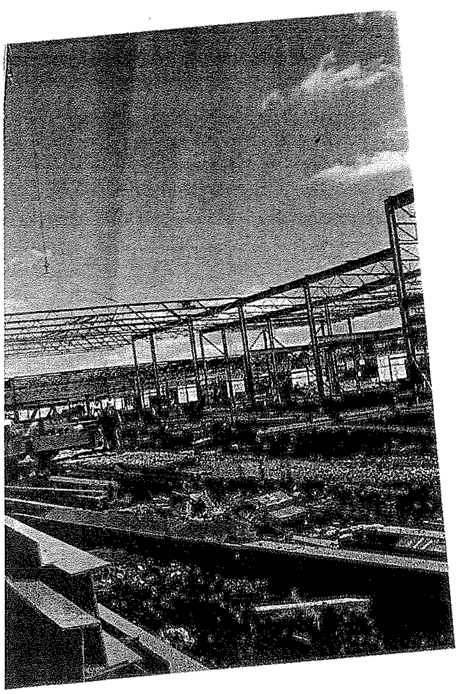
5-8 Solving Radical Equations and Inequalities

What YOU'LL LEARN

- To solve equations and inequalities containing radicals.

Why IT'S IMPORTANT

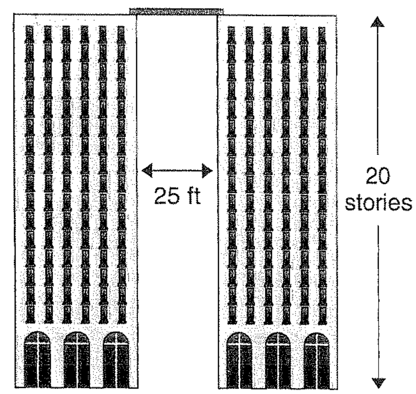
You can use radical equations to solve problems involving aviation and manufacturing.



Real World APPLICATION

Construction

At a building construction site, carts loaded with concrete must cross a 25-foot gap between two towers that are 20 stories high. The construction manager needs to select a beam strong enough to support a worker pushing a fully loaded cart. The beam must be at least 2 feet wide to accommodate the wheels of the cart. Will a beam 6 inches thick be able to safely support the load of the cart (865 lb) and the worker (250 lb maximum)?



The formula that expresses the relationship of the safe load s of a beam to its width w in feet and depth d in inches is $s = \frac{kwd^2}{\ell}$, where k is a constant equal to 576 and ℓ is the distance in feet between the supports. Since we wish to know the thickness or depth of the beam, let's solve the formula for d.

$$s = \frac{kwd^2}{\ell}$$

$$\ell s = kwd^2 \quad \textit{Multiply each side by } \ell.$$

$$\frac{\ell s}{kw} = d^2 \quad \textit{Divide each side by } kw.$$

$$\pm\sqrt{\frac{\ell s}{kw}} = d \quad \textit{Since } d \textit{ is squared, take the square root of each side.}$$

The safe load s is the sum of the weight of the cart (865 lb) and the maximum weight of a construction worker (250 lb) or 1115 pounds.

$$d = \sqrt{\frac{25 \cdot 1115}{576 \cdot 2}} \approx 4.92 \quad \ell = 25,\ s = 1115,\ k = 576, \text{ and } w = 2$$

A beam at least 5 inches thick would safely support the load across the gap. A 6-inch beam would give an extra safety margin.

Some equations contain irrational numbers expressed as radicals. You can use the properties of radicals to solve these equations.

Example 1 **Solve $x - 4 = x\sqrt{3}$.**

$$x - 4 = x\sqrt{3}$$

$$x - x\sqrt{3} = 4 \quad \textit{Isolate the variable on one side of the equation}$$

$$x(1 - \sqrt{3}) = 4 \quad \textit{Factor the GCF.}$$

$$x = \frac{4}{1 - \sqrt{3}}$$

(continued on the next page)

$$x = \frac{4}{(1 - \sqrt{3})} \cdot \frac{(1 + \sqrt{3})}{(1 + \sqrt{3})}$$ *Rationalize the denominator.*

$$x = \frac{4(1 + \sqrt{3})}{1 - 3}$$

$$x = \frac{4 + 4\sqrt{3}}{-2} \text{ or } -2 - 2\sqrt{3}$$

Check: $x - 4 = x\sqrt{3}$

$$-2 - 2\sqrt{3} - 4 \stackrel{?}{=} (-2 - 2\sqrt{3})(\sqrt{3})$$

$$-2\sqrt{3} - 6 = -2\sqrt{3} - 6 \checkmark$$

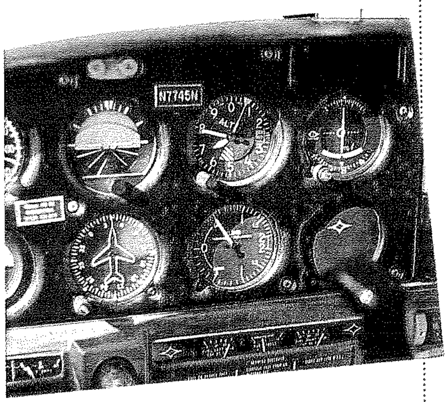
The solution is $-2 - 2\sqrt{3}$.

The Pythagorean theorem is often helpful when solving real-life problems.

Example 2

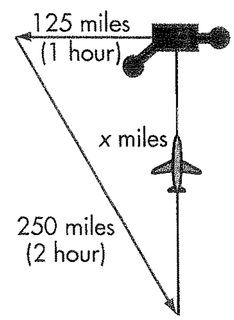
Real World APPLICATION

Aviation



A student pilot leaves the Tamiami Municipal Airport at 9:00 A.M. on her first solo flight. She is traveling due west at 125 mph. At 10:00 A.M., she turns and flies in a southeasterly direction at the same speed. At noon, she notices that the fuel gauge indicates a low level of fuel. At this time, she is directly south of the airport. How far is it to the airport?

Explore First draw a diagram of the flight path. The problem gives us the following information.



