volun

8. *Closing Off the First Quadrant* You are planning t off a corner of the first quadrant with a line segment long running from $(a, 0)$ to $(0, b)$. Show that the area triangle enclosed by the segment is largest when $a =$

9. *The Best Fencing Plan* A rectangular plot of farm will be bounded on one side by a river and on the ot sides by a single-strand electric fence. With 800 m o your disposal, what is the largest area you can enclos what are its dimensions?

10. *The Shortest Fence* A 216-m^2 rectangular pea pat be enclosed by a fence and divided into two equal pa another fence parallel to one of the sides. What dime for the outer rectangle will require the smallest total of fence? How much fence will be needed?

11. *Designing a Tank* Your iron works has contracted t design and build a 500-ft^3, square-based, open-top, r gular steel holding tank for a paper company. The ta be made by welding thin stainless steel plates togeth their edges. As the production engineer, your job is t dimensions for the base and height that will make th weigh as little as possible.

    (a) What dimensions do you tell the shop to use?

    (b) **Writing to Learn** Briefly describe how you to weight into account.

Again we have to be careful about what Theorem 7 does and does n
It does not say that there is a production level of minimum average co
says where to look to see if there is one. Look for production levels at
average cost and marginal cost are equal. Then check to see if any of
gives a minimum average cost.

Example 6 MINIMIZING AVERAGE COST

Suppose $c(x) = x^3 - 6x^2 + 15x$, where x represents thousands of un
Is there a production level that minimizes average cost? If so, what is i

Solution We look for levels at which average cost equals marginal co

Marginal cost: $c'(x) = 3x^2 - 12x + 15$

Average cost: $\dfrac{c(x)}{x} = x^2 - 6x + 15$

$$3x^2 - 12x + 15 = x^2 - 6x + 15 \qquad \text{Marginal cost = Average cost}$$
$$2x^2 - 6x = 0$$
$$2x(x - 3) = 0$$
$$x = 0 \quad \text{or} \quad x = 3$$

Since $x > 0$, the only production level that might minimize average co
$x = 3$ thousand units.

We use the second derivative test.

$$\frac{d}{dx}\left(\frac{c(x)}{x}\right) = 2x - 6$$
$$\frac{d^2}{dx^2}\left(\frac{c(x)}{x}\right) = 2 > 0$$

The second derivative is positive for all $x > 0$, so $x = 3$ gives an
absolute minimum.

Modeling Discrete Phenomena with Differentiable Functions

In case you are wondering how we can use differentiable functions $c(x)$ a
$r(x)$ to describe the cost and revenue that comes from producing a number
items x that can only be an integer, here is the rationale.

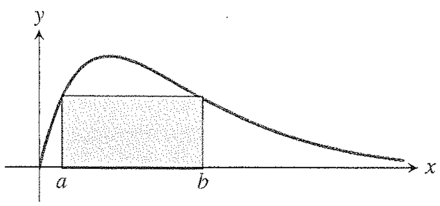
When x is large, we can reasonably fit the cost and revenue data w
smooth curves $c(x)$ and $r(x)$ that are defined not only at integer values of x b
at the values in between just as we do when we use regression equations. On
we have these differentiable functions, which are supposed to behave like t
real cost and revenue when x is an integer, we can apply calculus to draw co
clusions about their values. We then translate these mathematical conclusio
into inferences about the real world that we hope will have predictive valu
When they do, as is the case with the economic theory here, we say that tl
functions give a good model of reality.

What do we do when our calculus tells us that the best production level
a value of x that isn't an integer, as it did in Example 5? We use the neare
convenient integer. For $x \approx 3.414$ thousand units in Example 5, we might us
3414, or perhaps 3410 or 3420 if we ship in boxes of 10.

22. *Maximizing Volume* Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm. What is the maximum volume?

Exploration

23. The figure shows the graph of $f(x) = xe^{-x}$, $x \geq 0$.



(**a**) Find where the absolute maximum of f occurs.

(**b**) Let $a > 0$ and $b > 0$ be given as shown in the figure. Complete the following table where A is the area of the rectangle in the figure.

a	b	A
0.1		
0.2		
0.3		
⋮		
1		

(**c**) Draw a scatter plot of the data (a, A).

(**d**) Find the quadratic, cubic, and quartic regression equations for the data in (b), and superimpose their graphs on a scatter plot of the data.

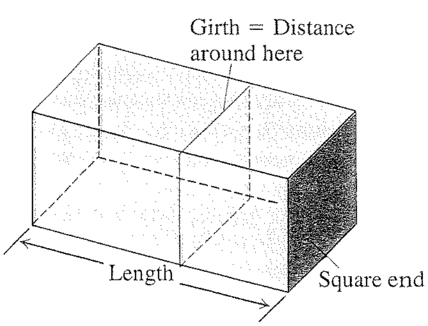
(**e**) Use each of the regression equations in (d) to estimate the maximum possible value of the area of the rectangle.

24. *Cubic Polynomial Functions*
Let $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$.

(**a**) Show that f has either 0 or 2 local extrema.

(**b**) Give an example of each possibility in (a).

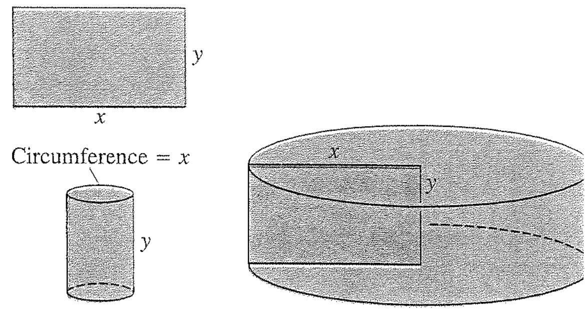
25. *Shipping Packages* The U.S. Postal Service will accept a box for domestic shipment only if the sum of its length and girth (distance around), as shown in the figure, does not exceed 108 in. What dimensions will give a box with a square end the largest possible volume?



26. *Constructing Cylinders* Compare the answers to the following two construction problems.

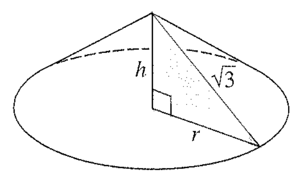
(**a**) A rectangular sheet of perimeter 36 cm and dimensi x cm by y cm is to be rolled into a cylinder as shown in (a) of the figure. What values of x and y give the larges volume?

(**b**) The same sheet is to be revolved about one of the si of length y to sweep out the cylinder as shown in part (the figure. What values of x and y give the largest volur



(a) (b)

27. *Constructing Cones* A right triangle whose hypoten $\sqrt{3}$ m long is revolved about one of its legs to generat right circular cone. Find the radius, height, and volume the cone of greatest volume that can be made this way.

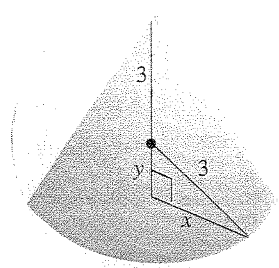


28. *Finding Parameter Values* What value of a makes $f(x) = x^2 + (a/x)$ have (**a**) a local minimum at $x = 2$? (**b**) a point of inflection at $x = 1$?

29. *Finding Parameter Values* Show that $f(x) = x^2 + ($ cannot have a local maximum for any value of a.

30. *Finding Parameter Values* What values of a and b m $f(x) = x^3 + ax^2 + bx$ have (**a**) a local maximum at x = and a local minimum at $x = 3$? (**b**) a local minimum a $x = 4$ and a point of inflection at $x = 1$?

31. *Inscribing a Cone* Find the volume of the largest rig circular cone that can be inscribed in a sphere of radius



12. *Catching Rainwater* A 1125-ft^3 open-top rectangular tank with a square base x ft on a side and y ft deep is to be built with its top flush with the ground to catch runoff water. The costs associated with the tank involve not only the material from which the tank is made but also an excavation charge proportional to the product xy.

(a) If the total cost is

$$c = 5(x^2 + 4xy) + 10xy,$$

what values of x and y will minimize it?

(b) Writing to Learn Give a possible scenario for the cost function in (a).

13. *Designing a Poster* You are designing a rectangular poster to contain 50 in^2 of printing with a 4-in. margin at the top and bottom and a 2-in. margin at each side. What overall dimensions will minimize the amount of paper used?

14. *Vertical Motion* The height of an object moving vertically is given by

$$s = -16t^2 + 96t + 112,$$

with s in ft and t in sec. Find **(a)** the object's velocity when $t = 0$, **(b)** its maximum height and when it occurs, and **(c)** its velocity when $s = 0$.

15. *Finding an Angle* Two sides of a triangle have lengths a and b, and the angle between them is θ. What value of θ will maximize the triangle's area? (*Hint:* $A = (1/2)\ ab \sin \theta$.)

16. *Designing a Can* What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of 1000 cm^3? Compare the result here with the result in Example 2.

17. *Designing a Can* You are designing a 1000-cm^3 right circular cylindrical can whose manufacture will take waste into account. There is no waste in cutting the aluminum for the side, but the top and bottom of radius r will be cut from squares that measure $2r$ units on a side. The total amount of aluminum used up by the can will therefore be

$$A = 8r^2 + 2\pi rh$$

rather than the $A = 2\pi r^2 + 2\pi rh$ in Example 2. In Example 2 the ratio of h to r for the most economical can was 2 to 1. What is the ratio now?

In Exercises 18 and 19, *work in groups of two or three.*

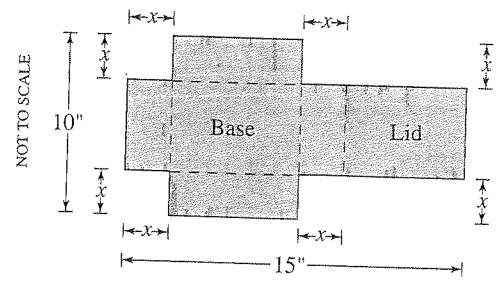
18. *Designing a Box with Lid* A piece of cardboard measures 10- by 15-in. Two equal squares are removed from the corners of a 10-in. side as shown in the figure. Two equal rectangles are removed from the other corners so that the tabs can be folded to form a rectangular box with lid.

(a) Write a formula $V(x)$ for the volume of the box.

(b) Find the domain of V for the problem situation and graph V over this domain.

(c) Use a graphical method to find the maximum volume and the value of x that gives it.

(d) Confirm your result in (c) analytically.



19. *Designing a Suitcase* A 24- by 36-in. sheet of cardl is folded in half to form a 24- by 18-in. rectangle as sl in the figure. Then four congruent squares of side leng are cut from the corners of the folded rectangle. The sl unfolded, and the six tabs are folded up to form a box sides and a lid.

(a) Write a formula $V(x)$ for the volume of the box.

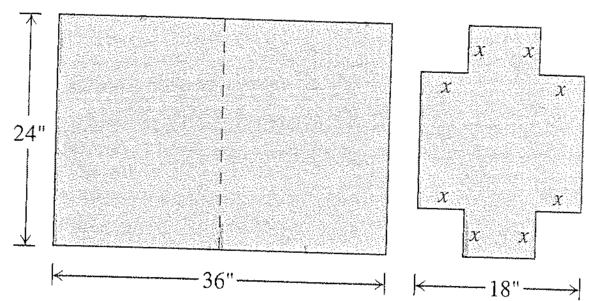
(b) Find the domain of V for the problem situation and graph V over this domain.

(c) Use a graphical method to find the maximum volun and the value of x that gives it.

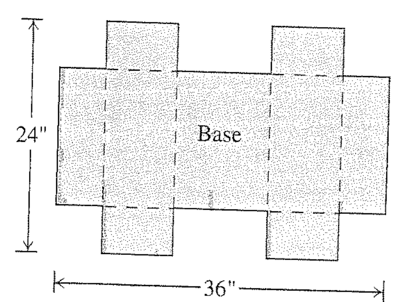
(d) Confirm your result in (c) analytically.

(e) Find a value of x that yields a volume of 1120 in^3.

(f) Writing to Learn Write a paragraph describing th issues that arise in part (b).



The sheet is then unfolded.



