Focus On
PREREQUISITE SKILLS

To be successful in this chapter, you'll need to understand these concepts and be able to apply them. Refer to the examples or lesson in parentheses if you need more review before beginning the chapter.

Rewrite a difference as a sum.

Example Rewrite $7 - 10$ as a sum.

$7 - 10 = 7 + (-10)$ *To subtract an integer, add its additive inverse.*

Example Rewrite $2x^2 - 4y^2$ as a sum.

$2x^2 - 4y^2 = 2x^2 + (-4y^2)$ *To subtract an expression, add its additive inverse.*

Rewrite each difference as a sum.

1. $-8 - 12$
2. $3ab - 7bc$
3. $6ax^2 - 15x^2y$
4. $10 - 2x - 3z$

Use the Distributive Property to simplify expressions. (Lesson 1–2)

Use the Distributive Property to rewrite each expression without parentheses.

5. $-3(5y^3 + 2y^2 - 5y)$
6. $-1(a + 1)$
7. $2.7x(a + 3b)$
8. $-\frac{2}{3}(4 + 3c)$

Determine the sets of numbers to which a given number belongs. (Lesson 1–2)

Find the value of each expression. Then name the sets of numbers to which each value belongs.

9. $3.8 + 4.1$
10. $26 \div (-2)$
11. $2^2 + 3^3$
10. $\sqrt{5 + 2}$

Focus On
READING SKILLS

In this chapter, you will learn about ***polynomials*** and ***radical expressions***. A polynomial is an algebraic expression that is the sum of one, two, three, or many monomials. The prefix "poly" means many. Some polynomials have specific names. A monomial has one term, since the prefix "mono" means one. A binomial has two terms, since the prefix "bi" means two. A trinomial has three terms, since the prefix "tri" means three. Mathematical terms often have prefixes that give you clues about the meanings of the terms.

5–1 Monomials

What YOU'LL LEARN

- To multiply and divide monomials,
- to represent numbers in scientific notation, and
- to multiply and divide expressions written in scientific notation.

Why IT'S IMPORTANT

You can use monomials to solve problems involving science and economics.

CONNECTION

Science

Exponents are used to express very large or very small numbers in **scientific notation.** A number is in scientific notation when it is in the form $a \times 10^n$, where $1 \leq a < 10$ and n is an integer.

The table below shows examples of numbers written in scientific notation.

Fact	Numeral	Scientific Notation
The temperatures produced in the center of a thermonuclear fusion bomb are as high as 400 million degrees Celsius.	400,000,000	4.0×10^{8}
The most powerful laser is the Nova at the Lawrence Hall of Science in California. It generates 100 trillion watts of power.	100,000,000,000,000	1.0×10^{14}
The most powerful microscope is capable of focusing to one-hundredth the diameter of an atom, measured in meters.	0.0000000003	3×10^{-10}

Notice that the last number in the table is written using negative exponents. Negative exponents are another way of expressing the inverse of a number. For example, $\frac{1}{x^2}$ can be written as x^{-2}.

Negative Exponents	For any real number a, and any integer n, where $a \neq 0$, $a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$.

Example 1 **Express each number in scientific notation.**

a. 2,340,000

$$2{,}340{,}000 = 2.34 \times 1{,}000{,}000$$
$$= 2.34 \times 10^6$$

b. 0.00012

$$0.00012 = 1.2 \times 0.0001$$
$$= 1.2 \times \frac{1}{10^4} \quad 0.0001 = \frac{1}{10{,}000} \text{ or } \frac{1}{10^4}$$
$$= 1.2 \times 10^{-4}$$

EXPLORATION — PROGRAMMING

Use the following program to explore how a TI-83 graphing calculator expresses small numbers.

```
PROGRAM: SMALLNOS
:For (N,1000,10000,1000)
:Disp "1/N IF N IS",N,"EQUALS",1/N
:Pause
:End
```

The :For statement tells the calculator to evaluate N for values from 1000 to 10,000 in increments of 1000.

The calculator will pause after each calculation. Press ENTER to continue for the next value.

Your Turn

a. How would you edit the program so that it will display N for values from 10,000 to 100,000 in increments of 1000? Rerun the program for these values.

b. How does your calculator display very small numbers? Give an example.

c. How might this differ from other scientific calculator displays?

TECHNOLOGY Tips

To enter a number that is already in scientific notation into a graphing calculator, use the [10ˣ] key. For example, 2.34×10^5 would be entered as 2.34 [×] [10ˣ] 5.

Exponents are also used in algebraic expressions called **monomials.** A monomial is an expression that is a number, a variable, or the product of a number and one or more variables. Some examples of monomials are $5c$, $-a$, 17, x^3, and $\frac{1}{2}x^4y^2$. Monomials cannot contain variables whose exponents cannot be written as whole numbers. Thus, expressions such as $\frac{1}{n^2}$ and $\sqrt{n}$ are not monomials. *Why?*

Constants are monomials that contain no variables. The numerical factor of a monomial is the **coefficient** of the variable. For example, the coefficient of m in $-6m$ is -6. The **degree** of a monomial is the sum of the exponents of its variables. For example, the degree of $12g^7h^4$ is $7 + 4$ or 11. The degree of a nonzero constant is 0.

The term <u>*power*</u> *is sometimes used to refer to the exponent itself.*

A **power** is an expression in the form of x^n. To **simplify** an expression containing powers means to rewrite the expression without parentheses or negative exponents.

Example 2 **Simplify $(2x^2y^3)(-5x^4y^2)$.**

$$(2x^2y^3)(-5x^4y^2) = (2 \cdot x \cdot x \cdot y \cdot y \cdot y)(-5 \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y)$$ *Definition of exponents*

$$= 2(-5) \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y$$ *Commutative property*

$$= -10x^6y^5$$

Example 2 suggests the following property of exponents.

Multiplying Powers	For any real number a and integers m and n, $a^m \cdot a^n = a^{m+n}$.

To multiply powers of the same variable, you add the exponents. Knowing this, it seems reasonable to expect that when dividing powers, you would subtract exponents. Consider $\frac{x^7}{x^4}$.

$$\frac{x^7}{x^4} = \frac{\overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{x}} \cdot x \cdot x \cdot x}{\underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{x}}} \qquad \textit{Remember that } x \textit{ cannot equal } 0.$$

$$= x \cdot x \cdot x$$

$$= x^3 \qquad \textit{Note that } x^3 = x^{7-4}.$$

It appears that our hypothesis is true. To divide powers of the same base, you subtract exponents. This is stated formally below.

Dividing Powers	**For any real number a, except $a = 0$, and integers m and n, $\frac{a^m}{a^n} = a^{m-n}$.**

In the next example, the check uses the definition of exponents to verify the rule for division of powers.

Example 3 **Simplify $\frac{p^5}{p^9}$. Assume that $p \neq 0$.**

$$\frac{p^5}{p^9} = p^{5-9} \qquad \textit{Dividing powers}$$

$$= p^{-4} \text{ or } \frac{1}{p^4} \qquad \textit{Remember that a simplified expression cannot contain negative exponents.}$$

Check: $$\frac{p^5}{p^9} = \frac{\overset{1}{\cancel{p}} \cdot \overset{1}{\cancel{p}} \cdot \overset{1}{\cancel{p}} \cdot \overset{1}{\cancel{p}} \cdot \overset{1}{\cancel{p}}}{\underset{1}{\cancel{p}} \cdot \underset{1}{\cancel{p}} \cdot \underset{1}{\cancel{p}} \cdot \underset{1}{\cancel{p}} \cdot \underset{1}{\cancel{p}} \cdot p \cdot p \cdot p \cdot p}$$

$$= \frac{1}{p^4} \text{ or } p^{-4}$$

Let's use the property of division of powers and the definition of exponents to simplify $\frac{y^5}{y^5}$.

Method 1

$$\frac{y^5}{y^5} = y^{5-5} \qquad \textit{Dividing powers}$$

$$= y^0$$

Method 2

$$\frac{y^5}{y^5} = \frac{\overset{1}{\cancel{y}} \cdot \overset{1}{\cancel{y}} \cdot \overset{1}{\cancel{y}} \cdot \overset{1}{\cancel{y}} \cdot \overset{1}{\cancel{y}}}{\underset{1}{\cancel{y}} \cdot \underset{1}{\cancel{y}} \cdot \underset{1}{\cancel{y}} \cdot \underset{1}{\cancel{y}} \cdot \underset{1}{\cancel{y}}}$$

$$= 1$$

Since $\frac{y^5}{y^5}$ cannot have two different values, we can conclude that $y^0 = 1$, where $y \neq 0$. In general, any nonzero number raised to the zero power is equal to 1.

The properties we have presented can be used to verify other properties of powers listed below.

Properties of Powers

Suppose m and n are integers and a and b are real numbers. Then the following properties hold.

Power of a Power: $(a^m)^n = a^{mn}$

Power of a Product: $(ab)^m = a^m b^m$

Power of a Quotient: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$ and

$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$ or $\frac{b^n}{a^n}, a \neq 0, b \neq 0$

Example 4 **Simplify each expression.**

a. $(a^4)^5$

$$(a^4)^5 = a^{4(5)} \quad \textit{Power of a power}$$
$$= a^{20}$$

b. $(-5p^2s^4)^3$

$$(-5p^2s^4)^3 = (-5)^3 \cdot (p^2)^3 \cdot (s^4)^3 \quad \textit{Power of a product}$$
$$= -125p^6s^{12}$$

c. $\left(\frac{-2m}{n}\right)^4$

$$\left(\frac{-2m}{n}\right)^4 = \frac{(-2m)^4}{n^4} \quad \textit{Power of a quotient}$$
$$= \frac{(-2)^4\, m^4}{n^4}$$
$$= \frac{16m^4}{n^4}$$

d. $\left(\frac{a}{3}\right)^{-2}$

$$\left(\frac{a}{3}\right)^{-2} = \left(\frac{3}{a}\right)^2 \quad \textit{Definition of negative exponent}$$
$$= \frac{3^2}{a^2} \text{ or } \frac{9}{a^2} \quad \textit{Power of a quotient}$$

To simplify some expressions, you must use several of the properties of powers.

Example 5 **Simplify** $\left(\frac{-4x^{2n}}{x^{3n}z^2}\right)^3$.

Method 1

$$\left(\frac{-4x^{2n}}{x^{3n}z^2}\right)^3 = \frac{(-4x^{2n})^3}{(x^{3n}z^2)^3} \quad \textit{Power of a quotient}$$
$$= \frac{(-4)^3(x^{2n})^3}{(x^{3n})^3(z^2)^3} \quad \textit{Power of a product}$$
$$= \frac{-64x^{6n}}{x^{9n}z^6} \quad \textit{Power of a power}$$
$$= \frac{-64x^{6n-9n}}{z^6} \quad \textit{Dividing powers}$$
$$= \frac{-64x^{-3n}}{z^6} \text{ or } \frac{-64}{x^{3n}z^6}$$

Method 2

Simplify the fraction first before cubing.

$$\left(\frac{-4x^{2n}}{x^{3n}z^2}\right)^3 = \left(\frac{-4x^{2n-3n}}{z^2}\right)^3$$
$$= \left(\frac{-4}{x^nz^2}\right)^3$$
$$= \frac{(-4)^3}{(x^n)^3(z^2)^3}$$
$$= \frac{-64}{x^{3n}z^6}$$

You can also multiply and divide expressions involving numbers written in scientific notation. A calculator provides the most efficient method for finding the product or quotient.

Example 6 **Use a scientific calculator to evaluate each expression.**

a. $(3.69 \times 10^{-5})(4.1 \times 10^{8})$

Estimate: $(4 \times 10^{-5})(4 \times 10^{8}) = 16 \times 10^{3}$ *or 16,000*

3.69 [EXP] [+/−] 5 [×] 4.1 [EXP] 8 [=] *15129*

b. $\dfrac{7.6 \times 10^{2}}{3.2 \times 10^{-6}}$

Estimate: $\dfrac{7 \times 10^{2}}{3.5 \times 10^{-6}} = 2 \times 10^{8}$ *or 200,000,000*

7.6 [EXP] 2 [÷] 3.2 [EXP] [+/−] 6 [=] *237500000*

Scientists use scientific notation because they often deal with very large or very small quantities.

Example 7

CONNECTION
Chemistry

GLOBAL CONNECTIONS

Primo Levi (1919–1987) was born in Turin, Italy. His autobiography, *The Periodic Table*, is based on his life as a chemist with each chapter named after a different element. This award-winning author was a survivor of Auschwitz.

