Graph each equation.

71. $b = 2[a] - 3$ (Lesson 2–6)

72. $b = 2a - 3$ (Lesson 2–2)

73. **Time Management** The chart below illustrates the time in hours per week that unmarried young people between the ages of 12 and 17 and the ages of 18 and 29 spend on various activities. (Lesson 1–4)

Activity	12–17 Years	18–29 Years
Leisure activities:		
Eating	8.1	7.5
Sleeping	62.6	55.9
Attending sports events	0.8	0.3
Attending cultural events	0.3	0.4
Going to movies	0.6	0.7
Visiting, socializing	4.4	7.8
Participating in sports	3.3	1.8
Watching TV	17.7	14.2
Reading	1.3	1.9
Work/school activities:		
Attending classes	18.1	4.1
Cleaning, laundry	2.2	3.2
Doing homework	3.1	3.4
Washing, dressing	6.4	6.8
Working	2.8	29.0
Yardwork, repairs	1.9	1.2

a. Find the mean and median of the leisure activities for both age groups to the nearest tenth.

b. Find the mean and median of the work/school activities for both age groups to the nearest tenth.

c. Does the distribution of the data say anything to you about the priorities of these two groups? What can you say about the differences between the ways the two groups of people spend their time?

For Extra Practice, see page 887.

SELF TEST

1. **Astronomy** The distance from Earth to the sun is approximately 93,000,000 miles. Write this number in scientific notation. (Lesson 5–1)

2. Write $8mn^{-5}$ without using negative exponents. (Lesson 5–1)

Simplify. (Lessons 5–1, 5–2, and 5–3)

3. $(-3x^2y)^3(2x)^2$

4. $(9x + 2y) + (7x + 3y)$

5. $(n + 2)(n^2 - 3n + 1)$

6. $(2d^4 + 2d^3 - 9d^2 - 3d + 9) \div (2d^2 - 3)$

7. Use synthetic division to find $(m^3 - 4m^2 - 3m - 7) \div (m - 4)$. (Lesson 5–3)

Factor completely. (Lesson 5–4)

8. $ax^2 + 6ax + 9a$

9. $8r^3 - 64s^6$

Simplify. (Lesson 5–5)

10. $\sqrt[3]{-64a^6b^9}$

11. $\sqrt{4n^2 + 12n + 9}$

5-6 Radical Expressions

What YOU'LL LEARN

- To simplify radical expressions,
- to rationalize the denominator of a fraction containing a radical expression, and
- to add, subtract, multiply, and divide radical expressions.

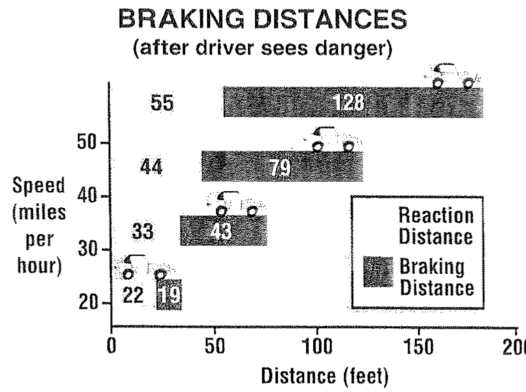
Why IT'S IMPORTANT

You can use radical expressions to solve problems involving sports and law enforcement.

Real World APPLICATION

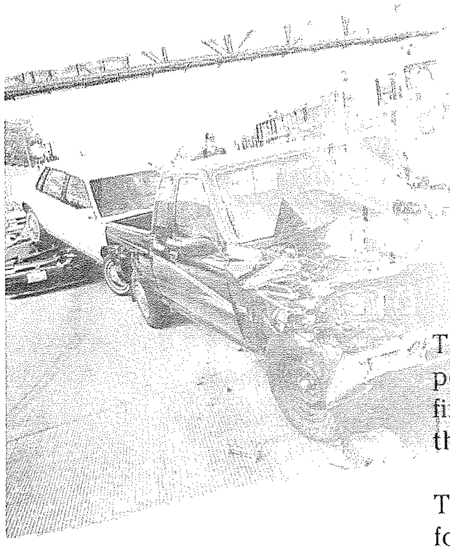
Law Enforcement

The graph at the right illustrates the distance a car travels after the driver sees danger and applies the brakes. These distances are for good weather conditions. If the road is wet or if the car has worn brakes, the car will travel farther than the distances shown.



After an accident, police investigators use the formula $s = 2\sqrt{5\ell}$ to estimate the speed s of a car in miles per hour. The variable ℓ represents the length in feet of the tire skid marks on the pavement. On one occasion, an accident scene investigation team measured skid marks 120 feet long. How fast was the car traveling?

Let's explore two ways of applying the formula.



Method 1

$$
\begin{aligned}
s &= 2\sqrt{5\ell} \\
&= 2\sqrt{5 \cdot 120} \\
&= 2\sqrt{5} \cdot \sqrt{120} \\
&\approx 2(2.2361)(10.9545) \\
&\approx 48.99
\end{aligned}
$$

The car was going about 49 miles per hour. Using this method, we first find each of the roots and then multiply.

Method 2

$$
\begin{aligned}
s &= 2\sqrt{5\ell} \\
&= 2\sqrt{5 \cdot 120} \\
&= 2\sqrt{600} \\
&\approx 2(24.4949) \\
&\approx 48.99
\end{aligned}
$$

The car was going about 49 miles per hour. Using this method, we find the root of the product.

The result is the same using either method. These examples demonstrate the following property of radicals.

Product Property of Radicals	For any real numbers a and b, and any integer n, $n > 1$, 1. if n is even, then $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ when a and b are both nonnegative, and 2. if n is odd, then $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.

When you simplify a square root, first write the prime factorization of the radicand. Then use the product property to isolate the perfect squares. Then simplify each radical.

Example 1 **Simplify $\sqrt{81p^4q^3}$.**

$$\sqrt{81p^4q^3} = \sqrt{9^2 \cdot (p^2)^2 \cdot q^2 \cdot q}$$ *Factor into squares if possible.*

$$= \sqrt{9^2} \cdot \sqrt{(p^2)^2}\sqrt{q^2} \cdot \sqrt{q}$$ *Product property of radicals*

$$= 9 \cdot p^2 |q| \sqrt{q}$$

However, in order for $\sqrt{q}$ to be defined, q must be positive. Therefore, the absolute value is unnecessary.

$$\sqrt{81p^4q^3} = 9p^2q\sqrt{q}$$

Simplifying nth roots is very similar to simplifying square roots. Find the factors that are nth powers and use the product property.