20. *Quickest Route* Jane is 2 mi offshore in a boat and wisl to reach a coastal village 6 mi down a straight shoreline from the point nearest the boat. She can row 2 mph and ca walk 5 mph. Where should she land her boat to reach the village in the least amount of time?

21. *Inscribing Rectangles* A rectangle is to be inscribed unc the arch of the curve $y = 4 \cos (0.5x)$ from $x = -\pi$ to $x = \pi$. What are the dimensions of the rectangle with largest area, and what is the largest area?

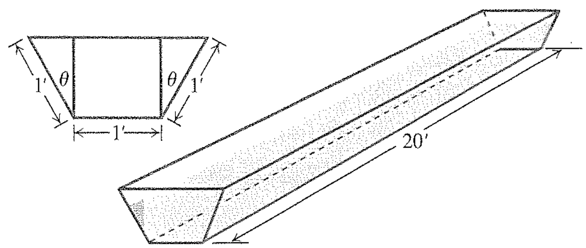
41. *Motion on a Line* The positions of two particles on the s-axis are $s_1 = \sin t$ and $s_2 = \sin(t + \pi/3)$, with s_1 and s_2 in meters and t in seconds.

(a) At what time(s) in the interval $0 \le t \le 2\pi$ do the particles meet?

(b) What is the farthest apart that the particles ever get?

(c) When in the interval $0 \le t \le 2\pi$ is the distance between the particles changing the fastest?

42. *Finding an Angle* The trough in the figure is to be made to the dimensions shown. Only the angle θ can be varied. What value of θ will maximize the trough's volume?

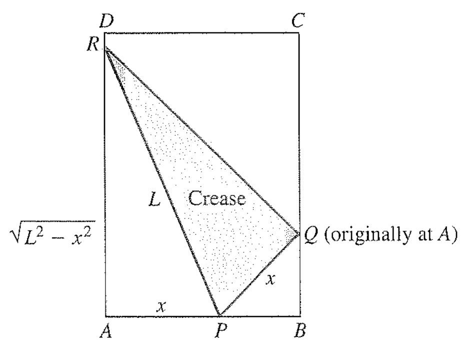


43. *Paper Folding* *Work in groups of two or three.* A rectangular sheet of 8 1/2- by 11-in. paper is placed on a flat surface. One of the corners is placed on the opposite longer edge, as shown in the figure, and held there as the paper is smoothed flat. The problem is to make the length of the crease as small as possible. Call the length L. Try it with paper.

(a) Show that $L^2 = 2x^3/(2x - 8.5)$.

(b) What value of x minimizes L^2?

(c) What is the minimum value of L?



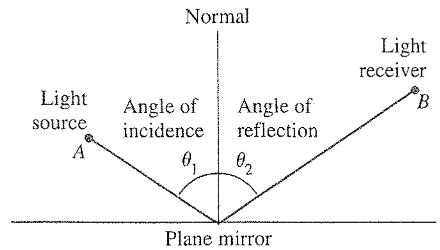
44. *Sensitivity to Medicine* *(continuation of Exercise 35, Section 3.3)* Find the amount of medicine to which the body is most sensitive by finding the value of M that maximizes the derivative dR/dM.

45. *Selling Backpacks* It costs you c dollars each to manufacture and distribute backpacks. If the backpacks sell at x dollars each, the number sold is given by

$$n = \frac{a}{x - c} + b(100 - x),$$

where a and b are certain positive constants. What selling price will bring a maximum profit?

46. *Fermat's Principle in Optics* Fermat's principle in optics states that light always travels from one point to another along a path that minimizes the travel time. Light from a source A is reflected by a plane mirror to a receiver at point B, as shown in the figure. Show that for the light to obey Fermat's principle, the angle of incidence must equal the angle of reflection, both measured from the line normal to the reflecting surface. (This result can also be derived without calculus. There is a purely geometric argument, which you may prefer.)



47. *Tin Pest* When metallic tin is kept below 13.2°C, it slowly becomes brittle and crumbles to a gray powder. Tin objects eventually crumble to this gray powder spontaneously if kept in a cold climate for years. The Europeans who saw tin organ pipes in their churches crumble away years ago called the change *tin pest* because it seemed to be contagious. And indeed it was, for the gray powder is a catalyst for its own formation.

A *catalyst* for a chemical reaction is a substance that controls the rate of reaction without undergoing any permanent change in itself. An *autocatalytic reaction* is one whose product is a catalyst for its own formation. Such a reaction may proceed slowly at first if the amount of catalyst present is small and slowly again at the end, when most of the original substance is used up. But in between, when both the substance and its catalyst product are abundant, the reaction proceeds at a faster pace.

In some cases it is reasonable to assume that the rate $v = dx/dt$ of the reaction is proportional both to the amount of the original substance present and to the amount of product. That is, v may be considered to be a function of x alone, and

$$v = kx(a - x) = kax - kx^2,$$

where

x = the amount of product,

a = the amount of substance at the beginning,

k = a positive constant.

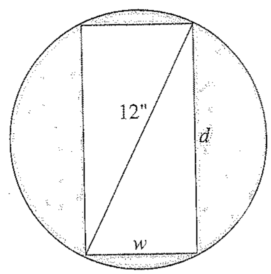
At what value of x does the rate v have a maximum? What is the maximum value of v?

32. *Strength of a Beam* The strength S of a rectangular wooden beam is proportional to its width times the square of its depth.

(a) Find the dimensions of the strongest beam that can be cut from a 12-in. diameter cylindrical log.

(b) Writing to Learn Graph S as a function of the beam's width w, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in (a).

(c) Writing to Learn On the same screen, graph S as a function of the beam's depth d, again taking $k = 1$. Compare the graphs with one another and with your answer in (a). What would be the effect of changing to some other value of k? Try it.



33. *Stiffness of a Beam* The stiffness S of a rectangular beam is proportional to its width times the cube of its depth.

(a) Find the dimensions of the stiffest beam that can be cut from a 12-in. diameter cylindrical log.

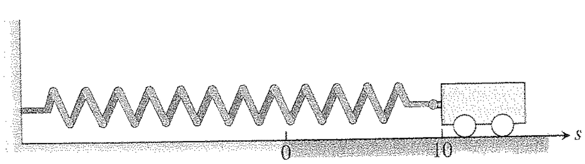
(b) Writing to Learn Graph S as a function of the beam's width w, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in (a).

(c) Writing to Learn On the same screen, graph S as a function of the beam's depth d, again taking $k = 1$. Compare the graphs with one another and with your answer in (a). What would be the effect of changing to some other value of k? Try it.

34. *Frictionless Cart* A small frictionless cart, attached to the wall by a spring, is pulled 10 cm from its rest position and released at time $t = 0$ to roll back and forth for 4 sec. Its position at time t is $s = 10 \cos \pi t$.

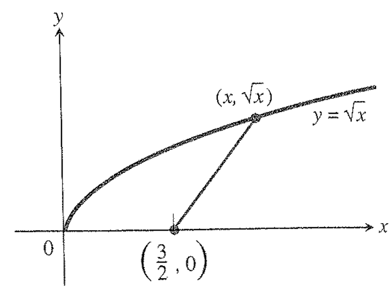
(a) What is the cart's maximum speed? When is the cart moving that fast? Where is it then? What is the magnitude of the acceleration then?

(b) Where is the cart when the magnitude of the acceleration is greatest? What is the cart's speed then?



35. *Electrical Current* Suppose that at any time t (sec) the current i (amp) in an alternating current circuit is $i = 2 \cos t + 2 \sin t$. What is the peak (largest magnitude) current for this circuit?

36. *Calculus and Geometry* How close does the curve $y = \sqrt{x}$ come to the point $(3/2, 0)$? (*Hint:* If you minimize the *square* of the distance, you can avoid square roots.)



37. *Calculus and Geometry* How close does the semicircle $y = \sqrt{16 - x^2}$ come to the point $(1, \sqrt{3})$?

38. Writing to Learn Is the function $f(x) = x^2 - x + 1$ ever negative? Explain.

39. Writing to Learn You have been asked to determine whether the function $f(x) = 3 + 4 \cos x + \cos 2x$ is ever negative.

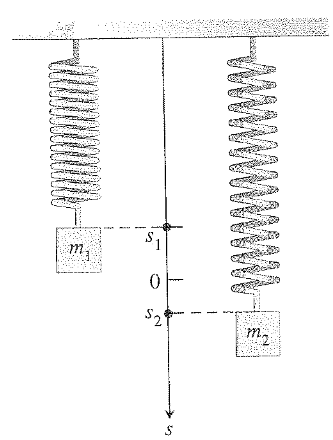
(a) Explain why you need to consider values of x only in the interval $[0, 2\pi]$.

(b) Is f ever negative? Explain.

40. *Vertical Motion* Two masses hanging side by side from springs have positions $s_1 = 2 \sin t$ and $s_2 = \sin 2t$, respectively, with s_1 and s_2 in meters and t in seconds.

(a) At what times in the interval $t > 0$ do the masses pass each other? (*Hint:* $\sin 2t = 2 \sin t \cos t$.)

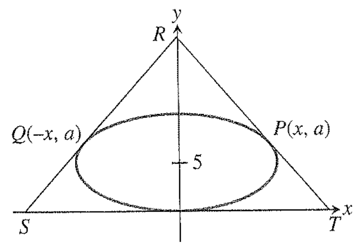
(b) When in the interval $0 \le t \le 2\pi$ is the vertical distance between the masses the greatest? What is this distance? (*Hint:* $\cos 2t = 2 \cos^2 t - 1$.)



55. *Circumscribing an Ellipse* Let $P(x, a)$ and $Q(-x, a)$ be two points on the upper half of the ellipse

$$\frac{x^2}{100} + \frac{(y-5)^2}{25} = 1$$

centered at $(0, 5)$. A triangle RST is formed by using the tangent lines to the ellipse at Q and P as shown in the figure.



(a) Show that the area of the triangle is

$$A(x) = -f'(x)\left[x - \frac{f(x)}{f'(x)}\right]^2,$$

where $y = f(x)$ is the function representing the upper half of the ellipse.

(b) What is the domain of A? Draw the graph of A. How are the asymptotes of the graph related to the problem situation?

(c) Determine the height of the triangle with minimum area. How is it related to the y-coordinate of the center of the ellipse?

(d) Repeat parts (a)–(c) for the ellipse

$$\frac{x^2}{C^2} + \frac{(y-B)^2}{B^2} = 1$$

centered at $(0, B)$. Show that the triangle has minimum area when its height is $3B$.

4.5 Linearization and Newton's Method

Linear Approximation • Newton's Method • Differentials • Estimating Change with Differentials • Absolute, Relative, and Percentage Change • Sensitivity to Change

Linear Approximation

The line $y = mx + b$ tangent to a curve $y = f(x)$ lies close to the curve near the point of tangency. You can observe this phenomenon by zooming in on the two graphs at the point of tangency, or by looking at tables of values for the difference $y = f(x) - (mx + b)$ near the x-coordinate of the point of tangency. Exploration 1 gives more details.

Exploration 1 **Approximating with Tangent Lines**

Let $f(x) = x^2$.

1. Show that the line tangent to the graph of f at the point $(1, 1)$ is $y = 2x - 1$.
2. Set $y_1 = x^2$ and $y_2 = 2x - 1$. Zoom in on the two graphs at $(1, 1)$.
3. For a more dramatic view, set $y_3 = y_1 - y_2$. Then turn off y_1 and y_2, zoom in on the point $(1, 0)$, and watch the difference collapse to zero. Explain why this gives a graphical measure of how well the tangent line approximates the function near $x = 1$.
4. Look at tables of values for y_3 with Δ Table $= 0.1, 0.01, 0.001$, and 0.0001. Explain why this gives a numerical measure of how well the tangent line approximates the function near $x = 1$.