A chemist works in a laboratory, doing research for a fertilizer company. For one batch of fertilizer, she needs 20 moles of sulfuric acid. A mole is a standard unit of measure in chemistry that contains 6.02×10^{23} molecules of a substance, Avogadro's constant. If she adds 1.08×10^{25} molecules of sulfuric acid, has she added enough to make one batch of fertilizer?

Divide the number of molecules she has by the number of molecules in a mole.

$$\frac{1.08 \times 10^{25} \text{ molecules}}{6.02 \times 10^{23} \text{ molecules/mole}} = \frac{1.08}{6.02} \cdot \frac{10^{25} \text{ molecules}}{10^{23} \text{ molecules/mole}}$$

$$\approx 0.179 \times 10^{2} \text{ or } 17.9 \text{ moles}$$ *Use a calculator.*

The chemist did not add enough sulfuric acid.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **Explain** why $a \neq 0$ in the expression $\frac{a^m}{a^n}$.
2. **Determine** if $4x^2$ and $(4x)^2$ are equivalent.
3. **You Decide** Is a negative number raised to the 777th power a negative or positive number? Explain.
4. Is $2(x^2y)^3$ in simplest form? Explain your answer.

Guided Practice

5. Write $3wz^{-4}$ without using negative exponents.

Simplify. Assume that no variable equals 0.

6. $y^5 \cdot y^7$
7. $(3a)^4$
8. $(m^2)^2(m^{-2})^2$
9. $\frac{40x^4}{-5x^2}$
10. $\frac{-2c^3d^6}{24c^2d^2}$
11. $\frac{16s^6t^5}{(2s^2t)^2}$
12. $\left(\frac{1}{x^3y^2}\right)^4$
13. $\left(\frac{bc}{2}\right)^{-3}$
14. $\left(\frac{-6y^5}{3y^2}\right)^{-2}$

Express each number in scientific notation.

15. 386,000
16. 0.000346

Evaluate. Express in both scientific and decimal notation.

17. $\frac{8 \times 10^{-1}}{16 \times 10^{-2}}$
18. $(3.42 \times 10^8)(1.1 \times 10^{-5})$

EXERCISES

Practice

Simplify. Assume that no variable equals 0.

19. $b^3 \cdot b^5$
20. $x^2 \cdot x \cdot x^3$
21. $(m^3)^3$
22. $(-3y)^3$
23. $\frac{an^6}{n^5}$
24. $\frac{-x^6y^6}{x^3y^4}$
25. $(a^3b^3)(ab)^{-2}$
26. $(4x^3y^{-4})(7xy^2)$
27. $(-3r^2s)^2(2rs^3)$
28. $(3a^3b)(-5a^2b^2)$
29. $(2mn^2)(5m^2n)$
30. $(-5x^2y)(-2x^4y^7)$
31. $\left(-\frac{3}{4}m^2n^3\right)\left(\frac{8}{9}mn^4\right)$
32. $2b^2(2ab)^3$
33. $4x(-3x)^3$
34. $4a^2(3b^3)(2a^2b)$
35. $5mn^2(m^3n)(-3p^2)$
36. $5x(6x^2y)(3xy^3)$
37. $\frac{-6x^2y^3z^3}{24x^2y^7z^3}$
38. $\frac{2x^5y^3z^3}{8x^3y^7z}$
39. $\frac{-15m^5n^8(m^3n^2)}{45m^4n}$
40. $\frac{2a^3b}{(-2ab^3)^{-2}}$
41. $\left(\frac{5a^3b}{10a^2b^2}\right)^4$
42. $\left(\frac{a}{b^{-1}}\right)^{-2}$
43. $\frac{40a^{-1}b^{-7}}{20a^{-5}b^{-9}}$
44. $\frac{5^{2x}}{5^{2x+2}}$
45. $\frac{8}{m^0+n^0}$

Express each number in scientific notation.

46. 810.4
47. 786,500,000
48. 0.0008742
49. 0.001250
50. 901,010,000
51. 0.03331

Evaluate. Express in both scientific and decimal notation.

52. $(6.23 \times 10^4)(2.0 \times 10^5)$
53. $(2 \times 10^{-3})(2.01 \times 10^{-2})$
54. $(45{,}000)(0.0025)$
55. $(9.5 \times 10^3)^2$
56. $(6.9 \times 10^3)(1.4 \times 10^3)^{-1}$
57. $\frac{(93{,}000{,}000)(0.005)}{0.0015}$

Find the value of *r* that makes each sentence true.

58. $y^{28} = y^{3r} \cdot y^7$
59. $2^{r+5} = 2^{2r-1}$
60. $2^{2r+1} = 32$
61. $(x^3 \cdot x^r)^5 = x^{30}$
62. $\frac{x^{2r}}{x^{-3r}} = x^{15}$
63. $\frac{m^r}{m^{15}} = (m^3)^{r+2}$

Critical Thinking

64. Which is greater, 100^{10} or 10^{100}? Explain your answer.

65. Express the quotient $\dfrac{x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7}{x^{-3} + x^{-4} + x^{-5} + x^{-6} + x^{-7} + x^{-8} + x^{-9}}$ in simplest form. Assume that x is not equal to zero. (*Hint:* Simplify the denominator first.)

Real World

Applications and Problem Solving

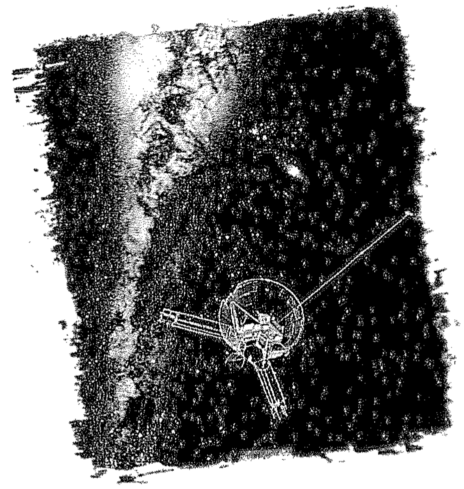
66. **Economics** In a recent year, the U.S. government received approximately \$468,000,000,000 in personal income taxes. The population of the country at that time was about 250,000,000. If everyone paid taxes, what was the average amount paid by each man, woman, and child?

Top Five List

Largest Planetary Moons (diameter)

1. Ganymede, Jupiter (3273 miles)
2. Titan, Saturn (3200 miles)
3. Callisto, Jupiter (2995 miles)
4. Io, Jupiter (2257 miles)
5. Moon, Earth (2159 miles)

67. **Astronomy** In April of 1983, the space probe *Pioneer 10* was as far from Earth as the planet Pluto. *Pioneer 10* sent radio signals that traveled at the speed of light, 3.00×10^5 kilometers per second. If *Pioneer 10* was 4.58×10^9 kilometers from Earth, how long would it take a tracking station to send a message indicating a mid-course correction in the space probe's travel course? (Use $t = \frac{d}{r}$, where d is distance and r is rate.)



Mixed Review

interNET CONNECTION

Data Update For more information on the U.S. population, visit: **www.algebra2.glencoe.com**

68. **Statistics** The table shows the percent of the population that is made up of people in their twenties for 18 large U.S. cities. (Lesson 4–8)
    a. Make a box-and-whisker plot of the data.
    b. List three reasons why you think someone would want this information.

City	Percent of Population
Boston, MA	26%
Columbus, OH	24
San Diego, CA	22
Seattle, WA	20
Washington, DC	20
Nashville, TN	19
Anchorage, AK	18
Baltimore, MD	18
Chicago, IL	18
Indianapolis, IN	18
Jacksonville, FL	18
Milwaukee, WI	18
Philadelphia, PA	18
Phoenix, AZ	18
New York, NY	17
Detroit, MI	16
Honolulu, HI	16
U.S. Average	16

69. Find M if $\begin{bmatrix} 3 & 6 & 1 \\ 2 & -1 & 0 \end{bmatrix} \cdot M = \begin{bmatrix} 3 & 6 & 1 \\ 2 & -1 & 0 \end{bmatrix}$. (Lesson 4–5)

70. Solve $\begin{vmatrix} 5a & 3 \\ a & 5 \end{vmatrix} = 7$ for a. (Lesson 4–4)

71. **SAT Practice** If $a = b + 1$ and $b \geq 1$, then which of the following must be equal to $a^2 - b^2$?
    A $(a - b)^2$ B $b^2 - 1$ C $a^2 - 1$ D $a + b$ E $a^2 - b - 1$

Solve each system algebraically or by graphing.

72. $8a - 3b + c = 7$
    $-2a + b - c = 0$
    $2a - 3b + 9c = -1$ (Lesson 3–7)

73. $x + y < 9$
    $x - y < 3$
    $y - x > 4$ (Lesson 3–4)

74. Describe the graphs of two linear equations that are dependent. (Lesson 3–1)

75. Write an equation of the line that passes through (2, 2) and is parallel to the line $4x - y + 3 = 0$. (Lesson 2–5)

$-y = -4x - 3$
$y = 4x + 3$

76. Find the x- and y-intercepts of $f(x) = 13x + 26$. (Lesson [illegible])

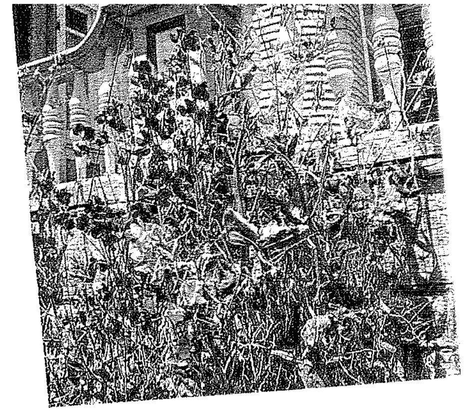
77. Graph $x - 3y = -3$. (Lesson 2–2)

78. Solve $-2(4 - 3x) > 4$. (Lesson [illegible])

79. Evaluate $15 - 3(2) \div 8 - 11$. (Lesson [illegible])

For Extra Practice, see page 885.

5-2 Polynomials



What YOU'LL LEARN

- To add, subtract, and multiply polynomials.

Why IT'S IMPORTANT

You can use polynomials to solve problems involving biology and genetics.

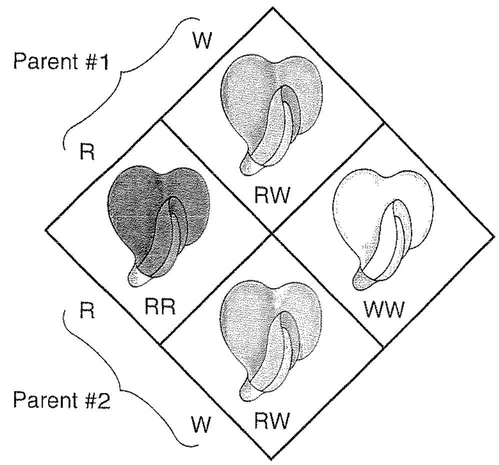
CONNECTION

Biology

Scientists can use algebraic expressions to summarize the possible outcomes in genetic breeding. Certain traits result from the pairing of two genes, one from the female parent and one from the male parent. For example, suppose a red-flowering, sweet pea plant has *genotype RR*, a white-flowering, sweet pea plant has genotype *WW*, and a pink-flowering, sweet pea plant has genotype *RW*. Each letter represents one of the two genes that make up the characteristic.

Suppose two pink-flowering plants are bred. The offspring can be expressed using algebra and a model called a *Punnett square.*

One gene from the mother pairs with one gene from the father for each possible offspring.



The sum of the possible results for four offspring can be written as $RR + RW + RW + WW$; that is, one red-, two pink-, and one white-flowering plants. Suppose we substitute x for R and y for W. The result would be a sum of four monomials, $xx + xy + xy + yy$, or $x^2 + 2xy + y^2$. The two monomials xy and xy can be combined because they are **like terms**. Like terms are two monomials that are the same, or differ only by their numerical coefficients.

Remember that a difference like $m^2 - 7mb - 12cd$ can be written as a sum, $m^2 + (-7mb) + (-12cd)$.

The expression $x^2 + 2xy + y^2$ is called a **polynomial.** A polynomial is a monomial or a sum of monomials. The monomials that make up the polynomial are called the **terms** of the polynomial. An expression like $m^2 - 7mb - 12cd$ with three unlike terms is called a **trinomial.** An expression like $xy + b^3$ with two unlike terms is called a **binomial.** The *degree* of a polynomial is the degree of the monomial with the greatest degree. Thus, the degree of $x^2 + 2xy + y^2$ is 2.