- She leaves the airport at 9:00 A.M.
- She flies west at 125 mph until 10:00 A.M.
- At 10:00, she turns southeast and flies until noon.
- At noon, the plane is directly south of the airport, and she turns north to return to the airport.

Plan Since the flight path makes a right triangle, we know we can use the Pythagorean theorem to find an expression for the length x of one of the legs of the triangle.

Solve

$$250^2 = 125^2 + x^2$$

$$250^2 - 125^2 = x^2$$ *Subtract 125^2 from each side.*

$$62{,}500 - 15{,}625 = x^2$$

$$46{,}875 = x^2$$

$$\pm\sqrt{46{,}875} = x$$ *Take the square root of each side.*

$$\pm 216.5 \approx x$$ *Use your calculator.*

Since distance cannot be negative, we know that the pilot is approximately 216.5 miles away from the airport.

Examine Use the Pythagorean theorem to examine the time involved.

$1^2 + y^2 = 2^2$ *Let y represent the unknown time flying due north.*

$$y^2 = 4 - 1 \text{ or } 3$$

$$y = \pm\sqrt{3}$$

The pilot flew $\sqrt{3}$ hours at 125 mph, or about 216.5 miles. ✓

Radical equations involving square roots are sometimes called square root equations.

Sometimes variables appear in the radicand. Equations with radicals like this are called **radical equations**. To solve this type of equation, you will need to raise each side of the equation to a power equal to the index to remove the variable from the radical.

Solving radical equations sometimes yields **extraneous solutions**, which are solutions that do not satisfy the original equation. You must check all the possible solutions in the *original* equation and disregard the extraneous solutions.

Example 3 **Solve $\sqrt{x-3} = \sqrt{2} - \sqrt{x}$.**

$$\sqrt{x-3} = \sqrt{2} - \sqrt{x}$$

$$(\sqrt{x-3})^2 = (\sqrt{2} - \sqrt{x})^2 \quad \textit{Square each side.}$$

$$x - 3 = 2 - 2\sqrt{2x} + x$$

$$-5 = -2\sqrt{2x} \quad \textit{Isolate the square root.}$$

$$(-5)^2 = (-2\sqrt{2x})^2 \quad \textit{Square each side again.}$$

$$25 = 8x$$

$$\frac{25}{8} = x$$

Check:

$$\sqrt{x-3} = \sqrt{2} - \sqrt{x}$$

$$\sqrt{\frac{25}{8} - 3} \stackrel{?}{=} \sqrt{2} - \sqrt{\frac{25}{8}}$$

$$\sqrt{\frac{1}{8}} \stackrel{?}{=} \sqrt{2} - \sqrt{\frac{25}{8}}$$

$$\frac{1}{2}\sqrt{\frac{1}{2}} \stackrel{?}{=} \sqrt{2} - \frac{5}{2}\sqrt{\frac{1}{2}} \quad \frac{1}{8} = \frac{1}{2^2 \cdot 2}, \frac{25}{8} = \frac{5^2}{2^2 \cdot 2}$$

$$3\sqrt{\frac{1}{2}} \stackrel{?}{=} \sqrt{2} \quad \textit{Add } \frac{5}{2}\sqrt{\frac{1}{2}} \textit{ to each side.}$$

$$3\sqrt{\frac{1}{2} \cdot \frac{2}{2}} \stackrel{?}{=} \sqrt{2} \quad \textit{Rationalize the denominator.}$$

$$\frac{3\sqrt{2}}{2} \neq \sqrt{2} \quad \textit{Why?}$$

The solution does not check. The equation has no real solution.

You can apply the same methods used in solving square root equations to solving equations of *n*th roots. Remember to undo a square root, you squared the expression. To undo the *n*th root, you must raise the expression to the *n*th power.

Example 4 **Solve $2(7n - 1)^{\frac{1}{3}} - 4 = 0$.**

In order to remove the cube root, we must first isolate it and then raise each side of the equation to the third power.

$$2(7n-1)^{\frac{1}{3}} - 4 = 0 \quad \textit{Remember that } (7n-1)^{\frac{1}{3}} = \sqrt[3]{7n-1}.$$

$$2(7n-1)^{\frac{1}{3}} = 4 \quad \textit{Add 4 to each side.}$$

$$(7n-1)^{\frac{1}{3}} = 2 \quad \textit{Isolate the cube root.}$$

$$\left[(7n-1)^{\frac{1}{3}}\right]^3 = 2^3 \quad \textit{Cube each side.}$$

$$7n - 1 = 8$$

$$7n = 9 \quad \textit{Add 1 to each side.}$$

$$n = \frac{9}{7} \quad \textit{Divide each side by 7.}$$

(continued on the next page)

Check:

$$2(7n-1)^{\frac{1}{3}} - 4 \stackrel{?}{=} 0$$
$$2\left[7\left(\tfrac{9}{7}\right) - 1\right]^{\frac{1}{3}} - 4 \stackrel{?}{=} 0$$
$$2(8)^{\frac{1}{3}} - 4 \stackrel{?}{=} 0$$
$$2(2) - 4 \stackrel{?}{=} 0$$
$$0 = 0 \ \checkmark$$

The solution is $\frac{9}{7}$.

You can use what you now know about solving square root equations to solve **square root inequalities.**

Example 5 **Solve $3 + \sqrt{4x - 5} \le 10$.**

Since the radicand of a radical expression must be greater than or equal to zero, first solve $4x - 5 \ge 0$.

$$4x - 5 \ge 0$$
$$4x \ge 5$$
$$x \ge \frac{5}{4}$$

Now solve $3 + \sqrt{4x - 5} \le 10$.

$3 + \sqrt{4x - 5} \le 10$

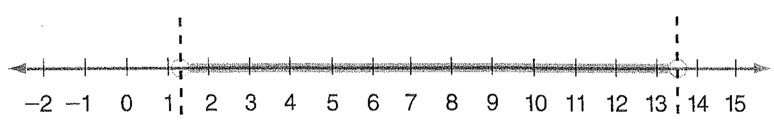
$\sqrt{4x - 5} \le 7$ *Subtract 3 from each side.*

$4x - 5 \le 49$ *Square each side.*

$4x \le 54$ *Add 5 to each side.*

$x \le 13\frac{1}{2}$ *Divide each side by 4 and simplify.*

It appears that $\frac{5}{4} \le x \le 13\frac{1}{2}$. Test values on each part of the number line below to check the result.