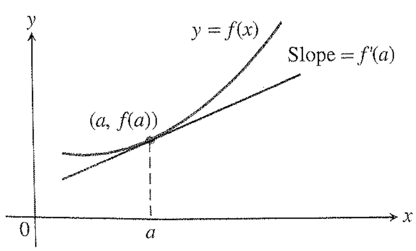


Figure 4.37 The tangent to the curve $y = f(x)$ at $x = a$ is the line $y = f(a) + f'(a)(x - a)$.

In general, the tangent to $y = f(x)$ at $x = a$ (Figure 4.37) passes through the point $(a, f(a))$, so its point-slope equation is

$$y = f(a) + f'(a)(x - a).$$

48. *How We Cough* When we cough, the trachea (windpipe) contracts to increase the velocity of the air going out. This raises the question of how much it should contract to maximize the velocity and whether it really contracts that much when we cough.

Under reasonable assumptions about the elasticity of the tracheal wall and about how the air near the wall is slowed by friction, the average flow velocity v (in cm/sec) can be modeled by the equation

$$v = c(r_0 - r)r^2, \quad \frac{r_0}{2} \le r \le r_0,$$

where r_0 is the rest radius of the trachea in cm and c is a positive constant whose value depends in part on the length of the trachea.

(a) Show that v is greatest when $r = (2/3)r_0$, that is, when the trachea is about 33% contracted. The remarkable fact is that X-ray photographs confirm that the trachea contracts about this much during a cough.

(b) Take r_0 to be 0.5 and c to be 1, and graph v over the interval $0 \le r \le 0.5$. Compare what you see to the claim that v is a maximum when $r = (2/3)r_0$.

49. *Tour Service* You operate a tour service that offers the following rates:

- $200 per person if 50 people (the minimum number to book the tour) go on the tour.
- For each additional person, up to a maximum of 80 people total, the rate per person is reduced by $2.

It costs $6000 (a fixed cost) plus $32 per person to conduct the tour. How many people does it take to maximize your profit?

50. *Wilson Lot Size Formula* One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where q is the quantity you order when things run low (shoes, radios, brooms, or whatever the item might be), k is the cost of placing an order (the same, no matter how often you order), c is the cost of one item (a constant), m is the number of items sold each week (a constant), and h is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security).

(a) Your job, as the inventory manager for your store, is to find the quantity that will minimize $A(q)$. What is it? (The formula you get for the answer is called the *Wilson lot size formula.*)

(b) Shipping costs sometimes depend on order size. When they do, it is more realistic to replace k by $k + bq$, the sum of k and a constant multiple of q. What is the most economical quantity to order now?

51. *Production Level* Show that if $r(x) = 6x$ and $c(x) = x^3 - 6x^2 + 15x$ are your revenue and cost functions, then the best you can do is break even (have revenue equal cost).

52. *Production Level* Suppose $c(x) = x^3 - 20x^2 + 20{,}000x$ is the cost of manufacturing x items. Find a production level that will minimize the average cost of making x items.

Extending the Ideas

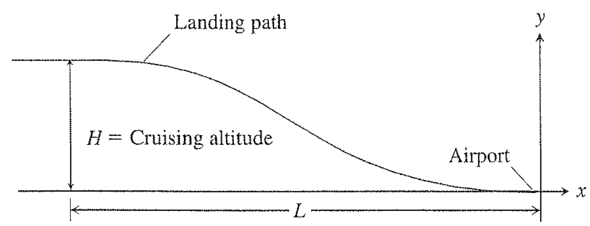
53. *Airplane Landing Path* An airplane is flying at altitude H when it begins its descent to an airport runway that is at horizontal ground distance L from the airplane, as shown in the figure. Assume that the landing path of the airplane is the graph of a cubic polynomial function $y = ax^3 + bx^2 + cx + d$ where $y(-L) = H$ and $y(0) = 0$.

(a) What is dy/dx at $x = 0$?

(b) What is dy/dx at $x = -L$?

(c) Use the values for dy/dx at $x = 0$ and $x = -L$ together with $y(0) = 0$ and $y(-L) = H$ to show that

$$y(x) = H\left[2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right].$$



In Exercises 54 and 55, you might find it helpful to use a CAS and to *work in groups of two or three.*

54. *Generalized Cone Problem* A cone of height h and radius r is constructed from a flat, circular disk of radius a in. as described in Exploration 1.

(a) Find a formula for the volume V of the cone in terms of x and a.

(b) Find r and h in the cone of maximum volume for $a = 4, 5, 6, 8$.

(c) **Writing to Learn** Find a simple relationship between r and h that is independent of a for the cone of maximum volume. Explain how you arrived at your relationship.

Example 2 FINDING A LINEARIZATION

The most important linear approximation for roots and powers is

$$(1 + x)^k \approx 1 + kx, \qquad (x \approx 0, \text{ any number } k).$$

Example 3 APPLYING EXAMPLE 2

The following approximations are consequences of Example 2.

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x \qquad k = 1/2$$

$$\frac{1}{1-x} = (1-x)^{-1} \approx 1 + (-1)(-x) = 1 + x \qquad k = -1; \text{ replace } x \text{ by } -x.$$

$$\sqrt[3]{1+5x^4} = (1+5x^4)^{1/3} \approx 1 + \frac{1}{3}(5x^4) = 1 + \frac{5}{3}x^4 \qquad k = 1/3; \text{ replace } x \text{ by } 5x^4.$$

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} \approx 1 + \left(-\frac{1}{2}\right)(-x^2) = 1 + \frac{1}{2}x^2 \qquad k = -1/2; \text{ replace } x \text{ by } -x^2.$$

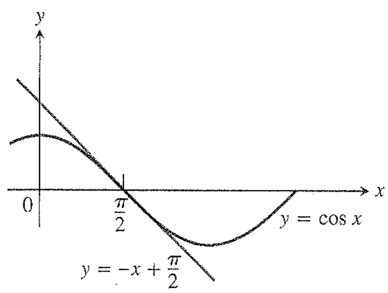


Figure 4.39 The graph of $f(x) = \cos x$ and its linearization at $x = \pi/2$. Near $x = \pi/2$, $\cos x \approx -x + (\pi/2)$. (Example 4)

Example 4 FINDING A LINEARIZATION

Find the linearization of $f(x) = \cos x$ at $x = \pi/2$ (Figure 4.39).

Solution Since $f(\pi/2) = \cos(\pi/2) = 0$, $f'(x) = -\sin x$, and $f'(\pi/2) = -\sin(\pi/2) = -1$, we have

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= 0 + (-1)\left(x - \frac{\pi}{2}\right) \\ &= -x + \frac{\pi}{2}. \end{aligned}$$

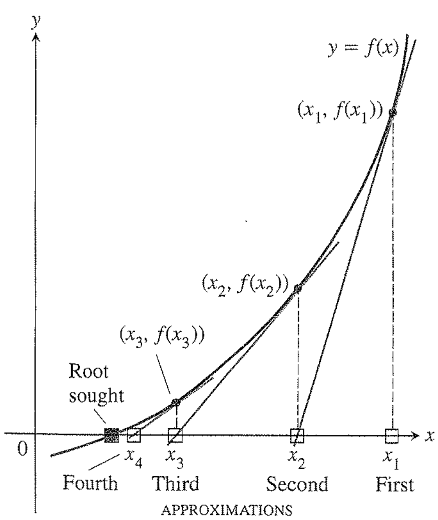


Figure 4.40 Usually the approximations rapidly approach an actual zero of $y = f(x)$.

Newton's Method

Newton's method is a numerical technique for approximating a zero of a function with zeros of its linearizations. Under favorable circumstances, the zeros of the linearizations *converge* rapidly to an accurate approximation. Many calculators use the method because it applies to a wide range of functions and usually gets results in only a few steps. Here is how it works.

To find a solution of an equation $f(x) = 0$, we begin with an initial estimate x_1, found either by looking at a graph or simply guessing. Then we use the tangent to the curve $y = f(x)$ at $(x_1, f(x_1))$ to approximate the curve (Figure 4.40). The point where the tangent crosses the x-axis is the next approximation x_2. The number x_2 is usually a better approximation to the solution than is x_1. The point where the tangent to the curve at $(x_2, f(x_2))$ crosses the x-axis is the next approximation x_3. We continue on, using each approximation to generate the next, until we are close enough to the zero to stop.

There is a formula for finding the $(n+1)$st approximation x_{n+1} from the nth approximation x_n. The point-slope equation for the tangent to the curve at $(x_n, f(x_n))$ is

$$y - f(x_n) = f'(x_n)(x - x_n).$$

Thus, the tangent is the graph of the linear function

$$L(x) = f(a) + f'(a)(x - a).$$

For as long as the line remains close to the graph of f, $L(x)$ gives a good approximation to $f(x)$.

Definition Linearization

If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a.

The approximation $f(x) \approx L(x)$ is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

Example 1 FINDING A LINEARIZATION

Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 0$ (Figure 4.38).

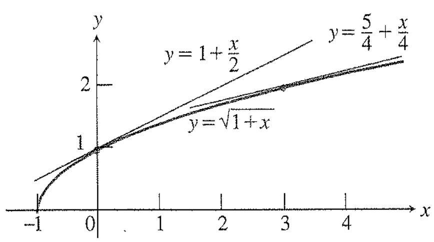


Figure 4.38 The graph of $f(x) = \sqrt{1 + x}$ and its linearization at $x = 0$ and $x = 3$. (Example 1)

Solution Since

$$f'(x) = \frac{1}{2}(1 + x)^{-1/2},$$

we have $f(0) = 1$, $f'(0) = 1/2$, and

$$L(x) = f(a) + f'(a)(x - a) = 1 + \frac{1}{2}(x - 0) = 1 + \frac{x}{2}.$$

Look at how accurate the approximation $\sqrt{1 + x} \approx 1 + (x/2)$ is for values of x near 0.

Approximation	|True Value − Approximation|
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	$< 10^{-2}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	$< 10^{-3}$
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	$< 10^{-5}$

As we move away from zero we lose accuracy. For example, for $x = 2$ the linearization gives 2 as the approximation for $\sqrt{3}$, which is not even accurate to one decimal place.

Do not be misled by the preceding calculations into thinking that whatever we do with a linearization is better done with a calculator. In practice, we would never use a linearization to find a particular square root. The utility of a linearization is its ability to replace a complicated formula by a simpler one over an entire interval of values. If we have to work with $\sqrt{1 + x}$ for x close to 0 and can tolerate the small amount of error involved, we can work with $1 + (x/2)$ instead. Of course, we then need to know how much error there is. We will have a full story on error in Chapter 9.

In Exercise 7 you will provide the details for the linearization in Example 2.

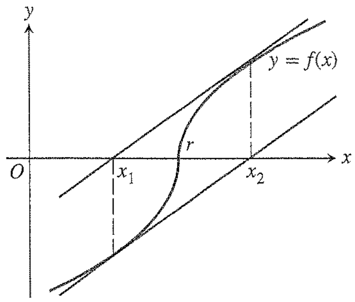


Figure 4.43 The graph of the function

$$f(x) = \begin{cases} -\sqrt{r-x}, & x < r \\ \sqrt{x-r}, & x \geq r. \end{cases}$$

If $x_1 = r - h$, then $x_2 = r + h$. Successive approximations go back and forth between these two values, and Newton's method fails to converge.

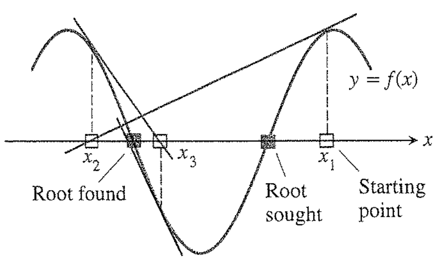


Figure 4.44 Newton's method may miss the zero you want if you start too far away.

Exploration 2 **Using Newton's Method on the Grapher**

Let $f(x) = x^3 + 3x + 1$. Here is a home screen program to perform the computations in Newton's method.

1. Let $y_1 = f(x)$ and $y_2 = \text{NDER}\, f(x)$.
2. Store $x_1 = -0.3$ into x.
3. Then store $x - (y_1/y_2)$ into x and press the "Enter" key over and over. Watch as the numbers converge to the zero of f.
4. Use different values for x_1 and repeat steps 2 and 3.
5. Write your own equation and use this approach to solve it using Newton's method. Compare your answer with the answer given by the built-in feature of your calculator that gives zeros of functions.