Example 2 **Simplify $7\sqrt[3]{27n^2} \cdot 4\sqrt[3]{8n}$.**

$$7\sqrt[3]{27n^2} \cdot 4\sqrt[3]{8n} = 7 \cdot 4 \cdot \sqrt[3]{27n^2 \cdot 8n}$$ *Product property of radicals*

$$= 28 \cdot \sqrt[3]{3^3 \cdot 2^3 \cdot n^3}$$ *Factor into cubes where possible.*

$$= 28 \cdot \sqrt[3]{3^3} \cdot \sqrt[3]{2^3} \cdot \sqrt[3]{n^3}$$ *Product property of radicals*

$$= 28 \cdot 3 \cdot 2 \cdot n \text{ or } 168n$$

Let's look at the radicals that involve division to see if there is a quotient property of radicals similar to the product property. Consider $\sqrt{\frac{25}{16}}$. First simplify the radical using the product property of radicals. Then try what you think the quotient property of radicals might be.

Method 1: Product Property

$$\sqrt{\frac{25}{16}} = \sqrt{25 \cdot \frac{1}{16}}$$

$$= \sqrt{25} \cdot \sqrt{\frac{1}{16}}$$

$$= \sqrt{5^2} \cdot \sqrt{\left(\frac{1}{4}\right)^2}$$

$$= 5 \cdot \frac{1}{4} \text{ or } \frac{5}{4}$$

Method 2: Quotient Property

$$\sqrt{\frac{25}{16}} = \frac{\sqrt{25}}{\sqrt{16}}$$

$$= \frac{5}{4}$$

These results suggest that there is a quotient property of radicals.

Quotient Property of Radicals

For real numbers a and b, $b \neq 0$, and any integer n, $n > 1$,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$, **if all roots are defined.**

A radical expression is simplified when the following conditions are met.

- The index, n, is as small as possible.
- The radicand contains no factors (other than 1) that are nth powers of an integer or polynomial.
- The radicand contains no fractions.
- No radicals appear in the denominator.

To eliminate radicals from a denominator or fractions from a radicand, you can use a process called **rationalizing the denominator**. To rationalize a denominator, you must multiply the numerator and denominator by a quantity so that the radicand has an exact root. Study the examples below.

Example 3 **Simplify each expression.**

a. $\dfrac{\sqrt{b^4}}{\sqrt{a^3}}$

$$\frac{\sqrt{b^4}}{\sqrt{a^3}} = \frac{\sqrt{b^2 \cdot b^2}}{\sqrt{a^2 \cdot a}}$$

$$= \frac{\sqrt{b^2} \cdot \sqrt{b^2}}{\sqrt{a^2} \cdot \sqrt{a}}$$

$$= \frac{b \cdot b}{a \cdot \sqrt{a}}$$

$$= \frac{b^2}{a\sqrt{a}} \cdot \frac{\sqrt{a}}{\sqrt{a}}$$ *Rationalize the denominator.*

$$= \frac{b^2\sqrt{a}}{a\sqrt{a^2}} \text{ or } \frac{b^2\sqrt{a}}{a^2}$$

b. $\sqrt[5]{\dfrac{3}{4s^2}}$

$$\sqrt[5]{\frac{3}{4s^2}} = \frac{\sqrt[5]{3}}{\sqrt[5]{4s^2}} \cdot \frac{\sqrt[5]{8s^3}}{\sqrt[5]{8s^3}}$$ *Why use* $\frac{\sqrt[5]{8s^3}}{\sqrt[5]{8s^3}}$?

$$= \frac{\sqrt[5]{24s^3}}{\sqrt[5]{2^5s^5}}$$

$$= \frac{\sqrt[5]{24s^3}}{2s}$$ $4 = 2^2,\ 8 = 2^3$

Can you add radicals in the same way you multiply them? In other words, if $\sqrt{a} \cdot \sqrt{a} = \sqrt{a \cdot a}$, does $\sqrt{a} + \sqrt{a} = \sqrt{a + a}$?

MODELING MATHEMATICS

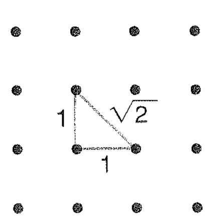
Adding Radicals

Materials: geometric dot paper, straightedge

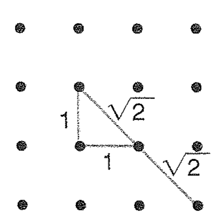
You can use geometric dot paper to show the sum of two like radicals, such as $\sqrt{2} + \sqrt{2}$. Is $\sqrt{2} + \sqrt{2} = \sqrt{2 + 2}$ or 2?

a. First find the length $\sqrt{2}$ units by using the Pythagorean theorem with the dot paper.

$$a^2 + b^2 = c^2$$
$$1^2 + 1^2 = c^2$$
$$2 = c^2$$
$$\sqrt{2} = c$$



b. Extend the segment to a length twice the length of $\sqrt{2}$ to represent $\sqrt{2} + \sqrt{2}$.



c. Is $\sqrt{2} + \sqrt{2} = \sqrt{2 + 2}$ or 2? Justify your answer.

Your Turn

Try this method to model other irrational numbers.

In the activity on the previous page, you discovered that you do not add radicals in the same manner as you multiply them. You add radicals in the same manner as adding monomials. That is, you can add only like terms or like radicals.

Two radical expressions are called **like radical expressions** if both the indices and the radicands are alike. Some examples of like and unlike radical expressions are given below.