Try $x = 0$.

$3 + \sqrt{4x - 5} \le 10$

$3 + \sqrt{-5} \le 10$

False; negative square roots are not defined in the set of real numbers.

Try $x = 7.5$.

$3 + \sqrt{4x - 5} \le 10$

$3 + \sqrt{30 - 5} \le 10$

$3 + 5 \le 10$ ✓ true

Try $x = 21.5$.

$3 + \sqrt{4x - 5} \le 10$

$3 + \sqrt{86 - 5} \le 10$

$3 + 9 \not\le 10$ false

So, $\frac{5}{4} \le x \le 13\frac{1}{2}$.

To solve inequalities that involve radicals, complete the following steps.

1. Identify the excluded values.
2. Solve the inequality.
3. Test values to find the solution.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **Explain** how you would solve $\sqrt{2x-3} - \sqrt{x+3} = 0$.
2. **Explain** why you need to isolate the nth root before raising it to the nth power to remove it.
3. **You Decide** Pedro and Rochelle were working in pairs to solve the equation $(x+5)^{\frac{1}{4}} = -4$. Pedro said he could tell that there was no solution without even working the problem. Rochelle said they should solve the equation and then they could test their results to see if there was a solution. Who is correct and why?

Guided Practice

Solve each equation. Be sure to check for extraneous solutions.

4. $\sqrt{3d+1} = 4$
5. $3 - (2-y)^{\frac{1}{2}} = 0$
6. $1 + x\sqrt{2} = 0$
7. $\sqrt{a-4} - 3 = 0$
8. $\frac{2}{3} \cdot (4m)^{\frac{1}{3}} = 4$
9. $\sqrt[3]{y+1} = 2$
10. $\sqrt{2x+3} - 4 \le -5$
11. $\sqrt{n+12} - \sqrt{n} > 2$
12. Solve $\sqrt{x+9} = 9 - \sqrt{x}$. Choose the best answer.
    a. 4 b. ± 4 c. 2 d. 16
13. Solve $y = \sqrt{r^2 + s^2}$ for r.

EXERCISES

Practice

Solve each equation. Be sure to check for extraneous solutions.

14. $\sqrt{x} = 3$
15. $\sqrt{q} - 8 = 0$
16. $x^{\frac{1}{2}} + 4 = 0$
17. $7 + 6n\sqrt{5} = 0$
18. $\sqrt[3]{r-1} = 3$
19. $\sqrt[3]{2p+1} = 3$
20. $\sqrt[4]{7+3z} = 2$
21. $x\sqrt{x} = 8$
22. $y\sqrt{3} - y = 7$
23. $\sqrt{2x-9} = -\frac{1}{3}$
24. $9 + \sqrt{4x+8} = 11$
25. $3 + \sqrt{4n-5} = 10$
26. $6 - \sqrt{2y+1} < 3$
27. $(7-5x)^{\frac{1}{2}} \ge 8$
28. $13 - 3r = r\sqrt{5}$
29. $3x + 5 = x\sqrt{3}$
30. $\sqrt{g-4} = \sqrt{2g-3}$
31. $\sqrt{2r-6} = \sqrt{3+r}$
32. $\sqrt{2} - \sqrt{x+6} \le -\sqrt{x}$
33. $\sqrt{k+9} - \sqrt{k} > \sqrt{3}$
34. $\sqrt{x-6} = 3 + \sqrt{x}$
35. $\sqrt{h+3} + \sqrt{h-1} = 5$
36. $\sqrt{x+21} - 1 = \sqrt{x+12}$
37. $\sqrt{4x+1} = 3 + \sqrt{4x-2}$
38. $(6g-5)^{\frac{1}{3}} + 2 = -3$
39. $\sqrt{a+1} = \sqrt{a+6} - 1$
40. $\sqrt{b+5} + \sqrt{b+10} > 2$
41. $\sqrt{a-5} - \sqrt{a+7} \le 4$
42. Explain why $\{25, 36\}$ is not the solution set for $x + \sqrt{x} - 30 = 0$.

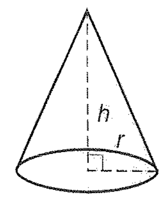
Solve for the variable indicated.

43. $T = 2\pi\sqrt{\frac{\ell}{g}}$ for ℓ

44. $t = \sqrt{\frac{2s}{g^2}}$ for s

45. $m^2 = \sqrt[3]{\frac{rp}{g^2}}$ for p

46. $r = \sqrt[3]{\frac{2mM}{c}}$ for c



47. **Geometry** The surface area of a cone can be found by using $S = \pi r\sqrt{r^2 + h^2}$, where r is the radius of the base and h is the height of the cone.

a. Solve the equation for h.

b. Find h if $S = 225$ and $r = 5$.

Critical Thinking

48. Is $\sqrt[n]{b^m} = (\sqrt[n]{b})^m$? Justify your answer.

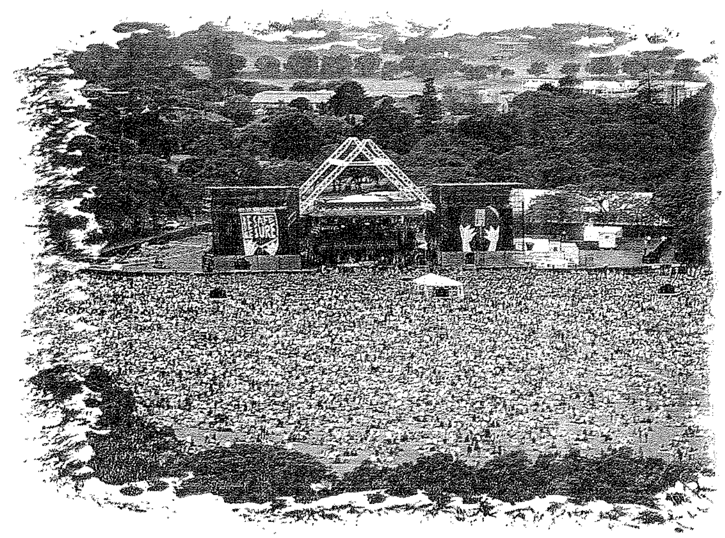
49. Is $\sqrt[km]{b} = \sqrt[k]{\sqrt[m]{b}}$? Justify your answer.