Newton's method does not work if $f'(x_1) = 0$. In that case, choose a new starting point.

Newton's method does not always converge. For instance (see Figure 4.43), successive approximations $r - h$ and $r + h$ can go back and forth between these two values, and no amount of iteration will bring us any closer to the zero r.

If Newton's method does converge, it converges to a zero of f. However, the method may converge to a zero that is different from the expected one if the starting value is not close enough to the zero sought. Figure 4.44 shows how this might happen.

Differentials

We sometimes use the notation dy/dx to represent the derivative y' of y with respect to x. Contrary to its appearance, it is not a ratio. We now introduce two new variables dx and dy with the property that if their ratio exists, it will be equal to the derivative.

Definition **Differentials**

Let $y = f(x)$ be a differentiable function. The **differential dx** is an independent variable. The **differential dy** is

$$dy = f'(x)\,dx.$$

Unlike the independent variable dx, the variable dy is always a dependent variable. It depends on both x and dx.

Example 6 FINDING THE DIFFERENTIAL *dy*

Find dy if

(a) $y = x^5 + 37x$. **(b)** $y = \sin 3x$.

Solution

(a) $dy = (5x^4 + 37)\,dx$ **(b)** $dy = (3 \cos 3x)\,dx$

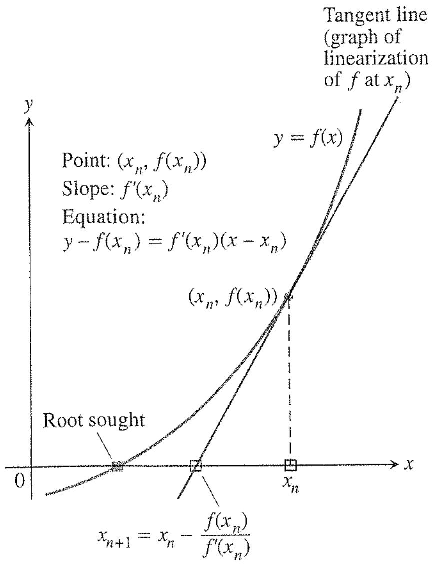
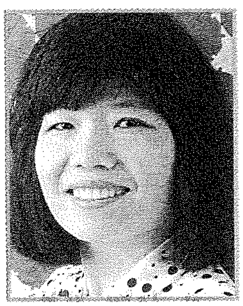


Figure 4.41 From x_n we go up to the curve and follow the tangent line down to find x_{n+1}.



Fan Chung Graham

1949–

"Don't be intimidated!" is Dr. Fan Chung Graham's advice to young women considering careers in mathematics. Fan Chung Graham came to the U.S. from Taiwan to earn a Ph.D. in Mathematics from the University of Pennsylvania. She worked in the field of combinatorics at Bell Labs and Bellcore, and then, in 1994, returned to her alma mater as a Professor of Mathematics. Her research interests include spectral graph theory, discrete geometry, algorithms, and communication networks.

We can find where it crosses the x-axis by setting $y = 0$ (Figure 4.41).

$$0 - f(x_n) = f'(x_n)(x - x_n)$$
$$-f(x_n) = f'(x_n) \cdot x - f'(x_n) \cdot x_n$$
$$f'(x_n) \cdot x = f'(x_n) \cdot x_n - f(x_n)$$
$$x = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{If } f'(x_n) \neq 0$$

This value of x is the next approximation x_{n+1}. Here is a summary of Newton's method.

Procedure for Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Example 5 USING NEWTON'S METHOD

Use Newton's method to solve $x^3 + 3x + 1 = 0$.

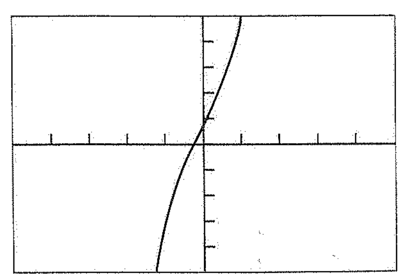
Solution Let $f(x) = x^3 + 3x + 1$, then $f'(x) = 3x^2 + 3$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + 3x_n + 1}{3x_n^2 + 3}.$$

The graph of f in Figure 4.42 suggests that $x_1 = -0.3$ is a good first approximation to the zero of f in the interval $-1 \leq x \leq 0$. Then,

$$x_1 = -0.3,$$
$$x_2 = -0.322324159,$$
$$x_3 = -0.3221853603,$$
$$x_4 = -0.3221853546.$$

The x_n for $n \geq 5$ all appear to equal x_4 on the calculator we used to do the computations. We conclude that the solution to the equation $x^3 + 3x + 1 = 0$ is about -0.3221853546.



[–5, 5] by [–5, 5]

Figure 4.42 Our graphing calculator's root finder reports -0.3221853546 as the zero of $f(x) = x^3 + 3x + 1$. (Example 5)

The corresponding change in L is

$$\begin{aligned}\Delta L &= L(a+dx) - L(a)\\ &= \underbrace{f(a) + f'(a)[(a+dx) - a]}_{L(a+dx)} - \underbrace{f(a)}_{L(a)}\\ &= f'(a)\,dx.\end{aligned}$$

Thus, the differential $df = f'(x)\,dx$ has a geometric interpretation: The value of df at $x = a$ is ΔL, the change in the linearization of f corresponding to the change dx.

Differential Estimate of Change

Let $f(x)$ be differentiable at $x = a$. The approximate change in the value of f when x changes from a to $a + dx$ is

$$df = f'(a)\,dx.$$

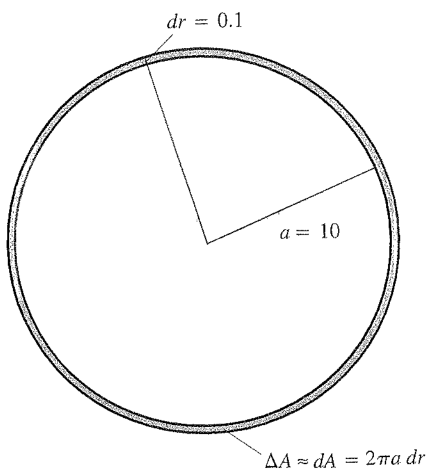


Figure 4.46 When dr is small compared with a, as it is when $dr = 0.1$ and $a = 10$, the differential $dA = 2\pi a\,dr$ gives a good estimate of ΔA. (Example 8)

Example 8 ESTIMATING CHANGE WITH DIFFERENTIALS

The radius r of a circle increases from $a = 10$ m to 10.1 m (Figure 4.46). Use dA to estimate the increase in the circle's area A. Compare this estimate with the true change ΔA.

Solution Since $A = \pi r^2$, the estimated increase is

$$dA = A'(a)\,dr = 2\pi a\,dr = 2\pi(10)(0.1) = 2\pi \text{ m}^2.$$

The true change is

$$\Delta A = \pi(10.1)^2 - \pi(10)^2 = (102.01 - 100)\pi = (\underbrace{2\pi}_{dA} + \underbrace{0.01\pi}_{\text{error}})\text{ m}^2.$$

Absolute, Relative, and Percentage Change

As we move from a to a nearby point $a + dx$, we can describe the change in f in three ways:

	True	Estimated
Absolute change	$\Delta f = f(a+dx) - f(a)$	$df = f'(a)\,dx$
Relative change	$\dfrac{\Delta f}{f(a)}$	$\dfrac{df}{f(a)}$
Percentage change	$\dfrac{\Delta f}{f(a)} \times 100$	$\dfrac{df}{f(a)} \times 100$

If $dx \neq 0$, then the quotient of the differential dy by the differential dx is equal to the derivative $f'(x)$ because

$$\frac{dy}{dx} = \frac{f'(x)\,dx}{dx} = f'(x).$$

We sometimes write

$$df = f'(x)\,dx$$

in place of $dy = f'(x)\,dx$, calling df the **differential of f.** For instance, if $f(x) = 3x^2 - 6$, then

$$df = d(3x^2 - 6) = 6x\,dx.$$

Every differentiation formula like

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{or} \quad \frac{d(\sin u)}{dx} = \cos u \frac{du}{dx}$$

has a corresponding differential form like

$$d(u+v) = du + dv \quad \text{or} \quad d(\sin u) = \cos u\,du.$$

Example 7 FINDING DIFFERENTIALS OF FUNCTIONS

(a) $d(\tan 2x) = \sec^2(2x)\,d(2x) = 2\sec^2 2x\,dx$

(b) $d\left(\dfrac{x}{x+1}\right) = \dfrac{(x+1)\,dx - x\,d(x+1)}{(x+1)^2} = \dfrac{x\,dx + dx - x\,dx}{(x+1)^2} = \dfrac{dx}{(x+1)^2}$

Estimating Change with Differentials

Suppose we know the value of a differentiable function $f(x)$ at a point a and we want to predict how much this value will change if we move to a nearby point $a + dx$. If dx is small, f and its linearization L at a will change by nearly the same amount (Figure 4.45). Since the values of L are simple to calculate, calculating the change in L offers a practical way to estimate the change in f.

In the notation of Figure 4.45, the change in f is

$$\Delta f = f(a + dx) - f(a).$$

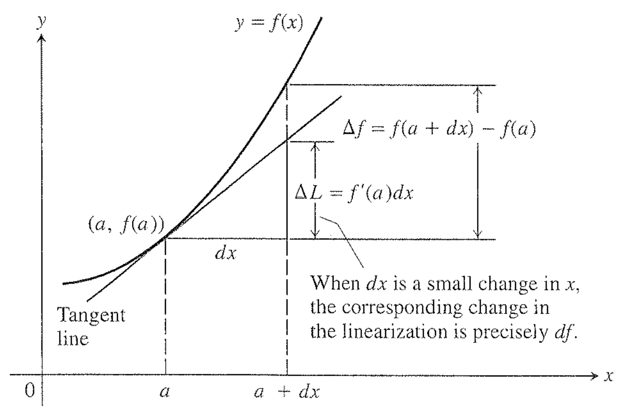
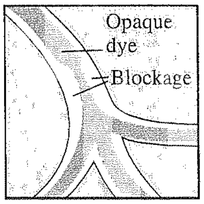


Figure 4.45 Approximating the change in the function f by the change in the linearization of f.

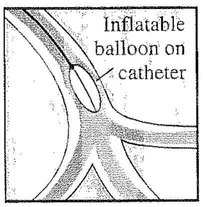
Angiography

An opaque dye is injected into a partially blocked artery to make the inside visible under X-rays. This reveals the location and severity of the blockage.



Angioplasty

A balloon-tipped catheter is inflated inside the artery to widen it at the blockage site.



Example 12 UNCLOGGING ARTERIES

In the late 1830s, the French physiologist Jean Poiseuille ("pwa-ZOY") discovered the formula we use today to predict how much the radius of a partially clogged artery has to be expanded to restore normal flow. His formula,

$$V = kr^4,$$

says that the volume V of fluid flowing through a small pipe or tube in a unit of time at a fixed pressure is a constant times the fourth power of the tube's radius r. How will a 10% increase in r affect V?

Solution The differentials of r and V are related by the equation

$$dV = \frac{dV}{dr}\, dr = 4kr^3\, dr.$$

The relative change in V is

$$\frac{dV}{V} = \frac{4kr^3 dr}{kr^4} = 4\frac{dr}{r}.$$

The relative change in V is 4 times the relative change in r, so a 10% increase in r will produce a 40% increase in the flow.

Sensitivity to Change

The equation $df = f'(x)\, dx$ tells how *sensitive* the output of f is to a change in input at different values of x. The larger the value of f' at x, the greater the effect of a given change dx.

Example 13 FINDING DEPTH OF A WELL

You want to calculate the depth of a well from the equation $s = 16t^2$ by timing how long it takes a heavy stone you drop to splash into the water below. How sensitive will your calculations be to a 0.1 sec error in measuring the time?