$\sqrt[3]{2}$ and $\sqrt{2}$ are not like expressions.	*Different indices*
$\sqrt[4]{3a}$ and $\sqrt[4]{3}$ are not like expressions	*Different radicands*
$3\sqrt[4]{2x}$ and $4\sqrt[4]{2x}$ are like expressions.	*Radicands are 2x; indices are 4.*

Example 4 **Simplify $2\sqrt{20} - 2\sqrt{45} + 3\sqrt{80}$.**

$2\sqrt{20} - 2\sqrt{45} + 3\sqrt{80}$

$= 2\sqrt{2^2 \cdot 5} - 2\sqrt{3^2 \cdot 5} + 3\sqrt{2^2 \cdot 2^2 \cdot 5}$ *Rewrite each radicand*

$= 2\sqrt{2^2} \cdot \sqrt{5} - 2\sqrt{3^2} \cdot \sqrt{5} + 3\sqrt{2^2} \cdot \sqrt{2^2} \cdot \sqrt{5}$ *using its factors.*

$= 2 \cdot 2\sqrt{5} - 2 \cdot 3\sqrt{5} + 3 \cdot 2 \cdot 2\sqrt{5}$

$= 4\sqrt{5} - 6\sqrt{5} + 12\sqrt{5}$ *These are like radicals.*

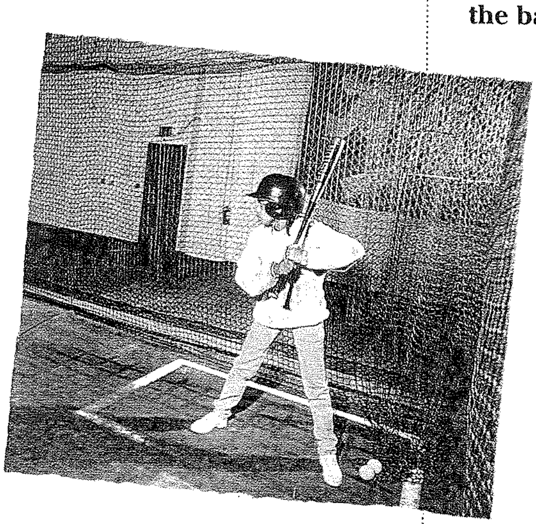
$= 10\sqrt{5}$

Example 5

Real World APPLICATION

Sports

As part of the tryout for a girls' softball team, each player must hit a series of balls in a batting cage. Ms. Johnson, the coach, determines the velocity of each hit with a speed gun. She uses the formula $d = v\sqrt{\frac{h}{4.9}}$ to estimate the distance the ball would have traveled if the ball had been hit in an open field. In the formula, v represents the velocity (in meters per second) of the baseball, and h is the height (in meters) from which the ball is hit. Zenobia hit two balls with speeds of 45 m/s and 47 m/s. If her bat is at a height of 0.8 meters from the ground, what is the difference between the distances these baseballs would have traveled?



Use the formula to express the differences of the two distances. Let D represent the difference of the distances, v_1 represent the velocity of the first hit, and v_2 represent the velocity of the second hit.

$D = v_2\sqrt{\frac{h}{4.9}} - v_1\sqrt{\frac{h}{4.9}}$

$= 47\sqrt{\frac{0.8}{4.9}} - 45\sqrt{\frac{0.8}{4.9}}$ *$h = 0.8$, $v_1 = 45$, and $v_2 = 47$*

$= (47 - 45)\sqrt{\frac{0.8}{4.9}}$ *Combine like radical expressions.*

$= 2\sqrt{\frac{0.8}{4.9}}$ *Estimate: Will the result be less than or greater than 2?*

Use a calculator to find an approximate value for this expression.

Enter: 2 [×] [(] 0.8 [÷] 4.9 [)] [√x] [=] 0.808122035

There is a difference of about 0.8 meters between the two hits.

Just as you can add and subtract radicals like monomials, you can multiply radicals using FOIL like you multiply binomials.

Example 6 **Simplify each expression.**

a. $(2\sqrt{6} - 3\sqrt{2})(3 + \sqrt{2})$

$$\begin{aligned}(2\sqrt{6} - 3\sqrt{2})(3 + \sqrt{2}) &= \overset{F}{2\sqrt{6}\cdot 3} + \overset{O}{2\sqrt{6}\cdot\sqrt{2}} - \overset{I}{3\sqrt{2}\cdot 3} - \overset{L}{3\sqrt{2}\cdot\sqrt{2}} \\ &= 6\sqrt{6} + 2\sqrt{2^2\cdot 3} - 9\sqrt{2} - 3\sqrt{2^2} \\ &= 6\sqrt{6} + 4\sqrt{3} - 9\sqrt{2} - 6\end{aligned}$$

b. $(3\sqrt{5} - 4)(3\sqrt{5} + 4)$

$$\begin{aligned}(3\sqrt{5} - 4)(3\sqrt{5} + 4) &= 3\sqrt{5}\cdot 3\sqrt{5} + 3\sqrt{5}\cdot 4 - 4\cdot 3\sqrt{5} - 4\cdot 4 \\ &= 9\sqrt{5^2} + 12\sqrt{5} - 12\sqrt{5} - 16 \\ &= 45 - 16 \\ &= 29\end{aligned}$$

Binomials like those in Example 6b, of the form $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$ where a, b, c, and d are rational numbers, are called **conjugates** of each other. The product of conjugates is always a rational number. We can use conjugates to rationalize denominators.

Example 7 **Simplify** $\frac{\sqrt{2} - 3}{\sqrt{2} + 7}$.

$$\begin{aligned}\frac{\sqrt{2} - 3}{\sqrt{2} + 7} &= \frac{(\sqrt{2} - 3)(\sqrt{2} - 7)}{(\sqrt{2} + 7)(\sqrt{2} - 7)} \\ &= \frac{(\sqrt{2} - 3)(\sqrt{2} - 7)}{(\sqrt{2})^2 - 49} \\ &= \frac{(\sqrt{2})^2 + \sqrt{2}(-7) + (-3)\sqrt{2} + (-3)(-7)}{-47} \\ &= \frac{2 - 7\sqrt{2} - 3\sqrt{2} + 21}{-47} \\ &= \frac{23 - 10\sqrt{2}}{-47}\end{aligned}$$

Multiply by $\frac{\sqrt{2} - 7}{\sqrt{2} - 7}$ because $\sqrt{2} - 7$ is the conjugate of $\sqrt{2} + 7$.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **Explain** how you would rationalize the denominator of $\frac{1}{\sqrt[3]{a^2b^3c}}$.
2. **Explain** why $7\sqrt{3} - 4\sqrt[3]{3} + 2\sqrt[3]{4}$ cannot be simplified further.
3. **Write** examples for each of the following.
   - **a.** two like radical expressions
   - **b.** two unlike radical expressions
   - **c.** two expressions that are conjugates of each other
4. **Explain** why the product of two conjugates is always a rational number.