Applications and Problem Solving

Real World

50. **Manufacturing** A company that manufactures ROM (Read Only Memory) chips for computers uses the formula $c = 100\sqrt[3]{n^2} + 1200$ to determine the cost of producing the chips, where c is the cost of production in dollars and n is the number produced. If a computer company places an order totaling \$10,000, how many chips will be produced?

51. **Concerts** The organizers of a rock concert are preparing for the arrival of 50,000 enthusiastic fans in the open field where the concert will take place. It is reasonable to allow each person 5 square feet of space, so the organizers need to rope off a circular area of 250,000 square feet. Using the formula $A = \pi r^2$, where A represents the area of the circular region and r represents the radius of the region, find the radius of this region.



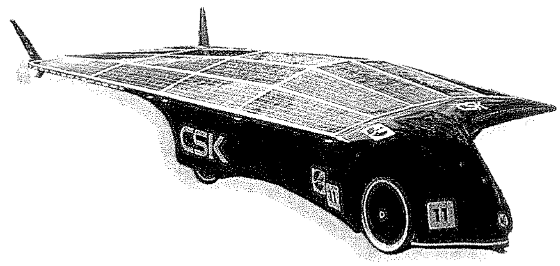
52. **History** Johann Kepler (1571–1630) is known mainly as a mathematician of the sky. He is credited with several laws of planetary motion. His third law states that the square of the time of revolution of each planet (its period) is proportional to the cube of its mean distance from the sun. This can be expressed as $\frac{T_a}{T_b} = \left(\frac{r_a}{r_b}\right)^{\frac{3}{2}}$, where T_a is the time it takes one planet to orbit the sun, r_a is that planet's average distance from the sun, and T_b and r_b are another planet's period and average distance from the sun.

a. Solve the formula for r_a.

b. Find the distance (to the nearest million miles) that Jupiter is from the sun, if its period is 12 years. Mercury is 36 million miles from the sun and has a period of 88 days.

53. **Aerospace Engineering** The radius of the orbit of a satellite is found by $r = \sqrt[3]{\frac{GMt^2}{4\pi^2}}$, where t represents the time it takes for the satellite to complete one orbit, G represents the constant of universal gravitation, and M is the mass of the central object. Solve the formula for t.

54. **Solar Energy** The energy of direct sunlight on a solar cell with an area of one square centimeter is converted into 0.01 watt of electrical energy. Suppose that a square solar cell must deliver 15 watts of energy. What should the dimensions of the cell be?



Solar-powered car

Mixed Review

55. How would you write *the seventh root of 5 cubed* using exponents? (Lesson 5–7)

Simplify.

56. $\sqrt{3}(\sqrt{6} - 2)$ (Lesson 5–6)

57. $\sqrt{x^2 + 10x + 25}$ (Lesson 5–5)

58. $(2t^3 - 2t - 3) \div (t - 1)$ (Lesson 5–3)

59. $(y^5)^2$ (Lesson 5–1)

60. Write an augmented matrix for the system of equations. Then solve the system. (Lesson 4–7)

$x + 5y + 2z = 10$

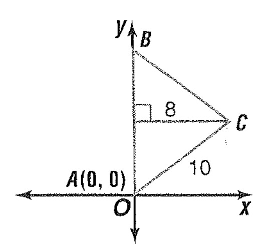
$3x - 3y + 2z = 2$

$2x + 4y - z = -15$

61. Evaluate $\begin{vmatrix} 4 & -2 \\ 3 & 7 \end{vmatrix}$. (Lesson 4–4)

62. **Business** Mr. Whitner bought 7 drums of two different cleaning fluids for his dry-cleaning business. One of the fluids cost $30 per drum, and the other was $20 per drum. The total cost of the supplies was $160. How much of each fluid did Mr. Whitner buy? Write a system of equations and solve by graphing. (Lesson 3–1)

63. **SAT Practice** In the figure, side $\overline{AC}$ of triangle ABC contains which of the following points?



A (3, 4) B (3, 5) C (4, 3)

D (4, 5) E (4, 6)

64. **Statistics** The cost per cup of several different types of juices are given below.

11¢	9¢	6¢	9¢	12¢	4¢	4¢	8¢	6¢	4¢	16¢
12¢	11¢	19¢	7¢	19¢	7¢	19¢	4¢	6¢	5¢	6¢

Make a stem-and-leaf plot of the costs. (Lesson 1–3)

For Extra Practice, see page 888.

5–9 Complex Numbers

What YOU'LL LEARN

- To simplify square roots containing negative radicands,
- to solve quadratic equations that have pure imaginary solutions, and
- to add, subtract, and multiply complex numbers.

Why IT'S IMPORTANT

You can use complex numbers to solve problems involving electricity and physics.

INTEGRATION
Number Theory

Keesha and Juanita formed a study group for their algebra class. Their teacher, Mrs. Rodriguez, gave them an assignment that included solving the equation $3x^2 + 15 = 0$. Keesha and Juanita solved the equation as follows.

$$3x^2 + 15 = 0$$
$$3x^2 = -15$$
$$x^2 = -5$$

They were puzzled at the last step because they knew that there is no real number x that when squared results in a negative number. They were stumped! After checking with a few of their classmates, they found that everyone had the same difficulty. Is there a solution to the equation?

About 400 years ago, a solution to this type of equation was proposed by French mathematician René Descartes (1596–1650). He proposed that the solution to the equation $x^2 = -1$ be represented by a number i, where i is not a real number. Thus, $i = \sqrt{-1}$.

Keesha's class could now solve the equation.