Solution The size of ds in the equation

$$ds = 32t\, dt$$

depends on how big t is. If $t = 2$ sec, the error caused by $dt = 0.1$ is only

$$ds = 32(2)(0.1) = 6.4 \text{ ft}.$$

Three seconds later at $t = 5$ sec, the error caused by the same dt is

$$ds = 32(5)(0.1) = 16 \text{ ft}.$$

Example 9 COMPUTING PERCENTAGE CHANGE

The estimated percentage change in the area of the circle in Example 8 is

$$\frac{dA}{A(a)} \times 100 = \frac{2\pi}{100\pi} \times 100 = 2\%.$$

The true percentage change is

$$\frac{\Delta A}{A(a)} \times 100 = \frac{2.01\pi}{100\pi} \times 100 = 2.01\%.$$

Usually it is not possible or easy to compute the exact (true) change as we did in Example 9. This is sometimes due to uncertainty in measurement, as shown in Example 10.

Example 10 ESTIMATING THE EARTH'S SURFACE AREA

Suppose the earth were a perfect sphere and we determined its radius to be 3959 ± 0.1 miles. What effect would the tolerance of ± 0.1 mi have on our estimate of the earth's surface area?

Note

If we underestimated the radius of the earth by 528 ft during a calculation of the earth's surface area, we would leave out an area the size of the state of Maryland.

Solution The surface area of a sphere of radius r is $S = 4\pi r^2$. The uncertainty in the calculation of S that arises from measuring r with a tolerance of dr miles is

$$dS = \left(\frac{dS}{dr}\right) dr = 8\pi r\, dr.$$

With $r = 3959$ and $dr = 0.1$, our estimate of S could be off by as much as

$$dS = 8\pi(3959)(0.1) \approx 9950 \text{ mi}^2,$$

to the nearest square mile, which is about the area of the state of Maryland.

Example 11 DETERMINING TOLERANCE

About how accurately should we measure the radius r of a sphere to calculate the surface area $S = 4\pi r^2$ within 1% of its true value?

Solution We want any inaccuracy in our measurement to be small enough to make the corresponding increment ΔS in the surface area satisfy the inequality

$$|\Delta S| \le \frac{1}{100} S = \frac{4\pi r^2}{100}.$$

We replace ΔS in this inequality by its approximation

$$dS = \left(\frac{dS}{dr}\right) dr = 8\pi r\, dr.$$

This gives

$$|8\pi r\, dr| \le \frac{4\pi r^2}{100}, \quad \text{or} \quad |dr| \le \frac{1}{8\pi r} \cdot \frac{4\pi r^2}{100} = \frac{1}{2} \cdot \frac{r}{100}.$$

We should measure r with an error dr that is no more than 0.5% of the true value.

23. $y = e^{\sin x}$, $x = \pi$, $dx = -0.1$

24. $y = 3 \csc \left(1 - \dfrac{x}{3}\right)$, $x = 1$, $dx = 0.1$

25. $y + xy - x = 0$, $x = 0$, $dx = 0.01$

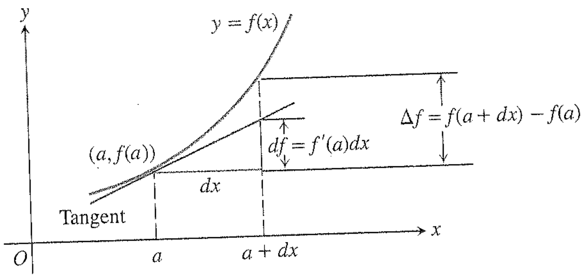
26. $y = \sec (x^2 - 1)$, $x = 1.5$, $dx = 0.05$

In Exercises 27–30, the function f changes value when x changes from a to $a + dx$. Find

(a) the absolute change $\Delta f = f(a + dx) - f(a)$.

(b) the estimated change $df = f'(a)\,dx$.

(c) the approximation error $|\Delta f - df|$.



27. $f(x) = x^2 + 2x$, $a = 0$, $dx = 0.1$

28. $f(x) = x^3 - x$, $a = 1$, $dx = 0.1$

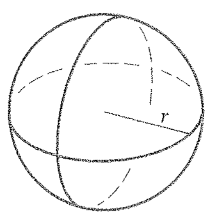
29. $f(x) = x^{-1}$, $a = 0.5$, $dx = 0.05$

30. $f(x) = x^4$, $a = 1$, $dx = 0.01$

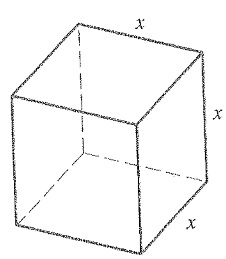
In Exercises 31–36, write a differential formula that estimates the given change in volume or surface area.

31. *Volume* The change in the volume $V = (4/3)\pi r^3$ of a sphere when the radius changes from a to $a + dr$

32. *Surface Area* The change in the surface area $S = 4\pi r^2$ of a sphere when the radius changes from a to $a + dr$



$V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$



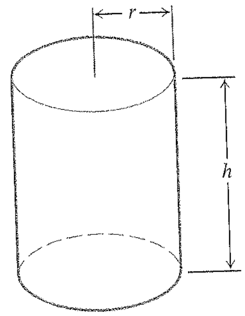
$V = x^3$, $S = 6x^2$

33. *Volume* The change in the volume $V = x^3$ of a cube when the edge lengths change from a to $a + dx$

34. *Surface Area* The change in the surface area $S = 6x^2$ of a cube when the edge lengths change from a to $a + dx$

35. *Volume* The change in the volume $V = \pi r^2 h$ of a right circular cylinder when the radius changes from a to $a + dr$ and the height does not change

36. *Surface Area* The change in the lateral surface area $S = 2\pi rh$ of a right circular cylinder when the height changes from a to $a + dh$ and the radius does not change



$V = \pi r^2 h$, $S = 2\pi rh$

37. *Linear Approximation* Let f be a function with $f(0) = 1$ and $f'(x) = \cos (x^2)$.

(a) Find the linearization of f at $x = 0$.

(b) Estimate the value of f at $x = 0.1$.

(c) **Writing to Learn** Do you think the actual value of f at $x = 0.1$ is greater than or less than the estimate in (b)? Explain.

38. *Expanding Circle* The radius of a circle is increased from 2.00 to 2.02 m.

(a) Estimate the resulting change in area.

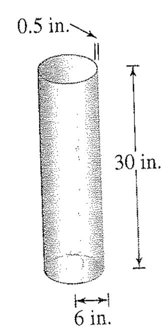
(b) Estimate as a percentage of the circle's original area.

39. *Growing Tree* The diameter of a tree was 10 in. During the following year, the circumference increased 2 in. About how much did the tree's diameter increase? the tree's cross section area?

40. *Percentage Error* The edge of a cube is measured as 10 cm with an error of 1%. The cube's volume is to be calculated from this measurement. Estimate the percentage error in the volume calculation.

41. *Percentage Error* About how accurately should you measure the side of a square to be sure of calculating the area to within 2% of its true value?

42. *Estimating Volume* Estimate the volume of material in a cylindrical shell with height 30 in., radius 6 in., and shell thickness 0.5 in.



Quick Review 4.5

In Exercises 1 and 2, find dy/dx.

1. $y = \sin(x^2 + 1)$
2. $y = \dfrac{x + \cos x}{x + 1}$

In Exercises 3 and 4, solve the equation graphically.

3. $xe^{-x} + 1 = 0$
4. $x^3 + 3x + 1 = 0$

In Exercises 5 and 6, let $f(x) = xe^{-x} + 1$. Write an equation for the line tangent to f at $x = c$.

5. $c = 0$
6. $c = -1$
7. Find where the tangent line in **(a)** Exercise 5 and **(b)** Exercise 6 crosses the x-axis.
8. Let $g(x)$ be the function whose graph is the tangent line to the graph of $f(x) = x^3 - 4x + 1$ at $x = 1$. Complete the table.

x	$f(x)$	$g(x)$
0.7		
0.8		
0.9		
1		
1.1		
1.2		
1.3		

In Exercises 9 and 10, graph $y = f(x)$ and its tangent line at $x = c$.

9. $c = 1.5, \quad f(x) = \sin x$
10. $c = 4, \quad f(x) = \begin{cases} -\sqrt{3 - x}, & x < 3 \\ \sqrt{x - 3}, & x \geq 3 \end{cases}$

Section 4.5 Exercises

In Exercises 1–6, **(a)** find the linearization $L(x)$ of $f(x)$ at $x = a$. **(b)** How accurate is the approximation $L(a + 0.1) \approx f(a + 0.1)$? See the comparisons following Example 1.

1. $f(x) = x^3 - 2x + 3, \quad a = 2$
2. $f(x) = \sqrt{x^2 + 9}, \quad a = -4$
3. $f(x) = x + \dfrac{1}{x}, \quad a = 1$
4. $f(x) = \ln(x + 1), \quad a = 0$
5. $f(x) = \tan x, \quad a = \pi$
6. $f(x) = \cos^{-1} x, \quad a = 0$
7. Show that the linearization of $f(x) = (1 + x)^k$ at $x = 0$ is $L(x) = 1 + kx$.
8. Use the linear approximation $(1 + x)^k \approx 1 + kx$ to find an approximation for the function $f(x)$ for values of x near zero.
   (a) $f(x) = (1 - x)^6$ **(b)** $f(x) = \dfrac{2}{1 - x}$
   (c) $f(x) = \dfrac{1}{\sqrt{1 + x}}$ **(d)** $f(x) = \sqrt{2 + x^2}$
   (e) $f(x) = (4 + 3x)^{1/3}$
   (f) $f(x) = \sqrt[3]{\left(1 - \dfrac{1}{2 + x}\right)^2}$
9. **Writing to Learn** Find the linearization of $f(x) = \sqrt{x + 1} + \sin x$ at $x = 0$. How is it related to the individual linearizations for $\sqrt{x + 1}$ and $\sin x$?
10. Use the linearization $(1 + x)^k \approx 1 + kx$ to approximate the following. State how accurate your approximation is.
    (a) $(1.002)^{100}$ **(b)** $\sqrt[3]{1.009}$

In Exercises 11–14, choose a linearization with center not at $x = a$ but at a nearby value at which the function and its derivative are easy to evaluate. State the linearization and the center.

11. $f(x) = 2x^2 + 4x - 3, \quad a = -0.9$
12. $f(x) = \sqrt[3]{x}, \quad a = 8.5$
13. $f(x) = \dfrac{x}{x + 1}, \quad a = 1.3$
14. $f(x) = \cos x, \quad a = 1.7$

In Exercises 15–18, use Newton's method to estimate all real solutions of the equation. Make your answers accurate to 6 decimal places.

15. $x^3 + x - 1 = 0$
16. $x^4 + x - 3 = 0$
17. $x^2 - 2x + 1 = \sin x$
18. $x^4 - 2 = 0$

In Exercises 19–26, **(a)** find dy, and **(b)** evaluate dy for the given value of x and dx.

19. $y = x^3 - 3x, \quad x = 2, \quad dx = 0.05$
20. $y = \dfrac{2x}{1 + x^2}, \quad x = -2, \quad dx = 0.1$
21. $y = x^2 \ln x, \quad x = 1, \quad dx = 0.01$
22. $y = x\sqrt{1 - x^2}, \quad x = 0, \quad dx = -0.2$

Extending the Ideas

53. *Formulas for Differentials* Verify the following formulas.