5. Why is the product property of radicals for odd indices different than the product property for even indices?

6. **Replace** the $\underline{\ ?\ }$ in $\sqrt[n]{\frac{1}{c}} \underline{\ ?\ } \frac{1}{\sqrt[n]{c}}$ with an $=$ or $\neq$ to make the statement true. Write a reason for your answer.

MODELING MATHEMATICS

7. Use geometric dot paper to draw segments that represent each length.

   a. $\sqrt{10}$ b. $\sqrt{13}$ c. $\sqrt{17}$ d. $\sqrt{29}$

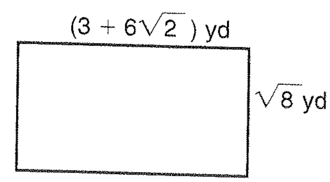
8. Describe how you could use models to show the sum $\sqrt{5} + \sqrt{10}$. Include a drawing in your explanation.

Guided Practice

Simplify.

9. $\sqrt{80}$
10. $4\sqrt{54}$
11. $\sqrt[3]{64x^6y^3}$
12. $\sqrt[4]{81m^4n^5}$
13. $(7\sqrt{6})(-3\sqrt{10})$
14. $\sqrt{3x^2z^3} \cdot \sqrt{15x^2z}$
15. $\frac{\sqrt[3]{81}}{\sqrt[3]{9}}$
16. $\sqrt{\frac{5}{12a}}$
17. $\sqrt{\frac{y^2}{y-3}}$
18. $\sqrt{2} + 5\sqrt[3]{2} + 7\sqrt{2} - 4\sqrt[3]{2}$
19. $5\sqrt[3]{135} - 2\sqrt[3]{81}$
20. $(5 + \sqrt{3})(2 - \sqrt{2})$
21. $(7 + \sqrt{11y})(7 - \sqrt{11y})$

22. **Geometry** Find the perimeter and area of the rectangle shown at the right.



EXERCISES

Practice

Simplify.

23. $\sqrt{32}$
24. $5\sqrt[3]{50}$
25. $\sqrt[3]{16}$
26. $\sqrt{98y^4}$
27. $\sqrt[3]{32}$
28. $\sqrt[4]{48}$
29. $\sqrt{y^3}$
30. $\sqrt{8a^2b^3}$
31. $5\sqrt{3} - 4\sqrt{3}$
32. $8\sqrt[3]{6} + 3\sqrt[3]{6}$
33. $\sqrt{90x^3y^4}$
34. $3\sqrt[3]{56a^6b^3}$
35. $(-3\sqrt{24})(5\sqrt{20})$
36. $\sqrt{26} \cdot \sqrt{39} \cdot \sqrt{14}$
37. $(4\sqrt{18})(2\sqrt{14})$
38. $8\sqrt{y^2} + 7\sqrt{y} - 4\sqrt{y}$
39. $\sqrt[3]{40} - 2\sqrt[3]{5}$
40. $(\sqrt{10} - \sqrt{6})(\sqrt{5} + \sqrt{3})$
41. $-3\sqrt{7}\,(2\sqrt{14} + 5\sqrt{2})$
42. $\sqrt{a}(\sqrt{b} + \sqrt{ab})$
43. $8\sqrt[3]{2x} + 3\sqrt[3]{2x} - 8\sqrt[3]{2x}$
44. $5\sqrt{20} + \sqrt{24} - \sqrt{180} + 7\sqrt{54}$
45. $(6 - \sqrt{2})(6 + \sqrt{2})$
46. $(5 + \sqrt{6})(5 - \sqrt{2})$
47. $(\sqrt{3} - \sqrt{5})^2$
48. $(x + \sqrt{y})^2$
49. $\sqrt{98} - \sqrt{72} + \sqrt{32}$
50. $\sqrt[4]{a^2} + \sqrt[4]{a^6}$
51. $\sqrt{\frac{a^4}{b^3}}$
52. $\sqrt[4]{\frac{2}{3}}$

Simplify.

53. $\sqrt{\frac{2}{5}} + \sqrt{40} + \sqrt{10}$

54. $\frac{7}{4 - \sqrt{3}}$

55. $\frac{\sqrt{6}}{5 + \sqrt{3}}$

56. $\frac{2 + \sqrt{6}}{2 - \sqrt{6}}$

57. $\frac{\sqrt{x + 1}}{\sqrt{x - 1}}$

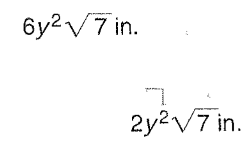
58. $\frac{1}{\sqrt{x^2 - 1}}$

59. $\sqrt[4]{x^4} + \sqrt[3]{x^6} + \sqrt{x^8}$

60. $(4\sqrt{5} - 3\sqrt{2})(2\sqrt{5} + 2\sqrt{2})$

61. Use the Pythagorean theorem to find the length of the hypotenuse of the right triangle shown at the right.

62. Find the radius, r, of a sphere whose surface area S is 2464 square inches. Use the formula $r = \frac{1}{2}\sqrt{\frac{S}{\pi}}$.



Critical Thinking

63. Under what conditions is the equation $\sqrt{x^3 y^2} = xy\sqrt{x}$ true?

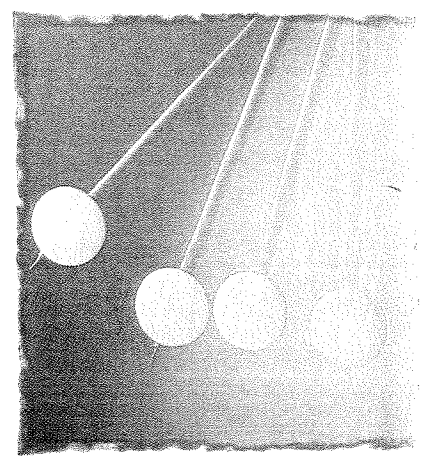
64. Under what conditions is $\sqrt[n]{(-x)^n} = x$?

Applications and Problem Solving

Real World

65. **Sports** Akikta and Francisco are in a weightlifting competition. They weigh 70 kilograms and 110 kilograms, respectively. Akikta's final and best lift is 190 kilograms, and Francisco's is 240 kilograms. The judges will use O'Carroll's formula for determining the superior weight lifter. The formula, $W = \frac{w}{\sqrt[3]{b - 35}}$, involves the weight b of the lifter and the weight w lifted. W represents the handicapped weight.
   a. Calculate each lifter's rating.
   b. Who will be the superior weight lifter, Akikta or Francisco?