$x^2 = -5$

$x = \pm\sqrt{-5}$

$x = \pm\sqrt{5} \cdot \sqrt{-1}$ *Extension of product property of radicals*

$x = \pm\sqrt{5} \cdot i$ $\quad \sqrt{-1} = i$

$x = \pm i\sqrt{5}$

To avoid $\sqrt{5} \cdot i$ being read as $\sqrt{5i}$, write $\sqrt{5} \cdot i$ as $i\sqrt{5}$.

Numbers such as $i\sqrt{5}$, $2i$, and $-5i$ are called **pure imaginary numbers**, and i is called the **imaginary unit**. Using i as you would any constant, you can define square roots of negative numbers.

Since $i = \sqrt{-1}$, it follows that $i^2 = -1$. Therefore,

$(3i)^2 = 3^2 \cdot i^2$ or $-9 \rightarrow \sqrt{-9} = \sqrt{9} \cdot \sqrt{-1}$ or $3i$

$(i\sqrt{3})^2 = i^2(\sqrt{3})^2$ or $-3 \rightarrow \sqrt{-3} = \sqrt{3} \cdot \sqrt{-1}$ or $i\sqrt{3}$

Definition of Pure Imaginary Numbers

For any positive real number b,
$\sqrt{-b^2} = \sqrt{b^2} \cdot \sqrt{-1}$ or bi,
where i is the imaginary unit, and bi is called a pure imaginary number.

Example 1 **Simplify each expression.**

a. $\sqrt{-12}$

$$\sqrt{-12} = \sqrt{2^2} \cdot \sqrt{3} \cdot \sqrt{-1}$$
$$= 2 \cdot \sqrt{3} \cdot i$$
$$= 2i\sqrt{3}$$

b. $\sqrt{-27x^3}$

$$\sqrt{-27x^3} = \sqrt{3^2} \cdot \sqrt{-3} \cdot \sqrt{x^2} \cdot \sqrt{x}$$
$$= 3x\sqrt{-3x}$$
$$= 3ix\sqrt{3x}$$

The commutative and associative properties for multiplication hold true for pure imaginary numbers.

Example 2 **Simplify each expression.**

a. $-3i \cdot 8i$

$$\begin{aligned} -3i \cdot 8i &= -24i^2 \\ &= -24(-1) \quad \textit{i^2 = -1} \\ &= 24 \end{aligned}$$

b. $\sqrt{-6} \cdot \sqrt{-10}$

$$\begin{aligned} \sqrt{-6} \cdot \sqrt{-10} &= i \cdot \sqrt{6}\, i \cdot \sqrt{10} \\ &= i^2 \sqrt{60} \\ &= -1 \cdot 2\sqrt{15} \text{ or } -2\sqrt{15} \end{aligned}$$

You can use the properties of powers to simplify powers of i.

Example 3 **Simplify i^{25}.**

LOOK BACK

You can review the properties of powers in Lesson 5-1.

$$\begin{aligned} i^{25} &= i \cdot i^{24} && \textit{Product of powers} \\ &= i \cdot (i^2)^{12} && \textit{Power of a power} \\ &= i \cdot (-1)^{12} && i^2 = -1 \\ &= i \cdot 1 \text{ or } i \end{aligned}$$

When you solve some equations, the answer may involve two pure imaginary numbers.

Example 4 **Solve $4x^2 + 36 = 0$.**

$$\begin{aligned} 4x^2 + 36 &= 0 \\ 4x^2 &= -36 && \textit{Subtract 36 from each side.} \\ x^2 &= -9 && \textit{Divide each side by 4.} \\ x &= \pm\sqrt{-9} && \textit{Take the square root of each side.} \\ x &= \pm 3i \end{aligned}$$

Check:

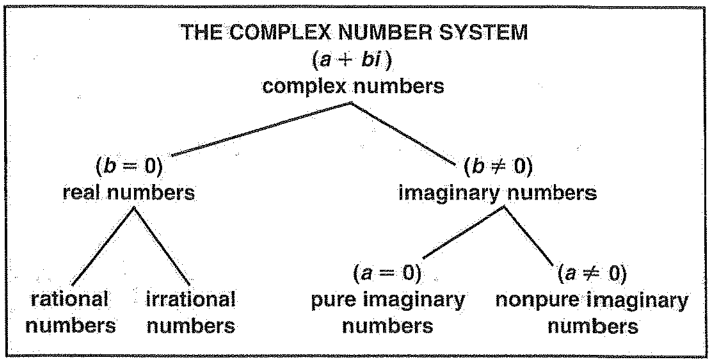
$$\begin{aligned} 4x^2 + 36 &= 0 \\ 4(3i)^2 + 36 &\stackrel{?}{=} 0 \\ 4 \cdot 9 \cdot i^2 + 36 &\stackrel{?}{=} 0 \\ -36 + 36 &\stackrel{?}{=} 0 \\ 0 &= 0 \ \checkmark \end{aligned}$$

$$\begin{aligned} 4x^2 + 36 &= 0 \\ 4(-3i)^2 + 36 &\stackrel{?}{=} 0 \\ 4 \cdot 9 \cdot i^2 + 36 &\stackrel{?}{=} 0 \\ -36 + 36 &\stackrel{?}{=} 0 \\ 0 &= 0 \ \checkmark \end{aligned}$$

Suppose you were asked to simplify the expression $5 + 2i$. Since 5 is a real number and $2i$ is a pure imaginary number, the terms are not like terms and cannot be combined. This type of expression is called a **complex number.**

Definition of a Complex Number	**A complex number is any number that can be written in the form *a* + *bi*, where *a* and *b* are real numbers and *i* is the imaginary unit; *a* is called the real part, and *bi* is called the imaginary part.**

The diagram below shows the complex number system.



- If $b = 0$, a real number results.
- If $b \neq 0$, the number is imaginary.
- If $a = 0$, the number is a pure imaginary number.

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

Definition of Equal Complex Numbers	$a + bi = c + di$ **if and only if** $a = c$ **and** $b = d$

To add or subtract complex numbers, combine like terms; that is, combine the real parts and combine the imaginary parts.