(a) $d(c) = 0$ (c a constant)

(b) $d(cu) = c\,du$ (c a constant)

(c) $d(u + v) = du + dv$

(d) $d(u \cdot v) = u\,dv + v\,du$

(e) $d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$

(f) $d(u^n) = nu^{n-1}\,du$

54. *Linearization* Show that the approximation of tan x by its linearization at the origin must improve as $x \to 0$ by showing that

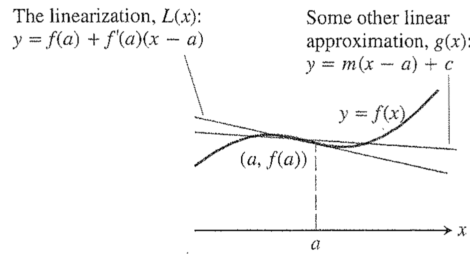
$$\lim_{x \to 0} \frac{\tan x}{x} = 1.$$

55. *The Linearization is the Best Linear Approximation* Suppose that $y = f(x)$ is differentiable at $x = a$ and that $g(x) = m(x - a) + c$ (m and c constants). If the error $E(x) = f(x) - g(x)$ were small enough near $x = a$, we might think of using g as a linear approximation of f instead of the linearization $L(x) = f(a) + f'(a)(x - a)$. Show that if we impose on g the conditions

i. $E(a) = 0$, The error is zero at $x = a$.

ii. $\lim_{x \to a} \frac{E(x)}{x - a} = 0$, The error is negligible when compared with $(x - a)$.

then $g(x) = f(a) + f'(a)(x - a)$. Thus, the linearization gives the only linear approximation whose error is both zero at $x = a$ and negligible in comparison with $(x - a)$.



4.6 Related Rates

Related Rate Equations • Solution Strategy • Simulating Related Motion

Related Rate Equations

Suppose that a particle $P(x, y)$ is moving along a curve C in the plane so that its coordinates x and y are differentiable functions of time t. If D is the distance from the origin to P, then using the Chain Rule we can find an equation that relates dD/dt, dx/dt, and dy/dt.

$$D = \sqrt{x^2 + y^2}$$

$$\frac{dD}{dt} = \frac{1}{2}(x^2 + y^2)^{-1/2}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)$$

Any equation involving two or more variables that are differentiable functions of time t can be used to find an equation that relates their corresponding rates.

Example 1 FINDING RELATED RATE EQUATIONS

Assume that the radius r and height h of a cone are differentiable functions of t and let V be the volume of the cone. Find an equation that relates dV/dt, dr/dt, and dh/dt.

Solution $V = \frac{\pi}{3}r^2h$ Cone volume formula

$$\frac{dV}{dt} = \frac{\pi}{3}\left(r^2 \cdot \frac{dh}{dt} + 2r\frac{dr}{dt} \cdot h\right) = \frac{\pi}{3}\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right)$$

43. *Estimating Volume* A surveyor is standing 30 ft from the base of a building. She measures the angle of elevation to the top of the building to be 75°. How accurately must the angle be measured for the percentage error in estimating the height of the building to be less than 4%?

44. *Tolerance* The height and radius of a right circular cylinder are equal, so the cylinder's volume is $V = \pi h^3$. The volume is to be calculated with an error of no more than 1% of the true value. Find approximately the greatest error that can be tolerated in the measurement of h, expressed as a percentage of h.

45. *Tolerance* **(a)** About how accurately must the interior diameter of a 10-m high cylindrical storage tank be measured to calculate the tank's volume to within 1% of its true value?

(b) About how accurately must the tank's exterior diameter be measured to calculate the amount of paint it will take to paint the side of the tank to within 5% of the true amount?

46. *Minting Coins* A manufacturer contracts to mint coins for the federal government. How much variation dr in the radius of the coins can be tolerated if the coins are to weigh within 1/1000 of their ideal weight? Assume the thickness does not vary.

47. *The Effect of Flight Maneuvers on the Heart* The amount of work done by the heart's main pumping chamber, the left ventricle, is given by the equation

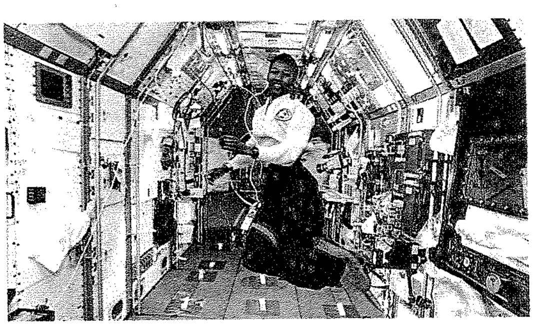
$$W = PV + \frac{V\delta v^2}{2g},$$

where W is the work per unit time, P is the average blood pressure, V is the volume of blood pumped out during the unit of time, δ ("delta") is the density of the blood, v is the average velocity of the exiting blood, and g is the acceleration of gravity.

When P, V, δ, and v remain constant, W becomes a function of g, and the equation takes the simplified form

$$W = a + \frac{b}{g} \ (a, b \text{ constant}).$$

As a member of NASA's medical team, you want to know how sensitive W is to apparent changes in g caused by flight maneuvers, and this depends on the initial value of g. As part of your investigation, you decide to compare the effect on W of a given change dg on the moon, where $g = 5.2$ ft/sec^2, with the effect the same change dg would have on Earth, where $g = 32$ ft/sec^2. Use the simplified equation above to find the ratio of dW_{moon} to dW_{Earth}.



48. *Measuring Acceleration of Gravity* When the length L of a clock pendulum is held constant by controlling its temperature, the pendulum's period T depends on the acceleration of gravity g. The period will therefore vary slightly as the clock is moved from place to place on the earth's surface, depending on the change in g. By keeping track of ΔT, we can estimate the variation in g from the equation $T = 2\pi(L/g)^{1/2}$ that relates T, g, and L.

(a) With L held constant and g as the independent variable, calculate dT and use it to answer (b) and (c).

(b) Writing to Learn If g increases, will T increase or decrease? Will a pendulum clock speed up or slow down? Explain.

(c) A clock with a 100-cm pendulum is moved from a location where $g = 980$ cm/sec^2 to a new location. This increases the period by $dT = 0.001$ sec. Find dg and estimate the value of g at the new location.

49. *Newton's Method* Suppose your first guess in using Newton's method is lucky in the sense that x_1 is a root of $f(x) = 0$. What happens to x_2 and later approximations?

50. *Oscillation* Show that if $h > 0$, applying Newton's method to

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ \sqrt{-x}, & x < 0 \end{cases}$$

leads to $x_2 = -h$ if $x_1 = h$, and to $x_2 = h$ if $x_1 = -h$. Draw a picture that shows what is going on.

51. *Approximations that Get Worse and Worse* Apply Newton's method to $f(x) = x^{1/3}$ with $x_1 = 1$, and calculate x_2, x_3, x_4, and x_5. Find a formula for $|x_n|$. What happens to $|x_n|$ as $n \to \infty$? Draw a picture that shows what is going on.

Exploration

52. *Quadratic Approximations*

(a) Let $Q(x) = b_0 + b_1(x - a) + b_2(x - a)^2$ be a quadratic approximation to $f(x)$ at $x = a$ with the properties:

i. $Q(a) = f(a)$,

ii. $Q'(a) = f'(a)$,

iii. $Q''(a) = f''(a)$.

Determine the coefficients b_0, b_1, and b_2.

(b) Find the quadratic approximation to $f(x) = 1/(1 - x)$ at $x = 0$.

(c) Graph $f(x) = 1/(1 - x)$ and its quadratic approximation at $x = 0$. Then zoom in on the two graphs at the point (0, 1). Comment on what you see.

(d) Find the quadratic approximation to $g(x) = 1/x$ at $x = 1$. Graph g and its quadratic approximation together. Comment on what you see.

(e) Find the quadratic approximation to $h(x) = \sqrt{1 + x}$ at $x = 0$. Graph h and its quadratic approximation together. Comment on what you see.

(f) What are the linearizations of f, g, and h at the respective points in (b), (d), and (e)?

Related Rate Problem Strategy

1. *Draw a picture and name the variables and constants.* Use t for time. Assume all variables are differentiable functions of t.
2. *Write down the numerical information* (in terms of the symbols you have chosen).
3. *Write down what we are asked to find* (usually a rate, expressed as a derivative).
4. *Write an equation that relates the variables.* You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
5. *Differentiate with respect to t.* Then express the rate you want in terms of the rate and variables whose values you know.
6. *Evaluate.* Use known values to find the unknown rate.

Example 3 A HIGHWAY CHASE

A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

Solution We carry out the steps of the strategy.

Step 1:

Picture and variables. We picture the car and cruiser in the coordinate plane, using the positive x-axis as the eastbound highway and the positive y-axis as the southbound highway (Figure 4.48). We let t represent time and set

x = position of car at time t,

y = position of cruiser at time t, and

s = distance between car and cruiser at time t.

We assume x, y, and s are differentiable functions of t.

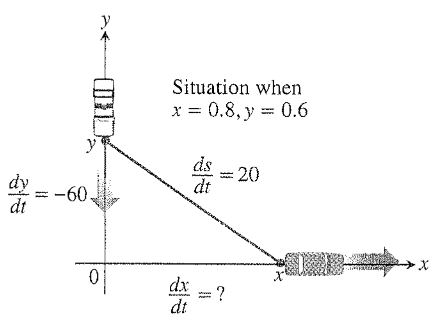


Figure 4.48 (Example 3)

Step 2:

Numerical information. At the instant in question,

$$x = 0.8 \text{ mi}, \quad y = 0.6 \text{ mi}, \quad \frac{dy}{dt} = -60 \text{ mph}, \quad \frac{ds}{dt} = 20 \text{ mph}.$$

Note dy/dt is negative because y is decreasing.

Step 3:

To find: dx/dt

Step 4:

How the variables are related: $s^2 = x^2 + y^2$
(We could also use $s = \sqrt{x^2 + y^2}$.)

Solution Strategy

How fast is a balloon rising at a given instant? How fast does the water level drop when a tank is drained at a certain rate? Questions like these ask us to calculate a rate that may be difficult to measure from a rate that we know, or is easy to measure. We begin by writing an equation that relates the variables involved and then differentiate it to get an equation that relates the rate we seek to the rates we know.

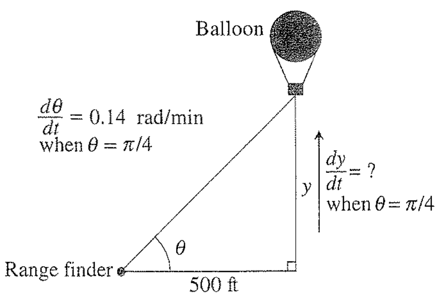


Figure 4.47 (Example 2)

Example 2 A RISING BALLOON

A hot-air balloon rising straight up from a level field is tracked by a range finder 500 ft from the lift-off point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

Solution We answer the question in six steps.

Step 1:

Draw a picture and name the variables and constants (Figure 4.47). The variables in the picture are

$\theta =$ the angle in radians the range finder makes with the ground,

$y =$ the height in feet of the balloon.

We let t represent time in minutes and assume θ and y are differentiable functions of t.

The one constant in the picture is the distance from the range finder to the lift-off point (500 ft). There is no need to give it a special symbol.

Step 2:

Write down the additional numerical information.

$$\frac{d\theta}{dt} = 0.14 \text{ rad/min} \quad \text{when} \quad \theta = \frac{\pi}{4}$$

Step 3:

Write down what we are to find. We want dy/dt when $\theta = \pi/4$.

Step 4:

Write an equation that relates the variables y and θ.

$$\frac{y}{500} = \tan\theta \quad \text{or} \quad y = 500 \tan\theta$$

Step 5:

Differentiate with respect to t using the Chain Rule. The result tells how dy/dt (which we want) is related to $d\theta/dt$ (which we know).

$$\frac{dy}{dt} = 500(\sec^2\theta)\frac{d\theta}{dt}$$

Step 6:

Evaluate with $\theta = \pi/4$ and $d\theta/dt = 0.14$ to find dy/dt.

$$\frac{dy}{dt} = 500(\sqrt{2})^2(0.14) = 140 \qquad \sec\frac{\pi}{4} = \sqrt{2}$$

Interpret

At the moment in question, the balloon is rising at the rate of 140 ft/min.