Example 1 **Determine whether or not each expression is a polynomial. Then state the degree of each polynomial.**

a. $\frac{2}{7}x^4y^3 - 21x^3$

This expression is a polynomial. The degree of the first term is 4 + 3 or 7, and the degree of the second term is 3. The degree of the polynomial is 7.

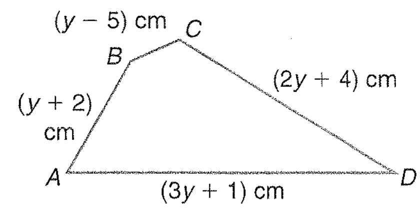
b. $9 + \sqrt{x} - 3$

This expression is not a polynomial because $\sqrt{x}$ is not a monomial.

To simplify a polynomial means to perform the operations indicated and combine like terms.

Example 2 INTEGRATION Geometry

Find the perimeter of quadrilateral *ABCD*.



The perimeter is the sum of the measures of the sides.

$$\begin{aligned} P &= AB + BC + CD + DA \\ &= (y + 2) + (y - 5) + (2y + 4) + (3y + 1) \\ &= y + y + 2y + 3y + 2 - 5 + 4 + 1 \\ &= 7y + 2 \end{aligned}$$

The perimeter of quadrilateral *ABCD* is $(7y + 2)$ cm.

Example 3 **Simplify $(4x^2 - 3x) - (x^2 + 2x - 1)$.**

$$\begin{aligned} (4x^2 - 3x) - (x^2 + 2x - 1) &= 4x^2 - 3x - x^2 - 2x + 1 \\ &= (4x^2 - x^2) + (-3x - 2x) + 1 \\ &= 3x^2 - 5x + 1 \end{aligned}$$

You can use the distributive property to multiply polynomials.

Example 4 **Find $3x(5x^4 - x^3 + 4x)$.**

$$\begin{aligned} 3x(5x^4 - x^3 + 4x) &= 3x(5x^4) + 3x(-x^3) + 3x(4x) \\ &= 15x^5 - 3x^4 + 12x^2 \end{aligned}$$

You can use algebra tiles to make a geometric model of the product of two binomials.

MODELING MATHEMATICS — Multiplying Binomials

Materials: algebra tiles

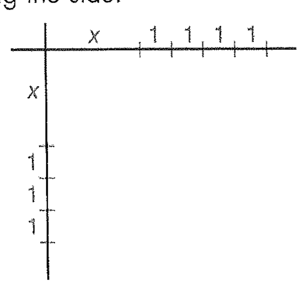
Use algebra tiles to find the product of $x + 4$ and $x + 3$.

Your Turn

a. Draw a 90° angle on your paper.

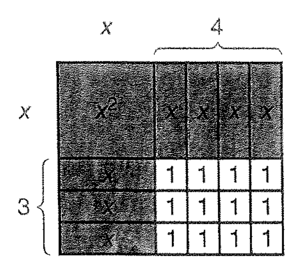
b. Use an x-tile and a 1-tile to mark off a length equal to $x + 4$ along the top.

c. Use the tiles to mark off a length equal to $x + 3$ along the side.



d. Draw lines to show the grid formed by these measures.

e. Fill in the lines with the appropriate tiles to show the area product. The model shows the polynomial $x^2 + 7x + 12$.



The area of the rectangle is the product of its length and width. Substituting the area, length, and width with the corresponding polynomials, we find that $x^2 + 7x + 12 = (x + 4)(x + 3)$.

In the following example, two different methods are used to multiply binomials.

Example 5 **Find $(4n + 3)(3n + 1)$.**

Method 1: Distributive Property

$$\begin{aligned}(4n+3)(3n+1) &= 4n(3n+1) + 3(3n+1)\\ &= 4n \cdot 3n + 4n \cdot 1 + 3 \cdot 3n + 3 \cdot 1\\ &= 12n^2 + 4n + 9n + 3\\ &= 12n^2 + 13n + 3\end{aligned}$$

Method 2: FOIL Method

$$\begin{aligned}(4n+3)(3n+1) &= \underbrace{4n \cdot 3n}_{\text{First terms}} + \underbrace{4n \cdot 1}_{\text{Outside terms}} + \underbrace{3 \cdot 3n}_{\text{Inside terms}} + \underbrace{3 \cdot 1}_{\text{Last terms}}\\ &= 12n^2 + 13n + 3\end{aligned}$$

The **FOIL method** is an application of the distributive property that makes the multiplication easier.

FOIL Method of Multiplying Polynomials

The product of two binomials is the sum of the products of

F the ***first*** **terms,**
O the ***outer*** **terms,**
I the ***inner*** **terms, and**
L the ***last*** **terms.**

Example 6 **Find $(k^2 + 3k + 9)(k + 3)$.**

$(k^2 + 3k + 9)(k + 3)$

$= k^2(k+3) + 3k(k+3) + 9(k+3)$ *Distributive property*

$= k^2 \cdot k + k^2 \cdot 3 + 3k \cdot k + 3k \cdot 3 + 9 \cdot k + 9 \cdot 3$ *Distributive property*

$= k^3 + 3k^2 + 3k^2 + 9k + 9k + 27$

$= k^3 + 6k^2 + 18k + 27$ *Combine like terms.*

CHECK FOR UNDERSTANDING

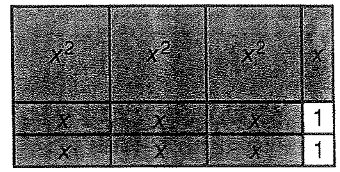
Communicating Mathematics

Study the lesson. Then complete the following.

1. **Demonstrate** the FOIL method by multiplying $(3a + 4b)$ and $(a - b)$.
2. **Show** another way to multiply the expression in Example 6 by distributing $(k^2 + 3k + 9)$ instead of $(k + 3)$.
3. **Write** a polynomial of degree 6 that has four terms.

MODELING MATHEMATICS

4. **Draw** a geometric representation of $2x^2 + 6x$.
5. **Write** two factors and a product for the model shown at the right.



Guided Practice

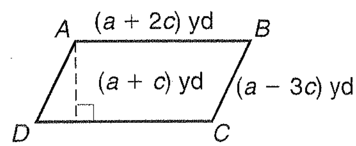
Determine whether each expression is a polynomial. Write *yes* or *no* and explain your reasoning. Then state the degree of each polynomial.

6. $14x + 3y$
7. $\frac{y^3}{4} - 7x$
8. $\frac{ax^2 + 6}{by^3 + 5}$

Simplify.

9. $(5x - 7y) + (6x + 8y)$
10. $(-2y^2 - 4y + 7) - (2y^2 + 4y - 7)$
11. $3y(2x + 6)$
12. $2m^2n(5mn - 3m^3n^2 + 4mn^4)$
13. $(x + 6)(x + 3)$
14. $(y - 10)(y + 7)$
15. $(3m - 1)(3m + 1)$
16. $(2p - 3s)^2$

17. **Geometry** Quadrilateral $ABCD$ is a parallelogram.
    a. Find the perimeter of $ABCD$.
    b. Find the area of $ABCD$.



EXERCISES

Practice

Determine whether each expression is a polynomial. Write *yes* or *no* and explain your reasoning. Then state the degree of each polynomial.

18. $x^2 + 2x + 3$
19. $y^3 + 3$
20. $\sqrt{s - 5}$
21. $\frac{3k^2}{2} + \frac{4k^7}{5}$
22. $\frac{4ab}{c} - \frac{2d}{x}$
23. $x\sqrt{3} + 8x^2y^4$

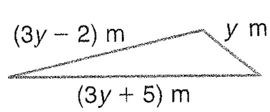
Simplify.

24. $(3r + s) - (r - s) - (r + 3s)$
25. $(z^2 - 6z - 10) + (2z^2 + 4z - 11)$
26. $(-12y - 6y^2) + (-7y + 6y^2)$
27. $(3m^2 + 5m - 6) + (7m^2 - 9)$
28. $(10x^2 - 3xy + 4y^2) - (3x^2 + 5xy)$
29. $(8r^2 + 5r + 14) - (7r^2 + 6r + 8)$
30. $4a(3a^2b)$
31. $4f(gf - bh)$
32. $\frac{2}{3}x^2(6x + 9y - 12xy^2)$
33. $-5mn^2(-3m^2n + 6m^3n - 3m^4n^4)$
34. $(c^2 - 6cd - 2d^2) + (7c^2 - cd + 8d^2) - (-c^2 + 5cd - d^2)$
35. $(4x^2 - 3y^2 + 5xy) - (8xy + 6x^2 + 3y^2)$.

INTEGRATION Geometry

Find the perimeter of each figure.

36.



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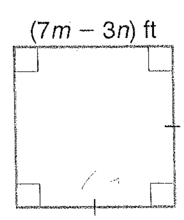

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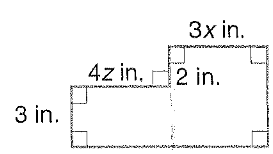
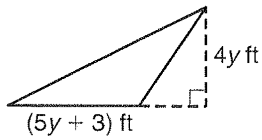


Simplify.

39. $(q - 7)(q + 5)$
40. $(m + 7)(m + 2)$
41. $(5 - r)(5 + r)$
42. $(2x + 7)(3x + 5)$
43. $(3y - 8)(2y + 7)$
44. $(x^3 - y)(x^3 + y)$
45. $g^{-3}(g^5 - 2g^3 + g^{-1})$
46. $x^{-3}y^2(yx^4 + y^{-1}x^3 + y^{-2}x^2)$
47. $(y - 3x)^2$
48. $(1 + 4m)^2$
49. $(2p + q^3)^2$
50. $(w^2 - 5)(2w^2 + 3)$

Find the area of each figure.

51.



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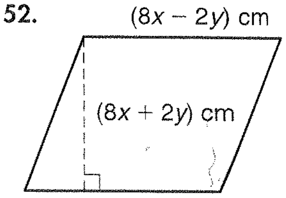
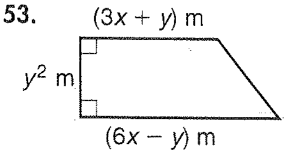


53.



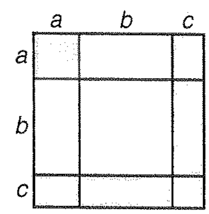
Simplify.

54. $(3y + 1)(3y - 1)(y + 2)$
55. $(x^2 + xy + y^2)(x - y)$
56. $(2q + 1)(q - 2)^2$
57. $(x - 2)(x + 2)(x^2 + 5)$
58. $(3b - c)^3$
59. $(y + x)^2(y - x)^2$

Critical Thinking

60. Draw a geometric representation of $(x + 3)(x - 3)$. Explain your results.

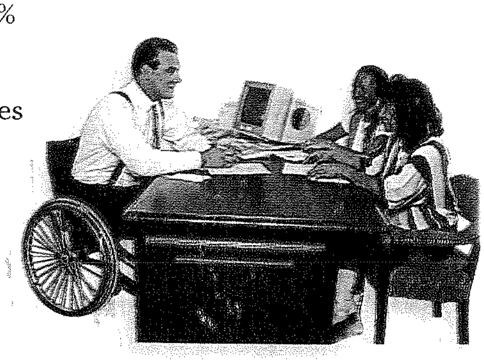
61. a. Write two different polynomial expressions that represent the area of the figure at the right.
    b. Write a polynomial for the perimeter of the figure.



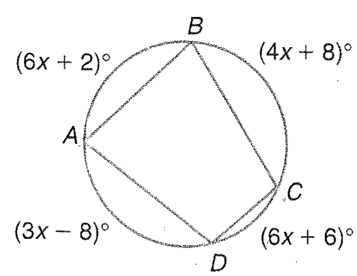
Applications and Problem Solving

Real World

62. **Personal Finance** Abey has $1500 to invest. She would like to have a return of at least $100 a year on her investment. She can invest some of the money in a mutual fund yielding 6% annually. She can also buy bonds yielding 7% annually. She decides to invest her money in both sources to diversify her investment.
    a. Express the amount of return Abey will make as a polynomial in one variable.
    b. How much money should Abey place in each of these investments to have the desired return?