Example 5 **Simplify each expression.**

a. $(8 - 5i) + (2 + i)$

$$(8 - 5i) + (2 + i)$$
$$= (8 + 2) + (-5i + i)$$
$$= 10 - 4i$$

b. $(4 + 7i) - (2 + 3i)$

$$(4 + 7i) - (2 + 3i)$$
$$= (4 - 2) + (7i - 3i)$$
$$= 2 + 4i$$

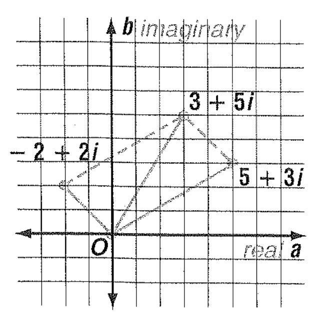
You can model the addition of complex numbers geometrically.

MODELING MATHEMATICS — Adding Complex Numbers

Materials: grid paper, straightedge

You can model the addition of complex numbers on a coordinate plane. The horizontal axis represents the real part a of the complex number and the vertical axis represents the coefficient b of the imaginary part. Use a coordinate plane to find $(5 + 3i) + (-2 + 2i)$.

- Create a coordinate plane and label the axes appropriately.
- Graph $5 + 3i$ by drawing a segment from the origin to $(5, 3)$ on the coordinate plane.
- Graph $-2 + 2i$ by drawing a segment from the origin to $(-2, 2)$ on the coordinate plane.
- Draw a parallelogram that has the two segments you drew as sides.
- The diagonal of the parallelogram drawn from the origin represents the sum of the two complex numbers. The endpoint of the diagonal is $(3, 5)$, which represents $3 + 5i$. So, $(5 + 3i) + (-2 + 2i) = (3 + 5i)$.



Your Turn

Model $(-2 + 3i) + (1 - 4i)$ on a coordinate plane.

You can multiply complex numbers by using the FOIL method.

Example 6 **Simplify $(4 + 2i)(3 - 5i)$.**

$$\begin{aligned}(4 + 2i)(3 - 5i) &= \overset{F}{4(3)} - \overset{O}{4(5i)} + \overset{I}{(2i)3} - \overset{L}{2i(5i)} \\ &= 12 - 20i + 6i - 10i^2 \\ &= 12 - 14i - 10(-1) \\ &= 22 - 14i\end{aligned}$$

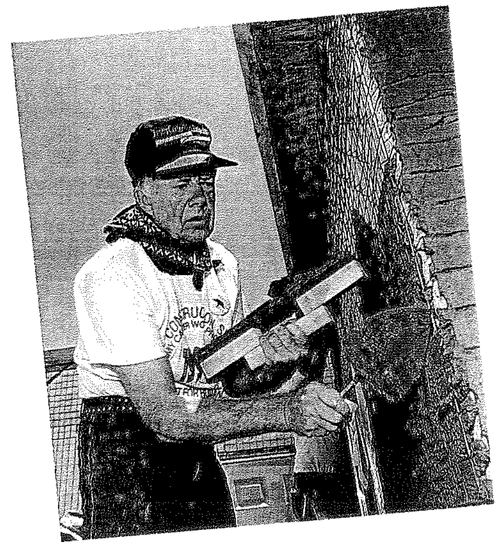
One of the real-world uses of imaginary numbers is in electricity. However, electrical engineers use j instead of i to represent the imaginary unit. This avoids any confusion with the I used as a symbol for current. Imaginary numbers are used to represent the impedance of a circuit. The impedance is the resistance to the flow of electricity through the circuit.

Example 7

Real World APPLICATION

Electricity

The Habitat for Humanity program utilizes volunteers to help build houses for low-income families who might otherwise not be able to afford the purchase of a home. At a recent site, Habitat workers built a small storage shed attached to the house. The electrical blueprints for the shed called for two AC circuits connected in series with a total voltage of 220 volts. One of the circuits must have an impedance of $7 - 10j$ ohms, and the other needs to have an impedance of $9 + 5j$. According to the building codes, the impedance cannot exceed $20 - 5j$ ohms. Will the circuits, as designed, meet the code?



Former President Jimmy Carter

Explore First we need to understand the electricity terms used in the problem. In a simplified electrical circuit, there are three basic components to be considered:

- the flow of the electrical current, I
- the resistance to that flow, Z, called impedance, and
- the electromotive force, E, called voltage.

The formula $E = I \cdot Z$ illustrates the relationship among these components.

Plan The total impedance is the sum of the individual impedances.

Solve

$$\begin{aligned}(7 - 10j) + (9 + 5j) &= 7 + 9 - 10j + 5j \\ &= 16 + (-10 + 5)j \\ &= 16 - 5j\end{aligned}$$

The total impedance is $16 - 5j$, which is less than $20 - 5j$. The circuits will meet the code.

Examine Since both numbers contain the same imaginary part, we can compare the real part. $16 < 20$, so the conclusion is correct.

Generally, you cannot order complex numbers that contain imaginary parts. However, if the imaginary parts of the two numbers are identical, you can compare the real number parts.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

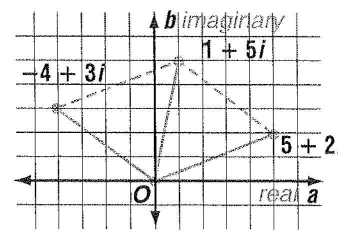
1. **Determine** if each statement is true or false.
   a. Every real number is a complex number.
   b. Every imaginary number is a complex number.

2. **Show** where each of the following lies on the complex coordinate plane.
   a. real numbers
   b. pure imaginary numbers

3. Which complex number is equivalent to $\sqrt{-50}$?
   a. $-5i$ b. $25i$ c. $5i\sqrt{2}$ d. $-5i\sqrt{2}$

MODELING MATHEMATICS

4. Identify the complex numbers and their sum shown in the graph at the right.



5. Graph the addends $(-2 + i)$ and $(4 + 4i)$ on the complex plane. Then find their sum geometrically.

Guided Practice

Simplify.