66. **Physics** Find the time that it takes a pendulum to complete a swing if its length is 10 inches. Use the formula $T = 2\pi\sqrt{\frac{L}{384}}$, where T represents time in seconds, and L represents the length of the pendulum in inches.



67. **Automotive Engineering** An automotive engineer is trying to design a safer car. The maximum force a road can exert on the tires of the car being redesigned is 2000 pounds. What is the maximum velocity v in ft/s at which this car can safely round a turn of radius 320 feet? Use the formula $v = \sqrt{\frac{F_c r}{100}}$, where F_c is the force the road exerts on the car and r is the radius of the turn.

Mixed Review

68. Simplify $\sqrt{(y + 2)^2}$.

69. Factor $a + b + 3a^2 - 3b^2$.

70. Find $(n^4 - 8n^3 + 54n + 105) \div (n - 5)$ by using synthetic division. (Lesson 5–3)

71. Simplify $(x^3 - 3x^2y + 4xy^2 + y^3) - (7x^3 + x^2y - 9xy^2 + y^3)$.

72. **Chemistry** Wavelengths of light are measured in Angstroms. An Angstrom is 10^{-8} centimeter. The wavelength of cadmium's green line is 5085.8 Angstroms. How many wavelengths of cadmium's green line are in one meter? (Lesson 5–1)

73. Find $\frac{2}{3}\begin{bmatrix} 9 & 0 \\ 12 & 15 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ -7 & -7 \end{bmatrix}$. (Lesson 4–2)

74. **SAT Practice** The sum of two positive consecutive integers is s. In terms of s, what is the value of the larger of these two integers?

A $\frac{s}{2} - 1$ B $\frac{s-1}{2}$ C $\frac{s}{2}$ D $\frac{s+1}{2}$ E $\frac{s}{2} + 1$

75. Use Cramer's rule to solve the system of equations. (Lesson 3–3)

$s + t = 5$

$3s - t = 3$

76. Graph $y + 3x > -1$. (Lesson 2–7)

77. Find $f(-3)$ if $f(x) = x^2 - 3x - 9$. (Lesson 2–1)

78. **Transportation** A San Antonio parking garage charges \$1.50 for the first hour and \$0.50 for each additional hour or part of an hour. For how many hours can you park your car if you only have \$4.50? (Lesson 1–7)

For Extra Practice, see page 887.

79. Evaluate $\frac{3ab}{cd}$ if $a = 3$, $b = 7$, $c = -2$, and $d = 0.5$. (Lesson 1–1)

Mathematics and SOCIETY

Divine Mathematics

Henry Wadsworth Longfellow (1807–1882) was one of America's most outstanding poets. He also was a lover of mathematics. Through one of the characters in his book *Kavanagh: A Tale*, published in 1965, Longfellow said the following.

> HOW DULL AND PROSAIC THE STUDY OF mathematics is made in our school-books; as if the grand science of numbers has been discovered and perfected merely to further the purpose of trade. There is something divine in the science of numbers . . . It holds the sea in the hollow of its hand. It measures the earth; it weighs the stars; it illumines the universe; it is law, it is order, it is beauty. And yet we imagine—that is, most of us—that its highest end and culminating point is book-keeping by double entry. It is our way of teaching it which makes it so prosaic. ■

1. Explain in your own words what Longfellow was trying to say in this passage.
2. Longfellow enjoyed including mathematical problems in his poetry and prose. See if you can solve the following problems posed by Longfellow.
   a. "A tree one hundred cubits high is distant from a well two hundred cubits; from this tree one monkey descends and goes to the well; another monkey takes a leap upwards, and then descends by the hypotenuse; and both pass over an equal space." What is the height of the leap?
   b. "Ten times the square root of a flock of geese, seeing the clouds collect, flew to the Manus lake; one-eighth of the whole flew from the edge of the water amongst a multitude of water lilies; and three couples were observed playing in the water." How many geese were there in the flock?

5–7 Rational Exponents

What YOU'LL LEARN

- To solve problems by identifying and achieving subgoals,
- to write expressions with rational exponents in simplest radical form and vice versa, and
- to evaluate expressions in either exponential or radical form.

Why IT'S IMPORTANT

You can use expressions with rational exponents to solve problems involving literature and music.

CONNECTION

Literature

In *The Pit and the Pendulum,* Edgar Allan Poe describes a situation in which a man is strapped to a wooden bench-like structure. A pendulum hangs from the ceiling that is "some thirty or forty feet overhead." The pendulum begins to swing, and with each swing the pendulum gets closer and closer to the man. The pendulum is at a right angle to his body, and a blade attached to the end of the pendulum is designed to cross right over his heart. As we read the story, we wonder if this man will be able to avoid being killed by the pendulum. Here is one way we can find out.

The formula describing the relationship of the length L of a pendulum to the time t it takes to make one swing (one full cycle back and forth) is given by $t = \sqrt{\frac{\pi^2 L}{8}}$. Suppose you wished to know how long it would take the pendulum to swing when it is 30 feet long and swinging right above the man's body.

Like a good mystery that unfolds little by little, page by page, a good problem solver approaches a problem using a series of small steps, or **subgoals.** By doing so, you make problem solving a simpler process.

Subgoal 1

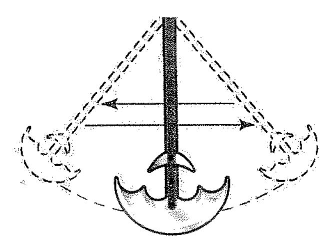
Replace L by the values given in the story to obtain the possible range of time it takes the pendulum to swing.

30 feet overhead	*40 feet overhead*
$t = \sqrt{\frac{\pi^2(30)}{8}}$	$t = \sqrt{\frac{\pi^2(40)}{8}}$
$t \approx 6.08$ seconds	$t \approx 7.02$ seconds

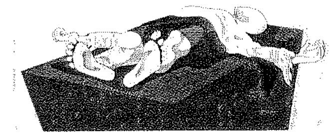
It would have taken about 6 or 7 seconds for the pendulum to swing one cycle.