63. **Geometry** Recall that the measure of an angle inscribed in a circle is half the measure of its intercepted arc. That is, $m\angle B = \frac{1}{2}m\widehat{ADC}$.
    a. Given a circle with inscribed quadrilateral $ABCD$ with the given arc measures, find the ratio of $m\angle A$ to $m\angle B$.
    b. If $m\widehat{AB} + m\widehat{BC} + m\widehat{CD} + m\widehat{AD} = 360°$, find the value of x.



64. **Make a Drawing** Suppose you were trying to model the product $(2x + 3y)(x + 5)$. Make a drawing of rectangles to represent each type of monomial in the product. Then make a drawing to illustrate the product and write the result.

65. **Genetics** Suppose p represents the ratio of dominant gene A in a population and q represents the ratio of recessive gene a in a population. The next generation is the result of $(p + q)^2$. This results in p^2 pure dominant genotypes (AA), q^2 pure recessive genotypes (aa), and $2pq$ hybrid genotypes (Aa). Since all members of the population must have at least one recessive gene or dominant gene present in their genotypes, $p + q = 1$.

Suppose that in the population of a certain village, the recessive left-handedness gene r had a frequency of 1:4 and the dominant right-handedness gene R had a frequency of 3:4. In the next generation, what would you predict the population genotypes to be?

Mixed Review

66. Simplify $2(rk)^2(5rt^2) - k(2rk)(2rt)^2$. (Lesson 5–1)

67. Solve the system of equations by using augmented matrices. (Lesson 4–7)

$4x - y + z = 6$
$2x + y + 2z = 3$
$3x - 2y + z = 3$

68. Write a matrix equation for the system of equations. (Lesson 4–6)

$3x - y = 5$
$2x + 33y = 29$

69. If $A = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 0 \\ 2 & 5 \end{bmatrix}$, find AB. (Lesson 4–3)

70. **SAT Practice** A 25-inch by 30-inch rectangular counter is to be completely covered with 1-inch square tiles, which cannot overlap one another and cannot overhang the counter. If white tiles are to cover the center of the counter and brown tiles are to form a 1-inch wide border along the edge of the counter, how many brown tiles will be used?

A 55 **B** 106 **C** 110 **D** 114 **E** 750

71. Solve the system of equations by using substitution. (Lesson 3–2)

$6x + 4y = 80$
$x - 7y = -2$

72. Solve the system of equations by graphing. (Lesson 3–1)

$2x + 2y = -12$
$3x - 2y = -3$

73. Name the points, (0, 0), (−1, −3), or (4, 0), that satisfy $4x - |y| \leq 12$. (Lesson 2–6)

74. Find $h(-7)$ if $h(x) = \frac{3 + x}{4}$. (Lesson 2–1)

75. **Statistics** Find the mean, median, and mode of {45, 49, 40, 39, 39, 46, 44, 41, 42}. (Lesson 1–4)

76. **Banking** Karen invests \$7500 in a certificate of deposit at the Lombard Bank. The simple interest rate is 7.2% per year. How much will she make in 5 years? (Lesson 1–1)

For Extra Practice, see page 886.

5-3 Dividing Polynomials

What **YOU'LL LEARN**

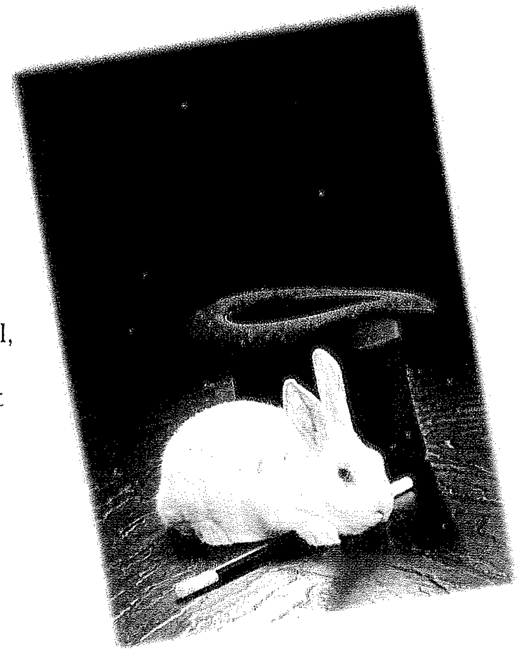
- To divide polynomials using long division, and
- to divide polynomials by binomials using synthetic division.

Why **IT'S IMPORTANT**

You can use polynomials to solve problems involving manufacturing and entertainment.

Real World APPLICATION

Cartography



Tionna, a senior at Franklin High School, spends one free period a day as a teacher's aide at the middle school that is located next to the high school. A magician was performing at the middle school and asked a member of the audience to participate in a number game. The magician said:

- Choose any number.
- Multiply your number by 3.
- Then add the sum of your number and 8 to the number you got when you multiplied.
- Now divide by the sum of your number and 2.

Then the magician said, "Without asking you what number you chose, I can tell you that your final result was . . . 4!"

Tionna wondered how the magician's trick worked. But magicians don't tell their secrets. So, she used her algebraic skills to write expressions that modeled the steps in the trick.

Choose a number.	x
Multiply by 3.	$3x$
Add the sum of your number and 8 to the previous result.	$3x + (x + 8)$ or $4x + 8$
Divide this result by your number plus 2.	$\frac{4x + 8}{x + 2}$

The final expression is $\frac{4x + 8}{x + 2}$. But how does this help Tionna solve the secret of the magician's trick? *You will be asked to solve the mystery in Exercise 3.*

In Lesson 5–1, you learned to divide monomials. You can divide a polynomial by a monomial by using those same skills.

Example 1 **Simplify $\frac{6r^2s^2 + 3rs^2 - 9r^2s}{3rs}$.**

This expression means that each term in the numerator shares a common denominator. Rewrite the expression as a sum of quotients.

$$\frac{6r^2s^2 + 3rs^2 - 9r^2s}{3rs} = \frac{6r^2s^2}{3rs} + \frac{3rs^2}{3rs} - \frac{9r^2s}{3rs}$$

$$= \frac{6}{3} \cdot r^{2-1}s^{2-1} + \frac{3}{3} \cdot r^{1-1}s^{2-1} - \frac{9}{3} \cdot r^{2-1}s^{1-1}$$

$$= 2rs + s - 3r \quad r^{1-1} = r^0 \text{ or } 1$$

You can use a process similar to long division of whole numbers to divide a polynomial by a polynomial. When doing the division, remember that you can only add and subtract like terms.

Example 2 **Simplify $\frac{c^2 - c - 30}{c - 6}$.**

In this lesson, assume that the denominator never equals zero.

$$\begin{array}{r} c \phantom{{}-c-30} \\ c - 6\overline{)c^2 - c - 30} \\ \underline{c^2 - 6c} \phantom{{}-30} \\ 5c - 30 \end{array} \quad \textit{-c - (-6c) = -c + 6c or 5c} \quad \Rightarrow \quad \begin{array}{r} c + 5 \\ c - 6\overline{)c^2 - c - 30} \\ \underline{c^2 - 6c} \phantom{{}-30} \\ 5c - 30 \\ \underline{5c - 30} \\ 0 \end{array}$$

Therefore, the quotient is $c + 5$.

Just as with the division of whole numbers, the division of two polynomials may result in a quotient with a remainder. Remember that $9 \div 4 = 2 + R1$ and is often written as $2\frac{1}{4}$. The division of polynomials with a remainder is presented in the same manner.

Example 3 **Simplify $(s^2 + 4s - 16)(6 - s)^{-1}$.**

$$(s^2 + 4s - 16)(6 - s)^{-1} = \frac{s^2 + 4s - 16}{6 - s}$$

$$\begin{array}{r} -s - 10 \\ -s + 6\overline{)s^2 + 4s - 16} \\ \underline{s^2 - 6s} \phantom{{}-16} \\ 10s - 16 \\ \underline{10s - 60} \\ 44 \end{array} \quad \textit{For ease in dividing, rewrite } 6 - s \textit{ as } -s + 6.$$

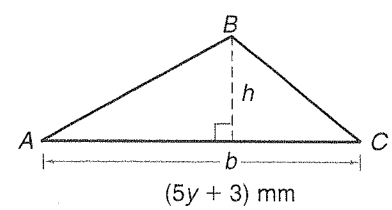
The quotient is $-s - 10 + \frac{44}{6 - s}$. *The remainder is* $\frac{44}{-s + 6}$ *or* $\frac{44}{6 - s}$.

In Example 4, solving for a variable involves dividing polynomials.

Example 4

INTEGRATION
Geometry

The area of triangle ABC is $(10y^2 + 6y)$ mm^2 and its base is $(5y + 3)$ mm. Find the measure of the altitude of the triangle.



An altitude of a triangle is a line segment from a vertex perpendicular to the line containing the opposite side. The measure of an altitude is the height h of the triangle.

$$A = \frac{1}{2}bh$$

$$10y^2 + 6y = \frac{1}{2}(5y + 3)(h) \quad \textit{A = 10y}^2 \textit{ + 6y, b = 5y + 3}$$

$$20y^2 + 12y = (5y + 3)h \quad \textit{Multiply each side by 2.}$$

$$\frac{20y^2 + 12y}{5y + 3} = h \quad \textit{Divide each side by } 5y + 3. \rightarrow \quad \begin{array}{r} 4y \phantom{{}+12y} \\ 5y + 3\overline{)20y^2 + 12y} \\ \underline{20y^2 + 12y} \\ 0 \end{array}$$

$$4y = h$$

The length of the altitude is $4y$ mm.

A simpler process called **synthetic division** has been devised to divide a polynomial by a binomial. Suppose we wanted to divide $(6x^3 - 19x^2 + x + 6)$ by $(x - 3)$. Long division would produce the following result.

$$\begin{array}{r} 6x^2 - 1x - 2 \\ x - 3\overline{)6x^3 - 19x^2 + x + 6} \\ \underline{(-)6x^3 - 18x^2} \qquad\qquad \\ -1x^2 + x + 6 \\ \underline{(-)-1x^2 + 3x} \quad \\ -2x + 6 \\ \underline{(-)-2x + 6} \\ 0 \end{array}$$

Compare the coefficients in this division with those in Example 5.

Study the next example.

Example 5 **Use synthetic division to find $(6x^3 - 19x^2 + x + 6) \div (x - 3)$.**

Write the terms of the polynomial so that the degrees of the terms are in descending order. Then write just the coefficients as shown at the right. *There must be a coefficient for every possible power of the variable.*

$$\begin{array}{cccc} 6x^3 & -\ 19x^2 & +\ x & +\ 6 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 6 & -19 & 1 & 6 \end{array}$$

Write the constant r of the divisor $x - r$ to the left.

In this case, $r = 3$.
Bring the first coefficient, 6, down as shown.

$$\begin{array}{c|cccc} 3 & 6 & -19 & 1 & 6 \\ \hline & 6 & & & \end{array}$$

Multiply the first coefficient by r: $3 \cdot 6 = 18$.
Write the product under the second coefficient.
Then add the product and the second coefficient:
$-19 + 18 = -1$.

$$\begin{array}{c|cccc} 3 & 6 & -19 & 1 & 6 \\ & & 18 & & \\ \hline & 6 & -1 & & \end{array}$$

Multiply the sum, -1, by r: $3(-1) = -3$.
Write the product under the next coefficient and add: $1 + (-3) = -2$.

$$\begin{array}{c|cccc} 3 & 6 & -19 & 1 & 6 \\ & & 18 & -3 & \\ \hline & 6 & -1 & -2 & \end{array}$$

Multiply the sum, -2, by r: $-2 \cdot 3 = -6$.
Write the product under the next coefficient and add: $6 + (-6) = 0$. The remainder is 0.

$$\begin{array}{c|cccc} 3 & 6 & -19 & 1 & 6 \\ & & 18 & -3 & -6 \\ \hline & 6 & -1 & -2 & 0 \end{array}$$

Writing the quotient is easy. The numbers along the bottom row are the coefficients of the powers of x in descending order. Start with the power that is one less than that of the dividend. Thus, the result of this division is $6x^2 - x - 2$.

Check this result. Does $(x - 3)(6x^2 - x - 2) = 6x^3 - 19x^2 + x + 6$?

To use synthetic division, the divisor must have a leading coefficient of 1. You can rewrite other division expressions so you can use synthetic division.

Example 6 **Use synthetic division to find $(4x^4 - 5x^2 + 2x + 4) \div (2x - 1)$.**

Use division to rewrite the divisor so it has a coefficient of 1.

$$\frac{4x^4 - 5x^2 + 2x + 4}{2x - 1} = \frac{(4x^4 - 5x^2 + 2x + 4) \div 2}{(2x - 1) \div 2} \text{ or } \frac{2x^4 - \frac{5}{2}x^2 + x + 2}{x - \frac{1}{2}}$$

Since the numerator does not contain all powers of x, you must include a 0 coefficient for the x^3 term.