The units in the formula for *dy/dt* (Step 6)

A radian is the ratio of two lengths, meaning its units are length/length. In Example 2, the units are ft/ft. Radians per minute are (ft/ft)/min = ft/(ft × min). The secant of an angle is also a ratio of lengths. The units of the secant in Example 2 are ft/ft. For the equation

$$\frac{dy}{dt} = 500 \times (\sec\theta)^2 \times \frac{d\theta}{dt} = 140,$$

we therefore have

$$\frac{dy}{dt} = \text{ft} \times \left(\frac{\text{ft}}{\text{ft}}\right)^2 \times \frac{\text{ft}}{\text{ft} \times \text{min}} = \frac{\text{ft}}{\text{min}}.$$

Because the units of radians (length/length) cancel, we usually regard radians as *dimensionless* and don't write anything in. The trigonometric ratios (length/length) are dimensionless, too. With this understanding, the units of radians/minute, instead of being (length/length)/minute, are 1/minute or "per minute," the secant's units do not appear at all, and the units of

$$500 \times (\sqrt{2})^2 \times (0.14) = 140$$

are calculated as

$$\text{ft} \times \frac{1}{\text{min}} = \frac{\text{ft}}{\text{min}}.$$

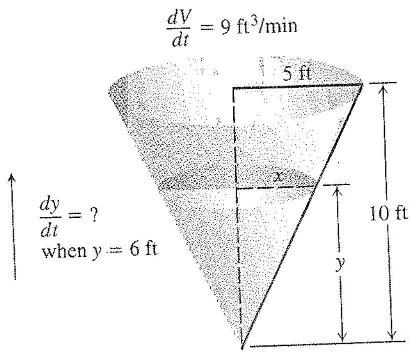


Figure 4.49 (Example 4)

This equation involves x as well as V and y. Because no information is given about x and dx/dt at the time in question, we need to eliminate x. The similar triangles in Figure 4.49 give us a way to express x in terms of y:

$$\frac{x}{y} = \frac{5}{10} \quad \text{or} \quad x = \frac{y}{2}.$$

Therefore,

$$V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}y^3.$$

Step 5:

Differentiate with respect to t.

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3y^2\frac{dy}{dt} = \frac{\pi}{4}y^2\frac{dy}{dt}$$

Step 6:

Evaluate. Use $y = 6$ and $dV/dt = 9$ to solve for dy/dt.

$$9 = \frac{\pi}{4}(6)^2\frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{1}{\pi} \approx 0.32$$

Interpret

At the moment in question, the water level is rising at about 0.32 ft/min.

Simulating Related Motion

Parametric mode on a grapher can be used to simulate the motion of moving objects when the motion of each can be expressed as a function of time.

Exploration 1 Sliding Ladder

A 13-ft ladder is leaning against a wall. Suppose that the base of the ladder slides away from the wall at the constant rate of 3 ft/sec.

1. Explain why the motion of the two ends of the ladder can be represented by the parametric equations:

$$x_1(t) = 3t, \quad y_1(t) = 0$$
$$x_2(t) = 0, \quad y_2(t) = \sqrt{13^2 - (3t)^2}.$$

2. What values of t make sense in this problem situation?
3. Use simultaneous mode. Give a viewing window that shows the action. (*Hint:* It may be helpful to "hide" the coordinate axes if your grapher has this feature. If not, adjust the parametric equations to move the action away from the axes.)
4. Use analytic methods to find the rates at which the top of the ladder is moving down the wall at $t = 0.5$, 1, 1.5, and 2 sec. In theory, how fast is the top of the ladder moving as it hits the ground?

Step 5:

Differentiate with respect to t.

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$= \frac{1}{\sqrt{x^2 + y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

Step 6:

Evaluate. Use $x = 0.8$, $y = 0.6$, $dy/dt = -60$, $ds/dt = 20$, and solve for dx/dt.

$$20 = \frac{1}{\sqrt{(0.8)^2 + (0.6)^2}}\left(0.8\frac{dx}{dt} + (0.6)(-60)\right)$$

$$\frac{dx}{dt} = \frac{20\sqrt{(0.8)^2 + (0.6)^2} + (0.6)(60)}{0.8} = 70$$

Interpret

At the moment in question, the car's speed is 70 mph.

Example 4 FILLING A CONICAL TANK

Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

Solution We carry out the steps of the strategy.

Step 1:

Picture and variables. Figure 4.49 on the next page shows a partially filled conical tank. The variables in the problem are

V = volume (ft^3) of the water in the tank at time t (min),

x = radius (ft) of the surface of the water at time t, and

y = depth (ft) of water in tank at time t.

We assume V, x, and y are differentiable functions of t. The constants are the dimensions of the tank.

Step 2:

Numerical information. At the time in question,

$$y = 6 \text{ ft} \quad \text{and} \quad \frac{dV}{dt} = 9 \text{ ft}^3/\text{min}.$$

Step 3:

To find: dy/dt

Step 4:

How the variables are related: The water forms a cone with volume

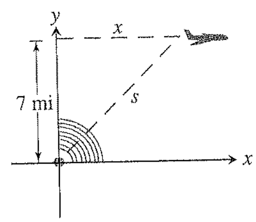
$$V = \frac{1}{3}\pi x^2 y.$$

10. *Changing Dimensions in a Rectangular Box* Suppose that the edge lengths x, y, and z of a closed rectangular box are changing at the following rates:

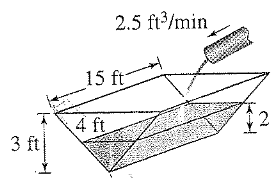
$$\frac{dx}{dt} = 1 \text{ m/sec}, \quad \frac{dy}{dt} = -2 \text{ m/sec}, \quad \frac{dz}{dt} = 1 \text{ m/sec}.$$

Find the rates at which the box's **(a)** volume, **(b)** surface area, and **(c)** diagonal length $s = \sqrt{x^2 + y^2 + z^2}$ are changing at the instant when $x = 4$, $y = 3$, and $z = 2$.

11. *Air Traffic Control* An airplane is flying at an altitude of 7 mi and passes directly over a radar antenna as shown in the figure. When the plane is 10 mi from the antenna ($s = 10$), the radar detects that the distance s is changing at the rate of 300 mph. What is the speed of the airplane at that moment?



12. *Filling a Trough* A trough is 15 ft long and 4 ft across the top as shown in the figure. Its ends are isosceles triangles with height 3 ft. Water runs into the trough at the rate of 2.5 ft³/min. How fast is the water level rising when it is 2 ft deep?

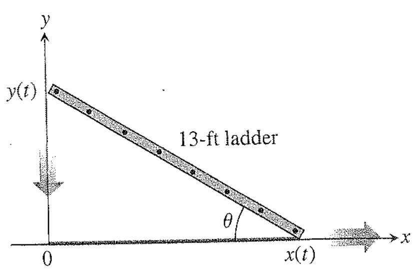


13. *Sliding Ladder* A 13-ft ladder is leaning against a house (see figure) when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.

(a) How fast is the top of the ladder sliding down the wall at that moment?

(b) At what rate is the area of the triangle formed by the ladder, wall, and ground changing at that moment?

(c) At what rate is the angle θ between the ladder and the ground changing at that moment?



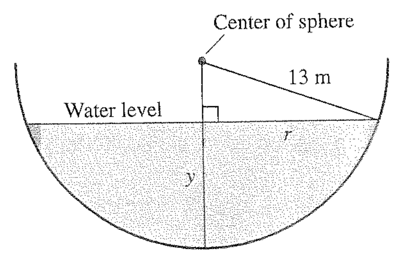
14. *Flying a Kite* Inge flies a kite at a height of 300 ft, the wind carrying the kite horizontally away at a rate of 25 ft/sec. How fast must she let out the string when the kite is 500 ft away from her?

15. *Boring a Cylinder* The mechanics at Lincoln Automotive are reboring a 6-in. deep cylinder to fit a new piston. The machine they are using increases the cylinder's radius one-thousandth of an inch every 3 min. How rapidly is the cylinder volume increasing when the bore (diameter) is 3.800 in.?

16. *Growing Sand Pile* Sand falls from a conveyor belt at the rate of 10 m³/min onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast are the **(a)** height and **(b)** radius changing when the pile is 4 m high? Give your answer in cm/min.

17. *Draining Conical Reservoir* Water is flowing at the rate of 50 m³/min from a concrete conical reservoir (vertex down) of base radius 45 m and height 6 m. **(a)** How fast is the water level falling when the water is 5 m deep? **(b)** How fast is the radius of the water's surface changing at that moment? Give your answer in cm/min.

18. *Draining Hemispherical Reservoir* Water is flowing at the rate of 6 m³/min from a reservoir shaped like a hemispherical bowl of radius 13 m, shown here in profile. Answer the following questions given that the volume of water in a hemispherical bowl of radius R is $V = (\pi/3)y^2(3R - y)$ when the water is y units deep.



(a) At what rate is the water level changing when the water is 8 m deep?

(b) What is the radius r of the water's surface when the water is y m deep?

(c) At what rate is the radius r changing when the water is 8 m deep?

19. *Growing Raindrop* Suppose that a droplet of mist is a perfect sphere and that, through condensation, the droplet picks up moisture at a rate proportional to its surface area. Show that under these circumstances the droplet's radius increases at a constant rate.

20. *Inflating Balloon* A spherical balloon is inflated with helium at the rate of 100π ft³/min.

(a) How fast is the balloon's radius increasing at the instant the radius is 5 ft?

(b) How fast is the surface area increasing at that instant?

Quick Review 4.6

In Exercises 1 and 2, find the distance between the points A and B.

1. $A(0, 5)$, $B(7, 0)$ **2.** $A(0, a)$, $B(b, 0)$

In Exercises 3–6, find dy/dx.

3. $2xy + y^2 = x + y$ **4.** $x \sin y = 1 - xy$

5. $x^2 = \tan y$ **6.** $\ln (x + y) = 2x$

In Exercises 7 and 8, find a parametrization for the line segment with endpoints A and B.

7. $A(-2, 1)$, $B(4, -3)$ **8.** $A(0, -4)$, $B(5, 0)$

In Exercises 9 and 10, let $x = 2 \cos t$, $y = 2 \sin t$. Find a parameter interval that produces the indicated portion of the graph.

9. The portion in the second and third quadrants, including the points on the axes.

10. The portion in the fourth quadrant, including the points on the axes.

Section 4.6 Exercises

In Exercises 1–41, assume all variables are differentiable functions of t.

1. *Area* The radius r and area A of a circle are related by the equation $A = \pi r^2$. Write an equation that relates dA/dt to dr/dt.

2. *Surface Area* The radius r and surface area S of a sphere are related by the equation $S = 4\pi r^2$. Write an equation that relates dS/dt to dr/dt.

3. *Volume* The radius r, height h, and volume V of a right circular cylinder are related by the equation $V = \pi r^2 h$.

(a) How is dV/dt related to dh/dt if r is constant?

(b) How is dV/dt related to dr/dt if h is constant?

(c) How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant?

4. *Electrical Power* The power P (watts) of an electric circuit is related to the circuit's resistance R (ohms) and current I (amperes) by the equation $P = RI^2$.

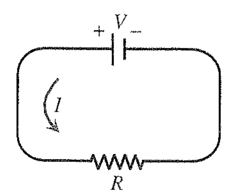
(a) How is dP/dt related to dR/dt and dI/dt?

(b) How is dR/dt related to dI/dt if P is constant?

5. *Diagonals* If x, y, and z are lengths of the edges of a rectangular box, the common length of the box's diagonals is $s = \sqrt{x^2 + y^2 + z^2}$. How is ds/dt related to dx/dt, dy/dt, and dz/dt?

6. *Area* If a and b are the lengths of two sides of a triangle, and θ the measure of the included angle, the area A of the triangle is $A = (1/2)\, ab \sin \theta$. How is dA/dt related to da/dt, db/dt, and $d\theta/dt$?

7. *Changing Voltage* The voltage V (volts), current I (amperes), and resistance R (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 volt/sec while I is decreasing at the rate of 1/3 amp/sec. Let t denote time in sec.



(a) What is the value of dV/dt?

(b) What is the value of dI/dt?

(c) Write an equation that relates dR/dt to dV/dt and dI/dt.

(d) **Writing to Learn** Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amp. Is R increasing, or decreasing? Explain.

8. *Heating a Plate* When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/sec. At what rate is the plate's area increasing when the radius is 50 cm?