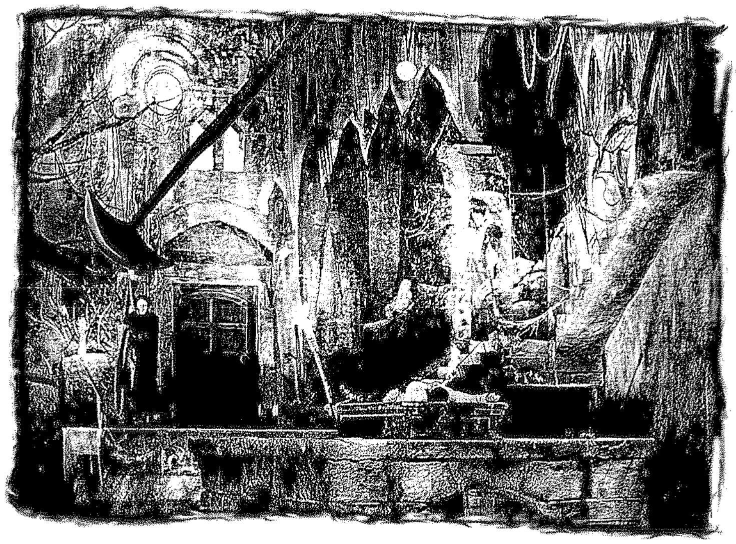
Subgoal 2

Determine if the man has enough time to escape the blade on the pendulum if the blade is just above his body and will strike him as it lowers for the next swing.



Since it takes at least 6 seconds for the pendulum to swing (that is, to move across to the other side and back to its original position), he would only have one-half the time to escape from the last safe pass of the blade over his body. One-half of 6 seconds is 3 seconds. Does this seem like enough time for him to escape?





You have learned that squaring a number and taking the square root of a number are inverse operations. But how would we evaluate an expression that contains a fractional exponent? Assume that fractional exponents behave as integral exponents.

For any number $a > 0$, $a^1 = a^{\frac{1}{2}(2)}$ or $\left(a^{\frac{1}{2}}\right)^2$.

So, $a^{\frac{1}{2}}$ is a number that when squared equals a. Since $(\sqrt{a})^2 = a$, it follows that $a^{\frac{1}{2}} = \sqrt{a}$. This suggests the following definition.

Definition of $b^{\frac{1}{n}}$	For any real number b and for any integer $n > 1$, $b^{\frac{1}{n}} = \sqrt[n]{b}$ except when $b < 0$ and n is even.

Example 1

PROBLEM SOLVING

Identify Subgoals

Economists refer to inflation as increases in the average cost of purchases. The formula $C = c(1 + r)^n$ can be used to predict the cost of consumer items at some projected time. In this formula, C represents the projected cost of the item at a given annual inflation rate, c the present cost of the item, r the rate of inflation (in decimal form), and n the number of years for the projection. Suppose a gallon of milk costs \$2.69 now. How much would the price increase in 6 months with an inflation rate of 5.3%?

Subgoal 1: Identify the known values.

$c = \$2.69$, $r = 5.3\%$ or 0.053, and $n = 6$ months or $\frac{1}{2}$ year

Subgoal 2: Find the value of C.

$$C = c(1 + r)^n$$
$$= 2.69(1 + 0.053)^{\frac{1}{2}}$$

Use a calculator to evaluate this expression. *Remember that $a^{\frac{1}{2}} = \sqrt{a}$.*

Enter: 2.69 [×] [(] 1 [+] .053 [)] [√x] [=] 2.760364704

The cost of the milk in 6 months will be \$2.76.

Subgoal 3: Find the increase in cost.

$C - c = \$2.76 - \$2.69$ or $\$0.07$

The increase in cost in 6 months is predicted to be 7 cents.

From the definition of $b^{\frac{1}{n}}$, we can say that $7^{\frac{1}{4}} = \sqrt[4]{7}$ and $(-8)^{\frac{1}{3}} = \sqrt[3]{-8}$ or -2. The expression $(-16)^{\frac{1}{4}}$ is not defined since $-16 < 0$ and 4 is even. Why do we need this restriction?

In Example 2, each expression is evaluated in two ways. Method 1 uses the definition of $b^{\frac{1}{n}}$. Method 2 uses the properties of powers.

Example 2 **Evaluate each expression.**

a. $81^{-\frac{1}{4}}$

Method 1

$$81^{-\frac{1}{4}} = \frac{1}{81^{\frac{1}{4}}} = \frac{1}{\sqrt[4]{81}} = \frac{1}{\sqrt[4]{3^4}} = \frac{1}{3}$$

Remember that $b^{-n} = \frac{1}{b^n}$.

Method 2

$$81^{-\frac{1}{4}} = (3^4)^{-\frac{1}{4}} = 3^{4\left(-\frac{1}{4}\right)} = 3^{-1} = \frac{1}{3}$$

b. $32^{\frac{3}{5}}$

Method 1

$$32^{\frac{3}{5}} = 32^{3\left(\frac{1}{5}\right)} = (32^3)^{\frac{1}{5}} = \sqrt[5]{32^3} = \sqrt[5]{(2^5)^3} = \sqrt[5]{2^5 \cdot 2^5 \cdot 2^5} = 2 \cdot 2 \cdot 2 \text{ or } 8$$

Method 2

$$32^{\frac{3}{5}} = (2^5)^{\frac{3}{5}} = 2^{5\left(\frac{3}{5}\right)} = 2^3 = 8$$

In part b of Example 2, Method 1 uses a combination of the definition of $b^{\frac{1}{n}}$ and the properties of powers. This example suggests the following general definition of rational exponents.

Definition of Rational Exponents	**For any nonzero real number b, and any integers m and n, with $n > 1$,** $$b^{\frac{m}{n}} = \sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$$ **except when $b < 0$ and n is even.**

When simplifying expressions containing rational exponents, it is usually easier to leave the exponent in rational form rather than to write the expression as a radical. To simplify an expression, you must write the expression with all positive exponents. Furthermore, any exponents in the denominator of a fraction must be positive *integers.* That is, it may be necessary to rationalize a denominator.

All of the properties of powers you learned in Lesson 5-1 apply to rational exponents.

Example 3 **Simplify each expression.**

a. $x^{\frac{2}{3}} \cdot x^{\frac{5}{3}}$

$$x^{\frac{2}{3}} \cdot x^{\frac{5}{3}} = x^{\left(\frac{2}{3}+\frac{5}{3}\right)}$$
$$= x^{\frac{7}{3}}$$

b. $y^{-\frac{5}{6}}$

$$y^{-\frac{5}{6}} = \frac{1}{y^{\frac{5}{6}}}$$
$$= \frac{1}{y^{\frac{5}{6}}} \cdot \frac{y^{\frac{1}{6}}}{y^{\frac{1}{6}}}$$

Why use $\frac{y^{\frac{1}{6}}}{y^{\frac{1}{6}}}$*?*

$$= \frac{y^{\frac{1}{6}}}{y^{\frac{6}{6}}}$$
$$= \frac{y^{\frac{1}{6}}}{y}$$

When simplifying a radical expression, always find the smallest index possible. Using rational exponents make this process easier.