Now use synthetic division.

$$\begin{array}{r|rrrrr} \frac{1}{2} & 2 & 0 & -\frac{5}{2} & 1 & 2 \\ & & 1 & \frac{1}{2} & -1 & 0 \\ \hline & 2 & 1 & -2 & 0 & |\,2 \end{array} \quad x - r = x - \frac{1}{2}$$

The quotient is $2x^3 + x^2 - 2x + \frac{2}{x - \frac{1}{2}}$ or $2x^3 + x^2 - 2x + \frac{4}{2x - 1}$.

Check: Divide using long division.

$$\begin{array}{r} 2x^3 + x^2 - 2x \\ 2x - 1\overline{)4x^4 + 0x^3 - 5x^2 + 2x + 4} \\ \underline{4x^4 - 2x^3} \\ 2x^3 - 5x^2 \\ \underline{2x^3 - x^2} \\ -4x^2 + 2x \\ \underline{-4x^2 + 2x} \\ 0 + 4 \end{array}$$

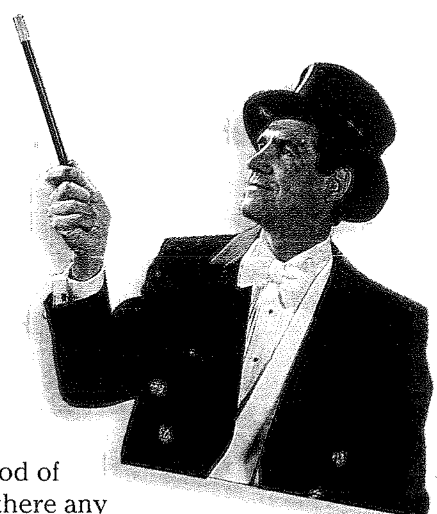
The quotient is $2x^3 + x^2 - 2x + \frac{4}{2x - 1}$. ✓

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **Show** how you would set up the synthetic division for $(5y^3 + y^2 - 7) \div (y + 1)$.
2. **Illustrate** why it is necessary to include terms with zero coefficients in the row of numbers for synthetic division.
3. **You Decide** Refer to the magician's trick at the beginning of the lesson. Jocelyn says that the answer is always 4, regardless of the number chosen. Marie says that the magician knows a special way to compute numbers quickly and the answer depends on the number chosen. Who is correct, and why?



MATH JOURNAL

4. **Assess Yourself** Compare the long division method of dividing polynomials with synthetic division. Are there any advantages to either method? Which do you prefer and why?

Guided Practice

Simplify.

5. $\frac{5xy^2 - 4xy + 7x^2y}{xy}$
6. $(6xy^2 - 3xy + 2x^2y)(xy)^{-1}$
7. $(a^2 - 10a - 24) \div (a + 2)$
8. $(9b^2 + 9b - 10) \div (3b - 2)$
9. $(a^3 + b^3) \div (a + b)$
10. $(y^5 - 3y^2 - 20) \div (y - 2)$

Use synthetic division to find each quotient.

11. $(3x^4 - 6x^3 - 2x^2 + x - 6) \div (x + 1)$

12. $(t^4 - 2t^3 + t^2 - 3t + 2)(t - 2)^{-1}$

13. $(12x^2 + 36x + 15) \div (6x + 3)$

14. $(x^3 + 13x^2 - 12x - 8) \div (x + 2)$

EXERCISES

Practice

Simplify.

15. $\frac{8x^2y^3 - 28x^3y^2}{4xy^2}$

16. $\frac{2mn^3 + 4m^2 - 9m^3n^2}{mn}$

17. $(12rs^3 + 9r^2s^2 - 15r^2s) \div (3rs)$

18. $(28k^3p - 42kp^2 + 56kp^3) \div (14kp)$

19. $(a^3b^2 - a^2b + 2a)(-ab)^{-1}$

20. $(b^3 + 8b^2 - 20b) \div (b - 2)$

21. $(x^2 - 12x - 45) \div (x + 3)$

22. $(n^3 + 2n^2 - 5n + 12) \div (n + 4)$

23. $(g^2 + 8g + 15)(g + 3)^{-1}$

24. $(2b^3 + b^2 - 2b + 3)(b + 1)^{-1}$

25. $(6t^3 + 5t^2 + 9) \div (2t + 3)$

26. $(50a^2 - 98b^2) \div (10a + 14b)$

27. $(5y^3 + y^2 - 7) \div (y + 1)$

28. $(2h^3 - 5h^2 + 22h + 51) \div (2h + 3)$

29. $(2y^2 + y - 16) \div (y - 3)$

30. $(x^3 - 4x^2) \div (x - 4)$

31. $(x^3 - 27) \div (x - 3)$

32. $(8x^3 - 1) \div (2x - 1)$

Use synthetic division to find each quotient.

33. $\frac{9d^3 + 5d - 8}{3d - 2}$

34. $\frac{m^3 - 7m + 3m^2 - 21}{m + 3}$

35. $(2c^3 - 3c^2 + 3c - 4) \div (c - 2)$

36. $(2x^3 - x^2 + 5x - 12) \div (2x - 3)$

37. $(w^2 - w^3)(w - 1)^{-1}$

38. $(6w^5 - 18w^2 - 120) \div (w - 2)$

39. $\frac{2m^4 - 5m^3 - 10m + 8}{m - 3}$

40. $\frac{a^4 - 5a^3 - 13a^2 + 53a + 60}{a + 1}$

41. $(y^5 + 32)(y + 2)^{-1}$

42. $(t^5 - 3t^2 - 20)(t - 2)^{-1}$

43. Solve $(b + 1)y = 2b^3 + b^2 - 2b + 3$ for y.

Critical Thinking

44. Use synthetic division to find each quotient. Then evaluate the dividend for the value stated.

    a. $\frac{3x^3 - 5x - 2}{x - 2}, x = 2$

    b. $\frac{2x^4 - 3x^2 + 1}{x + 1}, x = -1$

    c. Study the results of parts a and b. If the divisor represents $x - r$ and the dividend is $f(x)$, how is the division $\frac{f(x)}{x - r}$ related to $f(r)$?

45. Suppose the quotient resulting from dividing one polynomial by another is $r^2 - 6r + 9 - \frac{1}{r - 3}$. What two polynomials were divided?

Programming

46. The graphing calculator program below uses synthetic division to compute the coefficients of the quotient and the remainder when a polynomial is divided by a linear binomial. When prompted by a question mark, input the degree of the polynomial (N), the constant R for r in the divisor $x - r$, and the coefficients (C) of the polynomial. Press ENTER after each entry.

```
PROGRAM: SYNDIV
: ClrHome
: ClrList (L1,L2)
: Disp "LET N = DEGREE",
  "OF POLYNOMIAL"
: Disp "LET R = CONSTANT",
  "IN DIVISOR X-R"
: Prompt N,R
: Disp "ENTER COEFFICIENTS"
: For (X,1,N+1,1)
: Prompt C
: C →L1(X)
: End
: L1(1)→L2(1)
: Disp "COEFFICIENTS OF",
  "QUOTIENT ARE"
: For (X,1,N−1,1)
: L1(X+1)+R*L2(X)→
  L2(X+1)
: End
: Disp L2
: Disp "REMAINDER",
  L1(N+1)+R*L2(N)
```

Use the program to find each quotient.

a. $(2x^5 + 3x^4 - 6x^3 + 6x^2 - 8x + 3) \div (x + 1)$

b. $(a^4 + 2a^3 - 7a^2 + 2a - 8) \div (a - 3)$

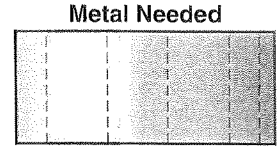
c. $(x^5 - 3x^2 - 20) \div (x - 2)$

Applications and Problem Solving

Real World

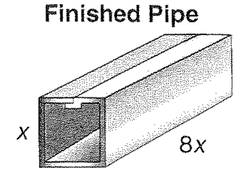
47. Manufacturing A machinist who makes square metal pipes found a formula for the amount of metal she needed to make a pipe. She found that to make a pipe $8x$ inches long, she needed $32x^2 + x$ square inches of metal. In figuring the area needed, the machinist allowed some fixed length of metal for overlap of the seam. If the width of the finished pipe will be x inches, how much did the machinist leave for the seam?

Metal Needed

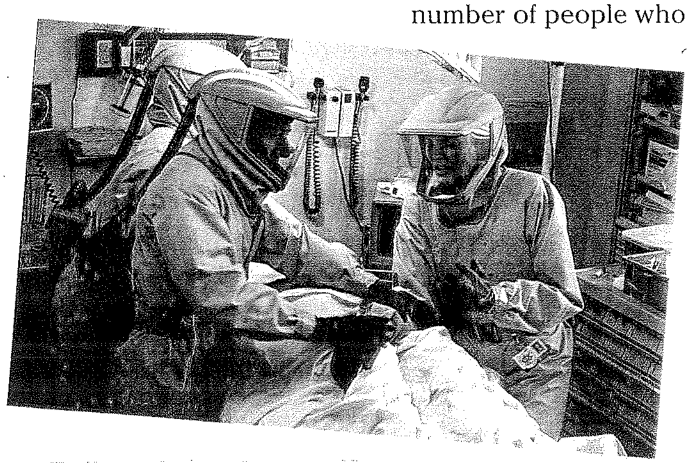


s = width of seam

Finished Pipe



48. Health In a 1995 film, a deadly disease threatens to wipe out the human race. Outbreaks of disease can become epidemics if there are no health preventatives available. The number of students and teachers at a large high school who will catch the flu during an outbreak in a certain year can be estimated by the formula $n = \frac{170t^2}{t^2 + 1}$, where t is the number of weeks from the beginning of the epidemic and n is the number of people who have become ill.



a. Perform the division indicated by $\frac{170t^2}{t^2 + 1}$.

b. Use the formula to estimate how many people will become ill during the first week.

c. Use your calculator to evaluate increasingly greater values of t. What happens to the value of n as t becomes very great?

Mixed Review

49. Find $(a - b)^2$. (Lesson 5–2)

50. **Astronomy** Earth is an average 1.496×10^8 kilometers from the sun. If light travels 3×10^5 kilometers per second, how long does it take sunlight to reach Earth? (Lesson 5–1)

51. **SAT Practice** If $a + b = c$ and $a = b$, then all of the following are true EXCEPT

    A $a - c = b - c$ **B** $a - b = 0$ **C** $2a + 2b = 2c$

    D $c - b = 2a$ **E** $a = \frac{c}{2}$

52. Find $\begin{bmatrix} -2 & 1 \\ 3 & -6 \\ 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & -3 & 7 \\ -3 & 2 & 9 & -1 \end{bmatrix}$. (Lesson 4–3)

53. Given $f(x, y) = 5x - 2y$, find $f(4, 1)$. (Lesson 3–5)

54. **Photography** The perimeter of a rectangular picture is 86 inches. Twice the width exceeds the length by 2 inches. What are the dimensions of the picture? (Lesson 3–2)

55. Find $h\left(-\frac{1}{2}\right)$ if $h(x) = [\![3x + 7]\!]$. (Lesson 2–6)

56. Write an equation in slope-intercept form for the line that has a slope of -2 and passes through the point at $(5, -3)$. (Lesson 2–4)

57. Solve $|x - 7| = 13$. (Lesson 1–6)

For **Extra Practice**, see page 886.

WORKING ON THE Investigation

Refer to the Investigation on pages 180–181.

3-2-1-Blast-Off!

You are interested in using mathematics to further evaluate the performance of the launchers during the shoot-off. There are four measures you can use to help calculate accuracy around the given target.

Error Range The longest shot minus the shortest shot is the error range.

Tolerance All data must lie within a tolerance distance around the target (target distance ± tolerance), Therefore, the tolerance is the distance between the target distance and the farthest shot from the target.

Relative Error The ratio of the tolerance to the target distance. This is expressed as a percent.

Average Error Using the target distance and each of the distance shots, find the absolute value of the difference between each of these values. Calculate the mean of those differences, by adding all the absolute differences and dividing the sum by 10 shots.