9. *Changing Dimensions in a Rectangle* The length ℓ of a rectangle is decreasing at the rate of 2 cm/sec while the width w is increasing at the rate of 2 cm/sec. When $\ell = 12$ cm and $w = 5$ cm, find the rates of change of

(a) the area, (b) the perimeter, and

(c) the length of a diagonal of the rectangle.

(d) **Writing to Learn** Which of these quantities are decreasing, and which are increasing? Explain.

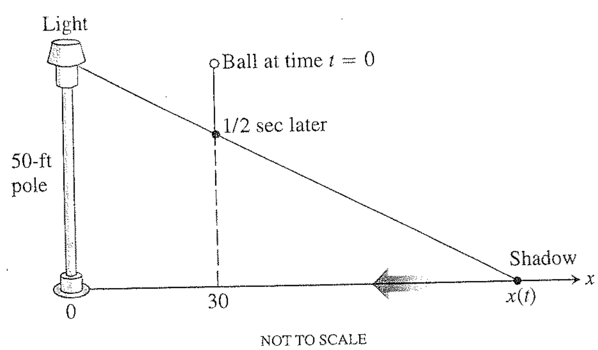
26. *Particle Motion* A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its x-coordinate (in meters) increases at a constant rate of 10 m/sec. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = 3$?

27. *Particle Motion* A particle moves from right to left along the parabolic curve $y = \sqrt{-x}$ in such a way that its x-coordinate (in meters) decreases at the rate of 8 m/sec. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = -4$?

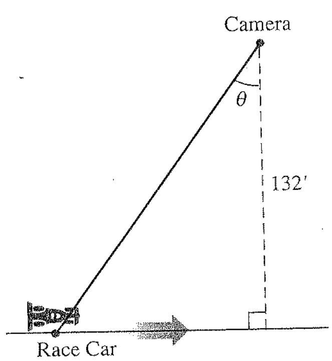
28. *Particle Motion* A particle $P(x, y)$ is moving in the coordinate plane in such a way that $dx/dt = -1$ m/sec and $dy/dt = -5$ m/sec. How fast is the particle's distance from the origin changing as it passes through the point (5, 12)?

29. *Moving Shadow* A man 6 ft tall walks at the rate of 5 ft/sec toward a streetlight that is 16 ft above the ground. At what rate is the length of his shadow changing when he is 10 ft from the base of the light?

30. *Moving Shadow* A light shines from the top of a pole 50 ft high. A ball is dropped from the same height from a point 30 ft away from the light as shown in the figure. How fast is the shadow of the ball moving along the ground 1/2 sec later? (Assume the ball falls a distance $s = 16t^2$ in t sec.)

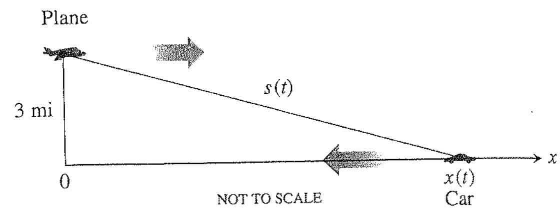


31. *Moving Race Car* You are videotaping a race from a stand 132 ft from the track, following a car that is moving at 180 mph (264 ft/sec) as shown in the figure. About how fast will your camera angle θ be changing when the car is right in front of you? a half second later?

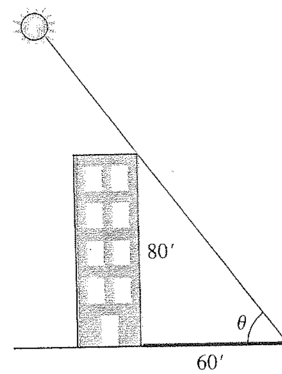


32. *Melting Ice* A spherical iron ball is coated with a layer of ice of uniform thickness. If the ice melts at the rate of 8 mL/min, how fast is the outer surface area of ice decreasing when the outer diameter (ball plus ice) is 20 cm?

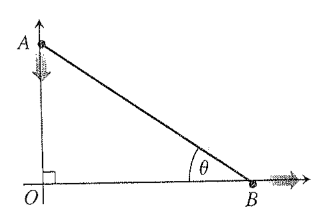
33. *Speed Trap* A highway patrol airplane flies 3 mi above a level, straight road at a constant rate of 120 mph. The pilot sees an oncoming car and with radar determines that at the instant the line-of-sight distance from plane to car is 5 mi the line-of-sight distance is decreasing at the rate of 160 mph. Find the car's speed along the highway.



34. *Building's Shadow* On a morning of a day when the sun will pass directly overhead, the shadow of an 80-ft building on level ground is 60 ft long as shown in the figure. At the moment in question, the angle θ the sun makes with the ground is increasing at the rate of 0.27°/min. At what rate is the shadow length decreasing? Express your answer in in./min, to the nearest tenth. (Remember to use radians.)



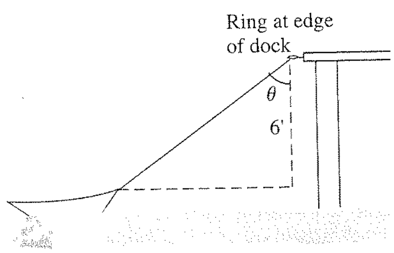
35. *Walkers* A and B are walking on straight streets that meet at right angles. A approaches the intersection at 2 m/sec and B moves away from the intersection at 1 m/sec as shown in the figure. At what rate is the angle θ changing when A is 10 m from the intersection and B is 20 m from the intersection? Express your answer in degrees per second to the nearest degree.



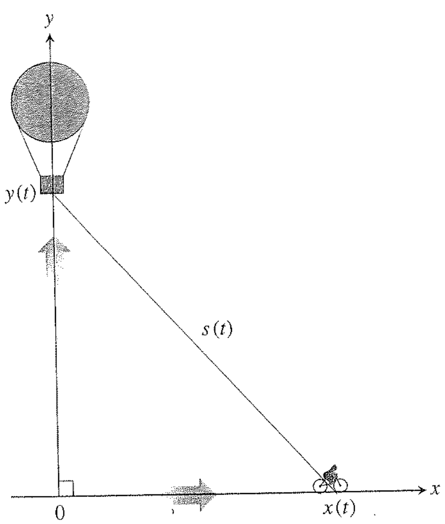
21. *Hauling in a Dinghy* A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 ft above the bow as shown in the figure. The rope is hauled in at the rate of 2 ft/sec.

(a) How fast is the boat approaching the dock when 10 ft of rope are out?

(b) At what rate is angle θ changing at that moment?



22. *Rising Balloon* A balloon is rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 ft above the ground, a bicycle moving at a constant rate of 17 ft/sec passes under it. How fast is the distance between the bicycle and balloon increasing 3 sec later (see figure)?



23. *Cost, Revenue, and Profit* A company can manufacture x items at a cost of $c(x)$ dollars, a sales revenue of $r(x)$ dollars, and a profit of $p(x) = r(x) - c(x)$ dollars (all amounts in thousands). Find dc/dt, dr/dt, and dp/dt for the following values of x and dx/dt.

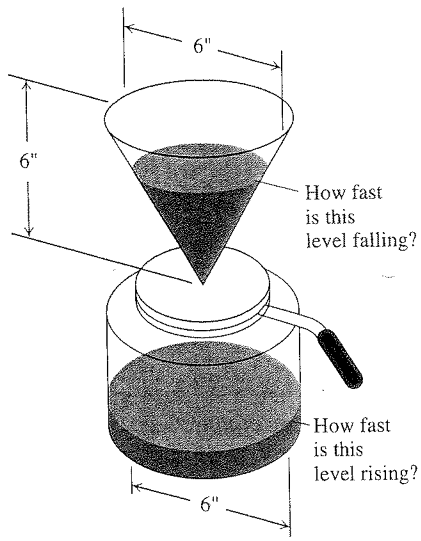
(a) $r(x) = 9x$, $c(x) = x^3 - 6x^2 + 15x$, and $dx/dt = 0.1$ when $x = 2$.

(b) $r(x) = 70x$, $c(x) = x^3 - 6x^2 + 45/x$, and $dx/dt = 0.05$ when $x = 1.5$.

24. *Making Coffee* Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 in^3/min.

(a) How fast is the level in the pot rising when the coffee in the cone is 5 in. deep?

(b) How fast is the level in the cone falling at that moment?



25. *Cardiac Output* *Work in groups of two or three.* In the late 1860s, Adolf Fick, a professor of physiology in the Faculty of Medicine in Würtzberg, Germany, developed one of the methods we use today for measuring how much blood your heart pumps in a minute. Your cardiac output as you read this sentence is probably about 7 liters a minute. At rest it is likely to be a bit under 6 L/min. If you are a trained marathon runner running a marathon, your cardiac output can be as high as 30 L/min.

Your cardiac output can be calculated with the formula

$$y = \frac{Q}{D},$$

where Q is the number of milliliters of CO_2 you exhale in a minute and D is the difference between the CO_2 concentration (mL/L) in the blood pumped to the lungs and the CO_2 concentration in the blood returning from the lungs. With $Q = 233$ mL/min and $D = 97 - 56 = 41$ mL/L,

$$y = \frac{233 \text{ mL/min}}{41 \text{ mL/L}} \approx 5.68 \text{ L/min},$$

fairly close to the 6 L/min that most people have at basal (resting) conditions. (Data courtesy of J. Kenneth Herd, M.D., Quillan College of Medicine, East Tennessee State University.)

Suppose that when $Q = 233$ and $D = 41$, we also know that D is decreasing at the rate of 2 units a minute but that Q remains unchanged. What is happening to the cardiac output?

36. *Moving Ships* Two ships are steaming away from a point O along routes that make a 120° angle. Ship A moves at 14 knots (nautical miles per hour; a nautical mile is 2000 yards). Ship B moves at 21 knots. How fast are the ships moving apart when $OA = 5$ and $OB = 3$ nautical miles?

In Exercises 37 and 38, a particle is moving along the curve $y = f(x)$.

37. Let $y = f(x) = \dfrac{10}{1 + x^2}$.

If $dx/dt = 3$ cm/sec, find dy/dt at the point where

(a) $x = -2$. **(b)** $x = 0$. **(c)** $x = 20$.

38. Let $y = f(x) = x^3 - 4x$.

If $dx/dt = -2$ cm/sec, find dy/dt at the point where

(a) $x = -3$. **(b)** $x = 1$. **(c)** $x = 4$.

Extending the Ideas

39. *Motion along a Circle* A wheel of radius 2 ft makes 8 revolutions about its center every second.

(a) Explain how the parametric equations

$$x = 2\cos\theta, \quad y = 2\sin\theta$$

can be used to represent the motion of the wheel.

(b) Express θ as a function of time t.

(c) Find the rate of horizontal movement and the rate of vertical movement of a point on the edge of the wheel when it is at the position given by $\theta = \pi/4$, $\pi/2$, and π.

40. *Ferris Wheel* A Ferris wheel with radius 30 ft makes one revolution every 10 sec.

(a) Assume that the center of the Ferris wheel is located at the point (0, 40), and write parametric equations to model its motion. (*Hint:* See Exercise 39.)

(b) At $t = 0$ the point P on the Ferris wheel is located at (30, 40). Find the rate of horizontal movement, and the rate of vertical movement of the point P when $t = 5$ sec and $t = 8$ sec.

41. *Industrial Production* **(a)** Economists often use the expression "rate of growth" in relative rather than absolute terms. For example, let $u = f(t)$ be the number of people in the labor force at time t in a given industry. (We treat this function as though it were differentiable even though it is an integer-valued step function.)

Let $v = g(t)$ be the average production per person in the labor force at time t. The total production is then $y = uv$. If the labor force is growing at the rate of 4% per year ($du/dt = 0.04u$) and the production per worker is growing at the rate of 5% per year ($dv/dt = 0.05v$), find the rate of growth of the total production, y.

(b) Suppose that the labor force in (a) is decreasing at the rate of 2% per year while the production per person is increasing at the rate of 3% per year. Is the total production increasing, or is it decreasing, and at what rate?

Chapter 4 Key Terms