6. $\sqrt{-64}$
7. $\sqrt{-98m^2n^2}$
8. $(4i)(-3i)$
9. $5\sqrt{-24} \cdot 3\sqrt{-18}$
10. $\sqrt{3} \cdot \sqrt{-27}$
11. i^{16}
12. $(15 + 10i) - (4 + 6i)$
13. $(4 + 2i) + (1 + 3i)$
14. $(4 - 3i)(5 + 7i)$
15. $(3 + 2i)(3 - 2i)$
16. Find the product $(2 - 4i)(3 + 9i)$.
17. Verify that $-2i\sqrt{2}$ is a solution for $5x^2 + 40 = 0$. Find the other solution.

Solve each equation.

18. $5x^2 + 30 = 5$
19. $a^2 + 16 = 0$

EXERCISES

Practice

Simplify.

20. $\sqrt{-169}$
21. $\sqrt{-\frac{4}{9}}$
22. $\sqrt{-49}$
23. $\sqrt{-100k^4}$
24. $\sqrt{-36m^4n^2}$
25. $\sqrt{-\frac{9x^3}{25y^8}}$
26. $(2i)^2$
27. $(-4i)(-5i)(3i)$
28. $5i(-2i)^2$
29. $(\sqrt{-11})(\sqrt{-22})$
30. $(2\sqrt{-50})(\frac{1}{8}\sqrt{-2})$
31. $\sqrt{-8} \cdot \sqrt{-18}$

32. $\sqrt{-5} \cdot \sqrt{20}$

33. $\sqrt{-8} \cdot \sqrt{6}$

34. $-2\sqrt{-x} \cdot -5\sqrt{-y}$

35. i^{17}

36. i^{59}

37. i^{34}

38. $2\sqrt{-18} + 3\sqrt{-2}$

39. $(4 - i) + (3 + 3i)$

40. $(8 - 5i) - (2 + i)$

41. $(7 - 6i) - (5 - 6i)$

42. $(2 - 4i) + (2 + 4i)$

43. $(11 - \sqrt{-3}) - (-4 + \sqrt{-5})$

44. $(4 + i)(4 - i)$

45. $(4 - i)(3 + 2i)$

46. $(3 - 4i)^2$

47. $(2 - \sqrt{-3})(2 + \sqrt{-3})$

48. $3(-5 - 2i) + 2(-3 + 2i)$

49. $(3 + 2i)^2 + (3 + 4i)^2$

Solve each equation.

50. $-6x^2 - 30 = 0$

51. $5x^2 + 40 = 0$

52. $3x^2 + 18 = 0$

53. $7x^2 + 84 = 0$

54. $\frac{2}{3}x^2 + 30 = 0$

55. $4x^2 + 5 = 0$

Simplify.

56. $(-6 + 2i)(7 - i)(4 + 3i)$

57. $(7 - 5i)(7 + 5i)(2 - 3i)$

58. $(7 - i)(4 + 2i)(5 + 2i)$

59. $(2 + i)(1 + 2i)(3 - 4i)$

Find the values of *m* and *n* that make each equation true.

60. $18 + 7i = 3m + 2ni$

61. $(2m + n) + (m - n)i = 7 - i$

62. $(m + 2n) + (2m - n)i = 5 + 5i$

63. $(2m - 3n)i + (m + 4n) = 13 + 7i$

64. Give an example to demonstrate that the product of two complex numbers in the form $a + bi$, where $a \neq 0$ and $b \neq 0$ may not result in a complex number of the same form.

Graphing Calculator

65. Refer to the equations in Exercises 50–55.
   a. Replace 0 with y in each equation and write the equation in the form $y = ax^2 + c$.
   b. Graph each equation and make a sketch of the graph.
   c. What characteristic(s) do all of the graphs have in common?
   d. How does the statement "the real roots of a function are its x-intercepts" relate to what you found in your graphs?

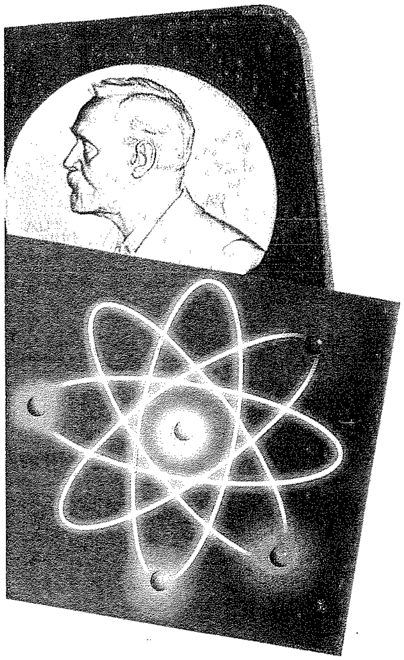
Critical Thinking

66. **Number Theory** Under which of the operations—addition, subtraction, or multiplication—is the set of imaginary numbers closed? Give examples to support your answer.

67. Show that $2 - 3i$ is a solution of $x^2 - 4x + 13 = 0$. Are there other solutions? Explain.

Applications and Problem Solving

68. **Electricity** Refer to the information in Example 7.
   a. A circuit has a current of $(7 + 3j)$ amps and an impedance of $(5 - j)$ ohms. What will the voltage of the circuit be?
   b. A circuit has been tested to have a current of $(10 + 5j)$ amps. The current needed to upgrade the circuit is $(50 + 20j)$ amps. How many additional amps are needed to upgrade the circuit?



69. **Quantum Mechanics** Wolfgang Pauli (1900–1958) won the 1945 Nobel Prize for physics for his discovery of the Pauli exclusion principle. The principle states that in an atom, no two electrons can have the same energy. In his study of electron spin, he used matrices that have become known as the Pauli spin matrices. These matrices are $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Suppose matrix $D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.
 a. Verify that AB equals the product of D and BA.
 b. Verify that CB equals the product of D and BC.