1. Find these four measures for each of the launchers used in your class' shoot-off.
2. Compare these measures for each of the launchers. How do these four measures help to determine which launcher was most accurate? What do each of these measures represent when analyzing the data? Does a higher number or a lower number show more accuracy for each of the measures? Explain for each measure.
3. Consider all the data. From that information, which launcher would you consider to be the most accurate? Explain your reasoning.

Add the results of your work to your Investigation Folder.

5-4

Factoring

What YOU'LL LEARN

- To factor polynomials, and
- to use factoring to simplify polynomial quotients.

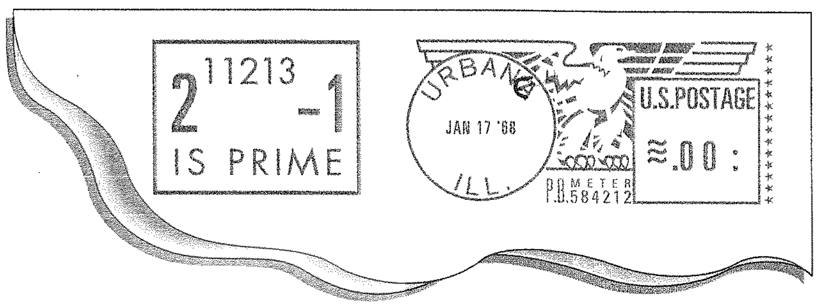
Why IT'S IMPORTANT

You can use factoring to solve problems involving history and geometry.

CONNECTION

History

You would think that someone would have found a formula for computing all the possible prime numbers, but no one has. A **prime number** is a whole number greater than 1 whose only **factors** are 1 and itself. In 1963, a computer at the University of Illinois calculated the largest prime number known at that time. In honor of this accomplishment, the postage meter at the mathematics department printed the number on their mail in 1968.



As of 1993, the largest prime number discovered was $391{,}581 \times 2^{216{,}193} - 1$. It was generated by a computer at the Amdahl Corporation in 1989 and has 65,087 digits.

Prime numbers can be used to factor whole numbers, as in the example $36 = 2 \cdot 2 \cdot 3 \cdot 3$. Many polynomials can also be factored. Their factors, however, are other polynomials. Polynomials that cannot be factored are called *prime*. The table below summarizes the most common factoring techniques used with polynomials.

Any Number of Terms	
Greatest Common Factor (GCF)	$a^3b^2 + 2a^2b - 4ab^2 = ab(a^2b + 2a - 4b)$
Two Terms	
Difference of Two Squares	$a^2 - b^2 = (a + b)(a - b)$
Sum of Two Cubes	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Difference of Two Cubes	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
Three Terms	
Perfect Square Trinomials	$a^2 + 2ab + b^2 = (a + b)^2$ $a^2 - 2ab + b^2 = (a - b)^2$
General Trinomials	$acx^2 + (ad + bc)x + bd = (ax + b)(cx + d)$
Four or More Terms	
Grouping	$ra + rb + sa + sb = r(a + b) + s(a + b)$ $= (r + s)(a + b)$

Whenever you factor a polynomial, always look for a common factor first. Then determine if the resulting polynomial factor can be factored again using one or more of the methods listed on the previous page.

Example 1 **Factor $5k^3p - 3kp^2 + k^3p^5$.**

$$5k^3p - 3kp^2 + k^3p^5 = (5 \cdot k \cdot k \cdot k \cdot p) - (3 \cdot k \cdot p \cdot p) + (k \cdot k \cdot k \cdot p \cdot p \cdot p \cdot p \cdot p)$$
$$= (kp \cdot 5k^2) - (kp \cdot 3p) + (kp \cdot k^2p^4)$$
$$= kp(5k^2 - 3p + k^2p^4)$$

The GCF is kp. The remaining polynomial is not factorable.

Check this result by finding the product.

The GCF is also used in grouping to factor a polynomial of four or more terms.

Example 2 **Factor $b^3 - 3b^2 + 4b - 12$.**

$$b^3 - 3b^2 + 4b - 12 = (b^3 - 3b^2) + (4b - 12)$$ *Group to find a GCF.*
$$= b^2(b - 3) + 4(b - 3)$$ *Factor the GCF of each binomial.*
$$= (b - 3)(b^2 + 4)$$ *Distributive property*

Check this result by finding the product.

You can use algebra tiles to model factoring a trinomial.

MODELING MATHEMATICS

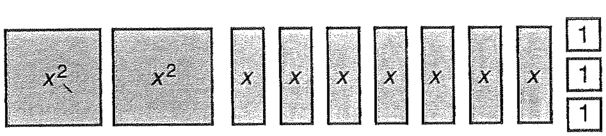
Factoring Trinomials

Materials: algebra tiles

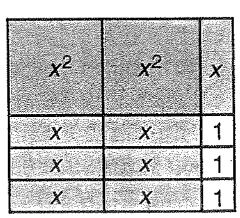
Use algebra tiles to factor $2x^2 + 7x + 3$.

Your Turn

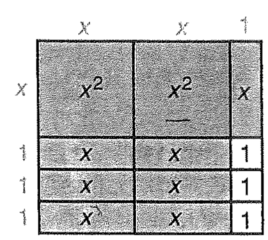
a. Use algebra tiles to model $2x^2 + 7x + 3$.



b. To find the product that resulted in this polynomial, arrange the tiles to form a rectangle.



c. Determine the dimensions of the rectangle.



The rectangle is $2x + 1$ units long and $x + 3$ units wide. The area of the rectangle can be expressed as $(2x + 1)(x + 3)$. Since the area of the rectangle is also $2x^2 + 7x + 3$, we can say that $2x^2 + 7x + 3 = (2x + 1)(x + 3)$. *Verify by using FOIL.*

You have used FOIL to multiply two binomials. Thinking about FOIL can help you factor a polynomial into the product of two binomials. Study the following example.

$$(ax + b)(cx + d) = \overbrace{ax \cdot cx}^{F} + \overbrace{ax \cdot d}^{O} + \overbrace{b \cdot cx}^{I} + \overbrace{b \cdot d}^{L}$$
$$= acx^2 + (ad + bc)x + bd$$

Notice that the product of the *coefficient* of x^2 and the *constant* term is *abcd*. The product of the two coefficients of the x terms, *bc* and *ad*, is also *abcd*.

Example 3 **Factor each polynomial.**

a. $\mathbf{7x^2 - 16x + 4}$

The product of the coefficient of the first term and the constant term is $7 \cdot 4$ or 28. So the two coefficients of the x terms must have a sum of -16 and a product of 28. You may need to use the guess-and-check strategy to find the two coefficients you need. The two coefficients must be -14 and -2 since $(-14)(-2) = 28$ and $-14 + (-2) = -16$.

Rewrite the expression using $-14x$ and $-2x$ in place of $-16x$ and factor by grouping.

$7x^2 - 16x + 4 = 7x^2 - 14x - 2x + 4$ *Substitute $-14x - 2x$ for $-16x$.*
$= (7x^2 - 14x) + (-2x + 4)$ *Associative property*
$= 7x(x - 2) - 2(x - 2)$ *Factor out the GCF of each group.*
$= (7x - 2)(x - 2)$ *Distributive property*

Check by using FOIL.

b. $\mathbf{2b^2x - 50x}$

$2b^2x - 50x = 2x(b^2 - 25)$ *Factor out the GCF.*
$= 2x(b + 5)(b - 5)$ *$b^2 - 25$ is the difference of two squares.*

c. $\mathbf{x^3y^3 + 64}$

$x^3y^3 = (xy)^3$ and $64 = 4^3$. Thus, this is the sum of two cubes.
In this example, $a = xy$ and $b = 4$.
$x^3y^3 + 64 = (xy + 4)[(xy)^2 - 4(xy) + 4^2]$
$= (xy + 4)(x^2y^2 - 4xy + 16)$

d. $\mathbf{a^6 - b^6}$

This polynomial could be considered the difference of two squares or the difference of two cubes. The difference of two squares should always be done before the difference of two cubes.

$a^6 - b^6 = (a^3 + b^3)(a^3 - b^3)$ *Difference of two squares*
$= (a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2)$ *Difference and sum of two cubes*

Check by using the distributive property.

You can use a graphing calculator to check that the factored form of a polynomial is correct.

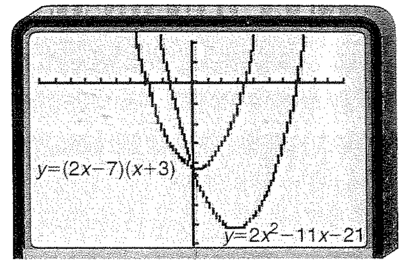
EXPLORATION GRAPHING CALCULATORS

Is the factored form of $2x^2 - 11x - 21$ equal to $(2x - 7)(x + 3)$? You can graph the polynomial and then the factored expression. If the two graphs coincide, the factored form is probably correct.

a. Press [Y=]. Enter $2x^2 - 11x - 21$ for Y1 and $(2x - 7)(x + 3)$ for Y2.

b. Set your viewing window for $[-10, 10]$ by $[-40, 10]$.

c. Press [GRAPH]. Notice that two different graphs appear. Thus, $2x^2 - 11x - 21 \neq (2x - 7)(x + 3)$.



Your Turn

You can use this method to check the factoring you do in this lesson.

a. Determine if $x^2 + 5x - 6 = (x - 3)(x - 2)$ is a true statement. If it is not correct, state the correct factorization.

b. Why doesn't this method guarantee a way to check the factored form of a polynomial?

In Lesson 5–3, you learned to simplify the quotient of two polynomials by using long division or synthetic division. You can determine if one polynomial is a factor of another by using division.

Example 4 **a. Show that $4 - w$ is a factor of $w^3 - 6w^2 + 13w - 20$.**
b. Find another factor of $w^3 - 6w^2 + 13w - 20$.

a.

$$\begin{array}{r} -w^2 + 2w - 5 \\ -w + 4\overline{)w^3 - 6w^2 + 13w - 20} \\ \underline{w^3 - 4w^2} \qquad\qquad\quad \\ -2w^2 + 13w \qquad \\ \underline{-2w^2 + 8w} \qquad \\ 5w - 20 \\ \underline{5w - 20} \\ 0 \end{array}$$

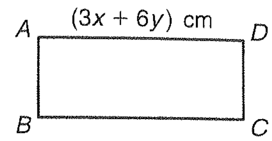
$4 - w = -w + 4$

Since the remainder is 0, $4 - w$ is a factor of $w^3 - 6w^2 + 13w - 20$.

b. The quotient in part a, $-w^2 + 2w - 5$, is another factor since $(4 - w)(-w^2 + 2w - 5) = w^3 - 6w^2 + 13w - 20$. Since $-w^2 + 2w - 5$ is not factorable, these are the only two factors of the polynomial.

Example 5 Find the width of rectangle $ABCD$ if its area is $(3x^2 + 9xy + 6y^2)$ cm^2.

INTEGRATION
Geometry



$A = \ell w$

$w = \frac{A}{\ell}$ *Solve for w*

$w = \frac{3x^2 + 9xy + 6y^2}{3x + 6y}$ *$A = 3x^2 + 9xy + 6y^2$, $\ell = 3x + 6y$*

$w = \frac{(3x + 6y)(x + y)}{3x + 6y}$ *Factor the numerator*

$w = \frac{\overset{1}{\cancel{(3x + 6y)}}(x + y)}{\underset{1}{\cancel{3x + 6y}}}$

$w = x + y$

The width of the rectangle is $(x + y)$ cm.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **Decide** which of the following is the complete factorization of $12x^2 - 8x - 15$.
   a. $(4x + 3)(3x - 5)$
   b. $(6x - 5)(2x + 3)$
   c. $(6x + 5)(2x - 3)$
   d. cannot be factored
2. **List** the cubes and squares of the numbers from 1 to 10. Explain how this list might help you in factoring.
3. **Explain** how to completely factor $2x^2 - 6x - 20$.
4. **Show** another way to group the terms to factor the polynomial in Example 2. Was the final result the same?
5. **Factor** the polynomial in Example 3d by using the difference of two cubes. Explain why it is better to factor a difference of squares first.

MODELING MATHEMATICS

6. Draw two different geometric models of $2x^2 + 6x$.
   a. What property is shown in each model?
   b. What is the completely-factored form of the polynomial?

Guided Practice

Factor completely. If the polynomial is not factorable, write *prime*.