Example 4 **Simplify each expression.**

a. $\frac{\sqrt[8]{16}}{\sqrt[6]{2}}$

$$\frac{\sqrt[8]{16}}{\sqrt[6]{2}} = \frac{16^{\frac{1}{8}}}{2^{\frac{1}{6}}}$$
$$= \frac{(2^4)^{\frac{1}{8}}}{2^{\frac{1}{6}}}$$
$$= \frac{2^{\frac{1}{2}}}{2^{\frac{1}{6}}}$$
$$= 2^{\left(\frac{1}{2}-\frac{1}{6}\right)}$$
$$= 2^{\frac{1}{3}} \text{ or } \sqrt[3]{2}$$

b. $\sqrt[4]{4n^2}$

$$\sqrt[4]{4n^2} = (4n^2)^{\frac{1}{4}}$$
$$= (2^2 \cdot n^2)^{\frac{1}{4}}$$
$$= 2^{2\left(\frac{1}{4}\right)} \cdot n^{2\left(\frac{1}{4}\right)}$$
$$= 2^{\frac{1}{2}} \cdot n^{\frac{1}{2}}$$
$$= \sqrt{2} \cdot \sqrt{n}$$
$$= \sqrt{2n}$$

c. $\frac{a^{\frac{1}{2}}+1}{a^{\frac{1}{2}}-1}$

$$\frac{a^{\frac{1}{2}}+1}{a^{\frac{1}{2}}-1} = \frac{a^{\frac{1}{2}}+1}{a^{\frac{1}{2}}-1} \cdot \frac{a^{\frac{1}{2}}+1}{a^{\frac{1}{2}}+1}$$

The conjugate of $a^{\frac{1}{2}} - 1$ *is* $a^{\frac{1}{2}} + 1$.

$$= \frac{a + 2a^{\frac{1}{2}} + 1}{a - 1}$$

In summary, an expression is simplified when the following conditions are met.

- It has no negative exponents.
- It has no fractional exponents in the denominator.
- It is not a complex fraction.
- The index of any remaining radical is the least number possible.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **Explain** how examining the subgoals helped to solve *The Pit and the Pendulum* application at the beginning of the lesson.
2. **Determine** if $3 \cdot 4^{\frac{1}{6}}$ is the simplest form of $2916^{\frac{1}{6}}$. If not, tell why not and write the expression in simplest form.
3. **Explain** why $(-81)^{\frac{1}{4}}$ is not defined.
4. **Write** a general rule for rationalizing expressions such as $\frac{1}{x^{\frac{n}{m}}}$.

MATH JOURNAL

5. Describe the steps you go through when solving a complicated word problem.

Guided Practice

Express using rational exponents.

6. $\sqrt{14}$
7. $\sqrt[4]{27}$
8. $\sqrt[6]{b^3}$
9. $\sqrt[3]{16a^5b^7}$

Evaluate.

10. $16^{\frac{1}{4}}$
11. $8^{-\frac{1}{3}}$
12. $64^{\frac{2}{3}}$
13. $\frac{16}{4^{\frac{3}{2}}}$

Simplify.

14. $\sqrt[6]{9x^3}$
15. $x^{\frac{3}{2}}y^{\frac{7}{3}}z^{\frac{9}{6}}$
16. $\frac{\sqrt[4]{125}}{\sqrt[4]{5}}$
17. $\frac{1}{5x^{\frac{1}{3}}}$
18. $(x^2y)^{-\frac{1}{3}}$
19. $c(a-4b)^{-\frac{1}{2}}$
20. $x^{\frac{1}{3}} \cdot x^{\frac{3}{4}}$
21. $\frac{y^{\frac{5}{6}}}{y^{\frac{1}{6}}}$
22. $\frac{x^3}{y^{\frac{1}{2}}} \cdot \frac{y}{x^{\frac{1}{3}}}$
23. To what power do you have to raise
    a. 4 to get 2?
    b. 32 to get 8?
    c. x^3 to get x?

EXERCISES

Practice

Evaluate.

24. $125^{\frac{1}{3}}$
25. $100^{-\frac{1}{2}}$
26. $16^{-\frac{3}{4}}$
27. $81^{\frac{2}{3}} \cdot 81^{\frac{3}{2}}$
28. $9^{\frac{5}{2}} \cdot 9^{\frac{3}{2}}$
29. $(-64)^{-\frac{2}{3}}$
30. $\left(\frac{16}{81}\right)^{\frac{1}{4}}$
31. $\left(\frac{1}{32}\right)^{-\frac{3}{5}}$
32. $\left(\frac{27}{64}\right)^{-\frac{1}{3}}$
33. $\frac{24}{6^{\frac{2}{3}}}$
34. $\frac{21}{7^{\frac{2}{3}}}$
35. $\frac{8}{3^{\frac{1}{2}}}$

Simplify.

36. $2^{\frac{5}{3}}a^{\frac{7}{3}}$
37. $(2m)^{\frac{1}{2}}m^{\frac{1}{2}}$
38. $11^{\frac{1}{3}}p^{\frac{7}{3}}q^{\frac{2}{3}}$
39. $\frac{1}{w^{\frac{4}{5}}}$
40. $\frac{1}{(u-v)^{\frac{1}{3}}}$
41. $x^{-\frac{5}{6}}$
42. $\frac{1}{b^{\frac{1}{2}}+1}$
43. $\frac{b^{-\frac{1}{2}}}{8b^{\frac{1}{3}} \cdot b^{-\frac{1}{4}}}$
44. $\frac{g^{\frac{3}{2}}+3g^{-\frac{1}{2}}}{g^{\frac{1}{2}}}$
45. $\frac{3x^{-\frac{1}{3}}+x^{\frac{5}{3}}y}{x^{\frac{2}{3}}}$
46. $\frac{2a^{\frac{1}{2}}+a^{\frac{3}{2}}}{a^{\frac{1}{2}}}$
47. $\frac{pq}{\sqrt[3]{r}}$