7. $-15x^2 - 5x$
8. $m^2 - 6m + 8$
9. $x^2 + xy + 3x$
10. $16r^2 - 169$
11. $y^2 - 3y - 10$
12. $a^2 + 5a + 6$
13. $3h^2 - 48$
14. $2r^3 - 16s^3$
15. $g^3 + 8000$
16. $21 - 7y + 3x - xy$
17. Determine if $3y - 2$ is a factor of $6y^3 - y^2 - 5y + 2$. If so, what is the other factor?

EXERCISES

Practice

Factor completely. If the polynomial is not factorable, write *prime.*

18. $3a^2bx + 15cx^2y + 25ad^3y$
19. $10a^3b - 12a^2b^2$
20. $w^2 + 10w + 9$
21. $16n^2 + 25m^2$
22. $3x^2 - 3y^2$
23. $y^2 - 12y + 20$
24. $12ab^3 - 8a^2b^2 + 10a^5b^3$
25. $y^2 + 7y + 6$
26. $x^2 - 5x + 4$
27. $x^4 - y^2$
28. $6m^2 + 13m + 6$
29. $3n^2 + 21n - 24$
30. $3ay^2 + 9a$
31. $3a^2 - 27b^2$
32. $a^2 + 8ab + 16b^2$
33. $5x - 14 + x^2$
34. $2x^2 + 3x + 1$
35. $5x^2 + 15x - 10$
36. $2a^2 + 13a - 7$
37. $3a^2 + 24a + 45$
38. $12z^2 - z - 6$
39. $m^2n^2 + mn + 1$
40. $8ax - 6x - 12a + 9$
41. $4ax + 14ay - 10bx - 35by$
42. $10w^2 - 14wv - 15w + 21v$
43. $81y^2 - 49$
44. $6a^2 + 27a - 15$
45. $2x^4 + 4x^3 + 2x^2$
46. $m^4 - 1$
47. $y^4 - 16$
48. $7mx^2 + 2nx^2 - 7my^2 - 2ny^2$
49. $8a^2 + 8ab + 8ac + 3a + 3b + 3c$
50. $5a^2x + 4aby + 3acz - 5abx - 4b^2y - 3bcz$
51. $3x^3 + 2x^2 - 5x + 9x^2y + 6xy - 15y$

Tami Slagle

52. Determine the value of k so that $x - 4$ is a factor of $x^2 + 8x + k$.
53. Use factoring to simplify $\left(\frac{n^2 + 2n - 15}{n^2 + 3n - 10}\right)\left(\frac{n^2 - 9}{n^2 - 9n + 14}\right)^{-1}$.
54. One factor of $m^2 - k^2 + 6k - 9$ is $(m - k + 3)$. What is the other factor?

Graphing Calculator

Use a graphing calculator to determine if each polynomial is factored correctly. For each correct factorization, sketch the graph shown on the screen. For each incorrect factorization, write the correct factorization and sketch the graph.

55. $x^2 + 6x + 9 = (x + 3)(x + 3)$
56. $3x^2 + 5x + 2 = (3x + 2)(x + 1)$
57. $x^3 + 8 = (x + 2)(x^2 - x + 4)$
58. $3x^2 - 48 = 3(x + 4)(x - 4)$
59. $2x^2 - 5x - 3 = (x - 1)(2x + 3)$

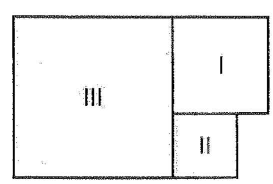
Critical Thinking

60. Factor $49p^{2n} + 14p^n + 1$.

Applications and Problem Solving

Real World

61. **Geometry** The figure is made of three squares.



   a. Suppose the area of square I is y^2 square units and the perimeter of square II is $4x$ units. Write a polynomial for the area of square III.
   b. Suppose the area of square I is 400 square units and the perimeter of square II is 44 units. Find the area of square III.

62. **Flags** The largest flag flown from a flagpole is a Brazilian national flag measuring 100 meters by 70 meters in Brasilia, the capital of Brazil.
   a. Find the area and the perimeter of the flag.
   b. Suppose a company made a flag 1 meter longer and wider than the Brazilian flag. What would be the area and the perimeter of that flag?
   c. Suppose a company makes a flag and increases the length and width by x feet. Find the area and perimeter of that flag. Write your answers as polynomials.

Mixed Review

63. Find $(t^3 - 3t + 2) \div (t + 2)$ by using synthetic division. (Lesson 5–3)
64. Use the FOIL method to find $(2x + 4)(7x - 1)$. (Lesson 5–2)
65. **Physics** Light from a laser travels about 300,000 kilometers per second. How many kilometers can it travel in a day? Write your answer in scientific notation. (Lesson 5–1)
66. Write the system of equations represented by $\begin{bmatrix} 5 & 4 \\ 3 & -5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ -24 \end{bmatrix}$. (Lesson 4–6)
67. Find $-2\begin{bmatrix} \frac{1}{2} & 3 \\ 5 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. (Lesson 4–2)
68. **SAT Practice** **Quantitative Comparison**

Column A	Column B
The area of a rectangle with length $3x$ and width x.	The area of a circle with radius x.

   A if the quantity in Column A is greater
   B if the quantity in Column B is greater
   C if the two quantities are equal
   D if the relationship cannot be determined from the information given

69. Solve the system of equations by using Cramer's Rule. (Lesson 3–3)
   $2x + 3y - 8 = 0$
   $3x + 2y - 17 = 0$
70. **Statistics** The table below shows the ideal weight for a man for a given height. (Lesson 2–5)

Height (inches)	66	68	70	72	74	76	78
Weight (pounds)	143	153	164	171	183	198	206

   a. Use the information in the chart to write a prediction equation for the relationship between a man's height and his ideal weight.
   b. Predict the ideal weight for a man who is 71 inches tall.
   c. Predict the height of a man who is at his ideal weight of 190 pounds.
71. Write an equation for the line that passes through the point (2, 2) that is perpendicular to the graph of $2x + 3y - 1 = 4$. (Lesson 2–4)
72. Write an algebraic expression to represent *the sum of a number and five times its square.* (Lesson 1–5)
73. State the property illustrated by $(4 + 11)6 = 4(6) + 11(6)$. (Lesson 1–2)

For Extra Practice, see page 886.

5-5 Roots of Real Numbers

What YOU'LL LEARN

- To simplify radicals having various indices, and
- to use a calculator to estimate roots of numbers.

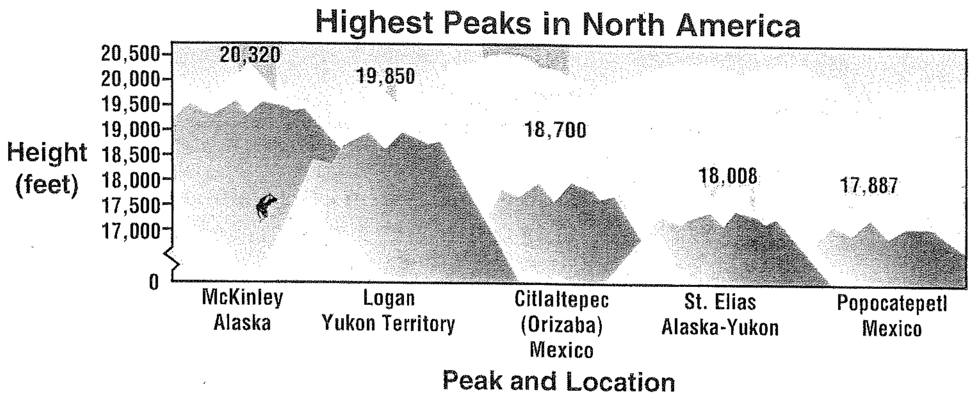
Why IT'S IMPORTANT

You can use the roots of real numbers to solve problems involving baseball and tourism.

Real World APPLICATION

Tourism

The graph shows the five highest mountain peaks in North America.



Source: *World Almanac*

Teresa and some of her friends are among a group of students visiting Mexico as a part of their study of Spanish. They are visiting Monte Albán in the Oaxaca Plateau of Mexico, which is near Orizaba. Monte Albán was the ancient religious center of the Zapotec Indians.

From an observation point at Monte Albán, Teresa and her friends can see a nearby river that is about 35 miles away. She guesses that the observation point is about 1000 feet high. Her friends guess the point to be higher than that. The formula for estimating the distance d (in miles) that can be seen from the top of a mountain of height h (in feet) is $d = 1.2\sqrt{h}$. Whose guess will be closer to the actual height of the observation point?

To determine the better estimate, we can rewrite the formula as $\sqrt{h} = \frac{d}{1.2}$. If $d = 35$ miles, then the following is true.

$$\sqrt{h} = \frac{35}{1.2} \quad \textit{Estimate: Will } \sqrt{h} \textit{ be greater or less than 35?}$$

$$\sqrt{h} = 29.1\overline{6} \quad \textit{Use a calculator.}$$

Now we need to determine a number h whose square root is $29.1\overline{6}$.

Finding the square root of a number and squaring a number are inverse operations. To find the square root of a number n, you must find a number whose square is n. For example, a square root of 36 is 6 since $6^2 = 36$. Since $(-6)^2 = 36$, -6 is also a square root of 36.

fabulous FIRSTS

Alison Hargreaves (1962–1995)

Alison Hargreaves, a Scottish mountaineer, was the first woman to climb Mt. Everest alone and without oxygen. She reached the summit of Everest (29,028 ft) on May 13, 1995.

Definition of Square Root

For any real numbers a and b, if $a^2 = b$, then a is a square root of b.

Using the definition for the situation above, $29.1\overline{6}$ is the square root of h. So, $h = (29.1\overline{6})^2$ or about 850.7 feet. Teresa's estimate of the height is fairly close.

Since finding the square root of a number and squaring a number are inverse operations, it makes sense that the inverse of raising a number to the nth power is finding the **nth root** of the number. For example, the table below shows the relationship between raising a number to a power and taking that root of the number.

Powers	Factors	Roots
$a^3 = 64$	$4 \cdot 4 \cdot 4 = 64$	4 is a cube root of 64
$a^4 = 16$	$2 \cdot 2 \cdot 2 \cdot 2 = 16$	2 is a fourth root of 16
$a^5 = 243$	$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 243$	3 is a fifth root of 243
$a^n = b$	$\underbrace{a \cdot a \cdot a \cdot a \cdot \cdots \cdot a}_{n \text{ factors of } a} = b$	a is an nth root of b

The pattern suggests the following formal definition of the nth root.

Definition of nth Root	**For any real numbers a and b, and any positive integer n, if $a^n = b$, then a is an nth root of b.**

$\sqrt[n]{98}$ is read "the nth root of 98."

The symbol $\sqrt[n]{\ }$ indicates an nth root.

index → $\sqrt[n]{98}$ ← *radical sign*

$\sqrt[n]{98}$ ← *radicand*

FYI

The radical sign was first introduced in 1525 by Christoff Rudolff in his algebra book *Die Coss*. This symbol was probably chosen because it resembled a small *r*, the first letter in the word *radix*, which means root.

Some numbers have more than one real nth root. For example, 49 has two square roots, 7 and -7. When there is more than one real root, the nonnegative root is called the **principal root**. When no index is given, as in $\sqrt{49}$, the radical sign indicates the principal square root. The symbol $\sqrt[n]{b}$ stands for the principal nth root of b. If n is odd and b is negative, there will be no nonnegative root. In this case, the principal root is negative.

$\sqrt{64} = 8$ — $\sqrt{64}$ indicates the principal square root of 64.

$-\sqrt{64} = -8$ — $-\sqrt{64}$ indicates the opposite of the principal square root of 64.

$\pm\sqrt{64} = \pm 8$ — $\pm\sqrt{64}$ indicates both square roots of 64.
$\pm$ means positive or negative.

$\sqrt[3]{-27} = -3$ — $\sqrt[3]{-27}$ indicates the principal cube root of -27.

$-\sqrt[4]{16} = -2$ — $-\sqrt[4]{16}$ indicates the opposite of the principal fourth root of 16.

The chart below gives a summary of the real nth roots of a number b.