48. $\left(m^{-\frac{2}{3}}\right)^{-\frac{1}{6}}$ 49. $b^{-\frac{1}{3}} - b^{\frac{1}{3}}$ 50. $\dfrac{z^{\frac{3}{2}}}{z^{\frac{1}{2}} + 2}$

51. $\left(\dfrac{m^{-2}n^{-6}}{121}\right)^{-\frac{1}{2}}$ 52. $\dfrac{8^{\frac{1}{6}} - 9^{\frac{1}{4}}}{\sqrt{3} + \sqrt{2}}$ 53. $\dfrac{a^{\frac{5}{3}} - a^{\frac{1}{3}}b^{\frac{4}{3}}}{a^{\frac{2}{3}} + b^{\frac{2}{3}}}$

54. $\sqrt[4]{49}$ 55. $r^{\frac{1}{2}}s^{\frac{1}{3}}$ 56. $\sqrt[6]{81p^4q^8}$

57. $\sqrt{13} \cdot \sqrt[3]{13^2}$ 58. $a^{\frac{5}{6}}b^{\frac{7}{3}}c^{\frac{3}{2}}$ 59. $\sqrt[3]{\sqrt{27}}$

Evaluate $f(x) = x^{-\frac{2}{3}} + x^{-3}$ for each value of x.

60. $f(-8)$ 61. $f(1000)$ 62. $f(0.001)$

Calculator

Evaluate each expression to the nearest hundredth by using your calculator.

63. $45^{0.33}$ 64. $2.75^{\frac{2}{3}}$ 65. $\left(4\frac{1}{2}\right)^{0.075}$

Critical Thinking

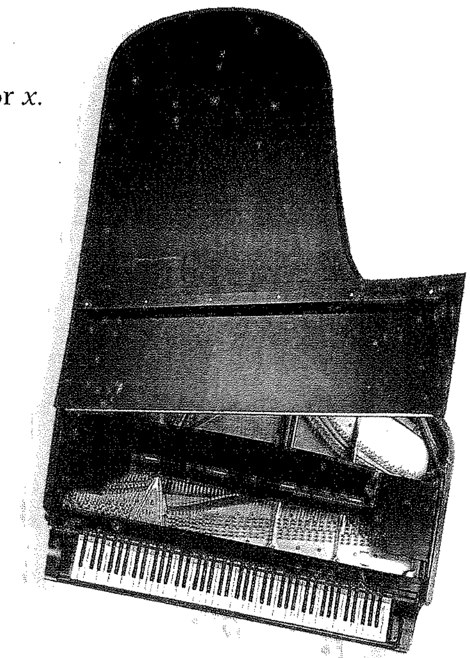
66. Explain how you would solve $9^x = 3^{x+\frac{1}{2}}$ for x.

Applications and Problem Solving

67. **Music** On a piano, the frequency of the A note above middle C should be set at 440 vibrations per second. The frequency f_n of a note that is n notes above A should be $f_n = 440(\sqrt[12]{2})^{n-1}$.

a. At what frequency should a piano tuner set the A that is one octave, or 12 notes, above the A above middle C?

b. Middle C is nine notes below A that has a frequency 440 vibrations per second. What should the frequency of middle C be?



68. **Environment** When estimating the amount of pollutants contained in the hot gases released by a smokestack, EPA agents must take into account the phenomenon that the velocity of the gas varies throughout the cylindrical smokestack. In such situations, the gas near the center of a cross section of the smokestack has a greater velocity than the gas near the perimeter. This can be described algebraically by the formula $V = V_{max}\left[1 - \left(\frac{r}{r_0}\right)^2\right]$, where V_{max} is the maximum velocity of the gas, r_0 is the radius of the smokestack, and V is the velocity of the gas at a distance r from the center of the circular cross section of the smokestack. A cross section of a smokestack has a radius of 25 feet. The gas flowing from it has a measured velocity of 10 mph, and the velocity of the gas at a distance r from the center is 7 mph.

a. Find the value of r.

b. Write the subgoals that you followed to solve the problem.

Mixed Review

69. Simplify $4\sqrt{(x-5)^2}$. (Lesson 5–6)

70. Simplify $\sqrt{\frac{9}{36}x^4}$. (Lesson 5–5)

71. Factor $a^3b^3 - 27$. (Lesson 5–4)

72. Simplify $\frac{-15r^5s^2}{5r^5s^{-4}}$. (Lesson 5–1)

73. Solve the system by using augmented matrices. (Lesson 4–7)

$$2x + y = 0$$
$$3x - 4y = 22$$

74. Can a 3×4 matrix have an inverse? Explain your answer. (Lesson 4–5)

75. **SAT Practice** If $12x + 7y = 19$ and $4x - y = 3$, then what is the value of $8x + 8y$?

A 2 B 8 C 16 D 22 E 32

76. Find a and b such that the solution of the system $2x + 3y = 7$ and $ax + by = -10$ is (2, 1). (Lesson 3–1)

77. **Statistics** A prediction equation in a study of the relationship between minutes spent studying s and test scores t is $t = 0.36s + 61.4$. Predict the score a student would receive if she spent 1 hour studying. (Lesson 2–5)

78. Find $h(m - 2)$ if $h(x) = \frac{x-5}{7}$. (Lesson 2–1)

79. Solve $4 < 2x - 2 < 10$. (Lesson 1–6)

80. Name the property illustrated by $6 + a = 6 + a$. (Lesson 1–2)

For Extra Practice, see page 887.

WORKING ON THE Investigation

Refer to the Investigation on pages 180–181.

3-2-1-Blast-Off!

Standard deviation is the most commonly used measure of variation. It is the average measure of how much each value in a set of data differs from the mean. To find the standard deviation, follow the steps below.

a. Find the mean of the set of data.
b. Find the difference between each value in the set of data and the mean.
c. Square each difference.
d. Find the mean of the squares.
e. Take the principal square root of this mean.

1. Find the standard deviation for each of the launchers used in your class shoot-off.
2. How does the standard deviation fit with the other measures you found for each of the launchers? Does its value change your analysis of which launcher is the most accurate? Explain.
3. Write a general formula for standard deviation. Also write it in a form with rational exponents.
4. Plot the data from each of the launchers used in your class shoot-off on box-and-whisker plots and relate the standard deviation to the length of the whiskers. Is there a correlation? Explain.

Add the results of your work to your Investigation Folder.