Real nth Roots of b, $\sqrt[n]{b}$, or $-\sqrt[n]{b}$			
n	$b > 0$	$b < 0$	$b = 0$
even	one positive root one negative root	no real roots	one real root, 0
odd	one positive root no negative roots	no positive roots one negative root	

Example 1 **Find each root.**

a. $\pm\sqrt{49x^8}$

$\pm\sqrt{49x^8} = \pm\sqrt{(7x^4)^2}$

$= \pm 7x^4$

The square roots of $49x^8$ are $\pm 7x^4$.

b. $-\sqrt{(a^2 + 1)^4}$

$-\sqrt{(a^2 + 1)^4} = -\sqrt{[(a^2 + 1)^2]^2}$

$= -(a^2 + 1)^2$

The opposite of the principal square root of $(a^2 + 1)^4$ is $-(a^2 + 1)^2$.

c. $\sqrt[5]{32x^{10}y^{15}}$

$\sqrt[5]{32x^{10}y^{15}} = \sqrt[5]{(2x^2y^3)^5}$

$= 2x^2y^3$

The principal fifth root of $32x^{10}y^{15}$ is $2x^2y^3$.

d. $\sqrt{-16}$

In this case of $\sqrt[n]{b}$, n is even and b is negative. Thus, $\sqrt{-16}$ has no real root.

When you find the nth root of an even power and an odd power is the result, you must take the absolute value of the result to ensure that the value is nonnegative.

$$\sqrt{(-3)^2} = |-3| \text{ or } 3 \qquad \sqrt{(-2)^{10}} = |(-2)^5| \text{ or } 32$$

If the result is an even power or you find the nth root of an odd power, there is no need to take the absolute value. *Why?*

Example 2 **Find each root.**

a. $\sqrt[6]{x^6}$

Since $x^6 = x \cdot x \cdot x \cdot x \cdot x \cdot x$, x is the sixth root of x^6. The index is even, so the principal root is nonnegative. However, since x could be negative, we must take the absolute value of x to identify the principal root.

$\sqrt[6]{x^6} = |x|$

b. $\sqrt[4]{16(x + 3)^{12}}$

$\sqrt[4]{16(x + 13)^{12}} = \sqrt[4]{2^4[(x + 3)^3]^4}$

Since the index is even and the power is odd, we must use the absolute value of $(x + 3)^3$.

$\sqrt[4]{16(x + 3)^{12}} = 2|(x + 3)^3|$

LOOK BACK

You can review real and irrational numbers in Lesson 2-8.

Recall that real numbers that cannot be expressed as terminating or repeating decimals are *irrational numbers*. $\sqrt{2}$ and $\sqrt{3}$ are examples of irrational numbers. Decimal approximations for irrational numbers, such as 3.14 for π, are often used in applications. You can use a calculator to find decimal approximations.

Example 3 **Use a calculator to find a decimal approximation for $\sqrt[5]{279}$.**

Use the root key. *It may be a second function key.*

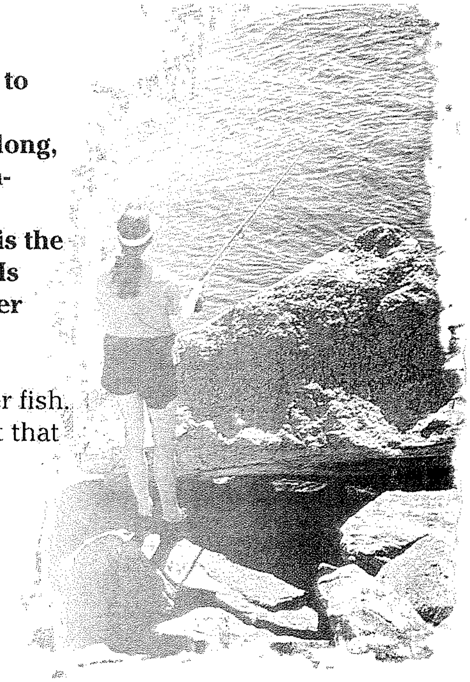
Enter: 279 [$\sqrt[x]{y}$] 5 [=] 3.084045954

Check: 3.084045954 [y^x] 5 [=] 279 ✓

Example 4 Real World APPLICATION

Fishing

Carla, upon returning from a fishing trip, bragged to her friends that she caught a Pacific halibut that weighed 23 kilograms and was at least 2.5 meters long, but it got away as she was hauling it in. The length-to-weight relationship for Pacific halibut can be estimated by the formula $L = 0.46\sqrt[3]{W}$, where W is the weight in kilograms and L is the length in meters. Is there a possibility that Carla told the truth, or is her story a little "fishy?"



Suppose that Carla is correct about the weight of her fish. Calculate the approximate length of a Pacific halibut that weighs 23 kilograms.

$$L = 0.46\sqrt[3]{W}$$
$$= 0.46\sqrt[3]{23}$$
$$= 0.46(2.84)$$
$$\approx 1.3064 \quad \textit{Use a calculator.}$$

Carla's fish probably was a little more than 1 meter long. So her story is a little "fishy."

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **Explain** why it is not always necessary to take the absolute value of a result to indicate the principal root.
2. **Describe** how you would check the nth root of a number using your calculator.
3. **Explain** whether or not $\sqrt[5]{-32}$ is a real number.
4. **Determine** if each statement is true, regardless of the value of x. Explain your answers.

   a. $\sqrt[4]{(-x)^4} = x$ **b.** $\sqrt[5]{(-x)^5} = x$

MATH JOURNAL

5. Copy and complete the table below. If there are things you do not understand about powers and roots, reread and study the lesson as necessary.

Powers	In words	Roots	In words
$2^3 = 2 \cdot 2 \cdot 2 = 8$	2 cubed is 8.	$\sqrt[3]{8} = \sqrt[3]{2 \cdot 2 \cdot 2} = 2$	The cube root of 8 is 2.
$8^4 = 4096$	8 to the fourth power is __?__ .		A fourth root of 4096 is 8.
			The fifth root of 16,807 is 7.
		$\sqrt[6]{729} =$ __?__	
$a^n = b$			

Guided Practice

Use a calculator to approximate each value to three decimal places.

6. $\sqrt{99}$
7. $-\sqrt[3]{23}$
8. $\sqrt[4]{64}$

Simplify.

9. $\sqrt{(-3)^2}$
10. $\sqrt[3]{27}$
11. $\sqrt[5]{-32}$
12. $\sqrt[4]{-10{,}000}$
13. $\sqrt[3]{m^3}$
14. $\sqrt[4]{x^4}$
15. $-\sqrt{25x^6}$
16. $\sqrt{25x^2y^4}$
17. $\sqrt{(3m-2n)^2}$
18. Find the principal fifth root of 32.
19. Find the principal third root of -125.

EXERCISES

Practice

Use a calculator to approximate each value to three decimal places.

20. $\sqrt{121}$
21. $-\sqrt{144}$
22. $\sqrt[3]{65}$
23. $\sqrt{0.81}$
24. $\sqrt[3]{-670}$
25. $\sqrt[4]{625}$
26. $\sqrt[7]{82{,}567}$
27. $\sqrt[6]{(345)^3}$
28. $\sqrt[4]{(3600)^2}$

Simplify.

29. $\sqrt{16}$
30. $\pm\sqrt{121}$
31. $\sqrt{196}$
32. $\sqrt{-(-6)^2}$
33. $\sqrt[4]{81}$
34. $-\sqrt{121}$
35. $\sqrt[4]{\left(-\frac{1}{2}\right)^4}$
36. $\sqrt[3]{-27}$
37. $\sqrt[3]{1000}$
38. $\sqrt{0.36}$
39. $\sqrt[3]{-0.125}$
40. $\sqrt[7]{-1}$
41. $\sqrt[3]{y^3}$
42. $\sqrt[4]{t^8}$
43. $-\sqrt[4]{x^4}$
44. $\sqrt{(5f)^4}$
45. $\sqrt{36g^6}$
46. $\sqrt{64h^8}$
47. $\sqrt[3]{m^6n^9z^{12}}$
48. $\sqrt[3]{8b^3c^3}$
49. $\sqrt[3]{-27a^9b^{12}}$
50. $\pm\sqrt{(a^3+2)^2}$
51. $\sqrt[3]{(s+t)^3}$
52. $\sqrt{(3x+y)^2}$
53. $-\sqrt{x^2+2x+1}$
54. $\sqrt{x^2+6x+9}$
55. $\pm\sqrt{s^2-2st+t^2}$
56. $\sqrt[5]{-(2m-3n)^5}$
57. $\sqrt{-4y^2-12y-9}$
58. $\pm\sqrt{16m^2-24mn+9n^2}$

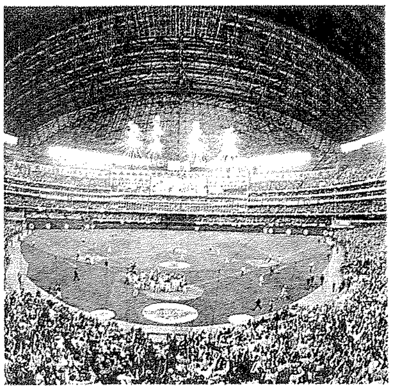
Critical Thinking

59. Under what condition does $\sqrt{x^2+y^2}=x+y$?

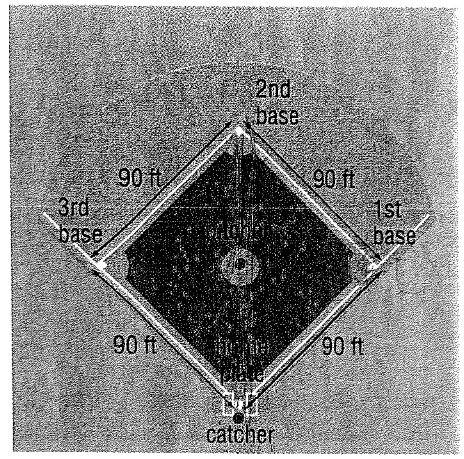
Applications and Problem Solving

Real World

60. **Aerospace Engineering** Scientists expect that in future space stations, artificial gravity will be created by rotating all or part of the space station. The formula $N=\frac{1}{2\pi}\sqrt{\frac{a}{r}}$ gives the number of rotations N required per second to maintain an acceleration of gravity of a meters per second squared on a satellite with a radius of r meters. The acceleration of gravity of Earth is 9.8 m/s^2. How many rotations per minute will produce an artificial gravity that is equal to half the acceleration of gravity on Earth in a space station with a 25-meter radius?



61. **Baseball** During a practice session between a baseball double-header, a catcher practiced throwing the baseball to each of the bases. If a standard baseball diamond has sides 90 feet long, how far did the catcher have to throw the ball to reach second base? (*Hint:* Draw a diagram and use the Pythagorean formula $c = \sqrt{a^2 + b^2}$, where c is the length of the hypotenuse of a right triangle and a and b are the lengths of the legs.)



62. **Physics** Rashad and Alberto are part of the staff in a fun house at their school carnival that is raising money for Muscular Dystrophy. Their job is to drop water-filled balloons from a 25-foot wall on unsuspecting students as they stand on a particular spot in the fun house. The formula for determining the time t it takes a water balloon to reach the ground is $t = \sqrt{\frac{2d}{g}}$, where d is the height in feet from which the object is dropped and g is the acceleration due to gravity, which equals 32.2 ft/s^2. Migdalia and Lawanda have told Rashad and Alberto that they can avoid getting hit by a balloon by moving quickly out of the way. How much time do the girls have to avoid getting wet? (Assume very little air friction and the girls are about 5 feet tall.)

Mixed Review

63. Simplify $\sqrt{(5b)^4}$. (Lesson 5–5)

64. Simplify $(2p + q^3)^2$. (Lesson 5–2)

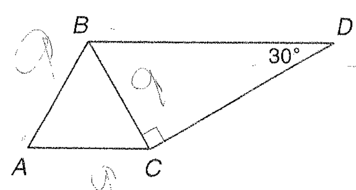
65. Simplify $y^2x^{-3}(yx^4 + y^{-1}x^3 + y^{-2}x^2)$. (Lesson 5–2)

66. **Astronomy** Moonlight takes 1.28 seconds to reach Earth. If the speed of light is 3×10^5 kilometers per second, how far is the moon from Earth? (Lesson 5–1)

67. Solve $\begin{vmatrix} 5 & 7 \\ -2 & 2x \end{vmatrix} = 54$. (Lesson 4–4)

68. Find $\begin{bmatrix} 5 & -2 \\ \frac{1}{2} & -3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. (Lesson 4–3)

69. **ACT Practice** In the figure, $\triangle ABC$ is an equilateral triangle with $\overline{AC}$ 9 units long. If $\angle BCD$ is a right angle and $\angle D$ measures 30°, what is the length BD, in units?



A 3 B 9

C $9\sqrt{2}$ D 18

E It cannot be determined from the information given.

70. Solve the system of equations by graphing. (Lesson 3–1)

$2x + 3y = -16$

$2y = 4x$