70. **Look for a Pattern** Find the values for all of the powers of i from i^0 to i^{12}.
 a. What pattern do you notice in the values?
 b. Describe how you would simplify i^n, where n is a positive integer.
 c. Evaluate each expression. Assume that k is an integer.
 (1) i^{4k} (2) i^{4k+1} (3) i^{4k+2} (4) i^{4k+3}
 d. Use your findings in part c to evaluate each expression.
 (1) i^{784} (2) i^{503} (3) $i^{8,413,634}$

Mixed Review

71. Solve $\sqrt{3y^2 + 11y - 5} = y\sqrt{3} + 1$. (Lesson 5–8)

72. **Identify Subgoals** What fraction of the perfect squares between 0 and 100 are odd? (Lesson 5–7)

73. **Physics** If a stone is dropped from a cliff, the equation $t = \frac{1}{4}\sqrt{d}$ represents the time t in seconds that it takes for the stone to reach the ground. If d represents the distance in feet that the stone falls, find how long it would take for a stone to fall from a 150-foot cliff. (Lesson 5–6)

74. Simplify $3p(p^2 - 2p + 3)$. (Lesson 5–2)

75. Solve the system of equations by using augmented matrices. (Lesson 4–7)
 $x + 3y - 2z = 9$
 $-x + 5y + 2z = 31$
 $2x - 9z = -32$

76. **SAT Practice** **Grid-in** If $3x + 2y = 36$ and $\frac{5y}{3x} = 5$, then $x = ?$

77. **Decorating** Carol and Frank are buying some new living room furniture. A sofa, loveseat, and coffee table cost \$2100. The sofa costs twice as much as the love seat. The sofa and the coffee table cost \$1510. What are the prices of each piece of furniture? (Lesson 3–7)

78. Solve the system of equations by using the elimination method. (Lesson 3–2)
 $3x - 2y = 7$
 $-3x + 9y = 14$

79. Write an equation in standard form for the line whose x-intercept is 6 and y-intercept is -5. (Lesson 2–4)

80. Graph $2x + 3y = 12$. (Lesson 2–2)

81. A number increased by 27 is 46. Find the number. (Lesson 1–4)

82. State the property illustrated by $3 + (a + b) = (a + b) + 3$. (Lesson 1–2)

For Extra Practice, see page 888.

5-10 Simplifying Expressions Containing Complex Numbers

What YOU'LL LEARN

- To simplify rational expressions containing complex numbers in the denominator.

Why IT'S IMPORTANT

You can use complex numbers to solve problems involving fractals and electricity.

Real World APPLICATION

Electricity

We take electricity for granted as a part of our everyday life. In most third-world countries, only large metropolitan areas have access to electricity. The International Foundation for the Promotion of New and Emerging Sciences and Technology (NEST) has been working to bring electricity to smaller rural communities.

In the last lesson, you used the formula $E = I \cdot Z$ to solve problems dealing with electrical circuits. Impedance Z is the resistance to the flow of electricity and is measured in ohms. Voltage E refers to the electrical potential in a circuit and is measured in volts. Current I is measured in amperes (amps). If a circuit of 110 volts were to be installed in a house, what would the impedance be for a current of $15 + 3\boldsymbol{j}$ amps?

$$E = I \cdot Z$$

$$110 = (15 + 3\boldsymbol{j})Z \quad \textit{E = 110, I = 15 + 3j}$$

$$\frac{110}{15 + 3\boldsymbol{j}} = Z \quad \textit{Divide each side by (15 + 3j).}$$

Remember that in electricity j is used instead of i to represent the imaginary unit.

Since $\boldsymbol{j}$ represents a radical, this expression is not in simplest form. *You will simplify this expression in Example 3.*

Remember that two radical expressions $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$ are conjugates. Since imaginary numbers also involve radicals, numbers of the form $a + b\boldsymbol{i}$ and $a - b\boldsymbol{i}$ are called **complex conjugates.** Recall that the product of two radical conjugates is a rational number. Let's investigate the product of two complex conjugates.

Example 1 **Simplify $(15 + 3i)(15 - 3i)$.**

$$\begin{aligned}(15 + 3\boldsymbol{i})(15 - 3\boldsymbol{i}) &= 15^2 - 9\boldsymbol{i}^2 && \textit{Difference of two squares}\\ &= 225 - (-9) && 9i^2 = 9(-1)\\ &= 234\end{aligned}$$

The product in Example 1 is rational. Let's explore the general case $(a + b\boldsymbol{i})(a - b\boldsymbol{i})$.

$$\begin{aligned}(a + b\boldsymbol{i})(a - b\boldsymbol{i}) &= a^2 - (b\boldsymbol{i})^2 && \textit{Difference of two squares}\\ &= a^2 - b^2\boldsymbol{i}^2\\ &= a^2 + b^2 && b^2i^2 = b^2(-1)\end{aligned}$$

Since a and b are real numbers, $a^2 + b^2$ will also be real. Thus, the product of complex conjugates is a real number.

We use conjugates of radicals to rationalize the denominators of expressions with radicals in the denominator. We can also use conjugates of complex numbers to rationalize denominators of expressions with complex numbers in the denominator.

Example 2 **Simplify each expression.**

LOOK BACK

You can review rationalizing the denominator in Lesson 5-6.

a. $\dfrac{8i}{1+3i}$

$$\frac{8i}{1+3i} = \frac{8i}{1+3i}\cdot\frac{1-3i}{1-3i} \quad \textit{1 + 3i and 1 − 3i are conjugates.}$$
$$= \frac{8i(1-3i)}{1+3^2}$$
$$= \frac{8i-24i^2}{10}$$
$$= \frac{8i+24}{10}$$
$$= \frac{4i+12}{5} \text{ or } \frac{12+4i}{5}$$

b. $\dfrac{2+i}{5i}$

$$\frac{2+i}{5i} = \frac{2+i}{5i}\cdot\frac{i}{i} \quad \textit{Why multiply by } \tfrac{i}{i} \textit{ instead of } \tfrac{5i}{5i}\textit{?}$$
$$= \frac{2i+i^2}{5i^2}$$
$$= \frac{2i-1}{-5} \text{ or } \frac{-1(1-2i)}{-1(5)}$$
$$= \frac{1-2i}{5}$$

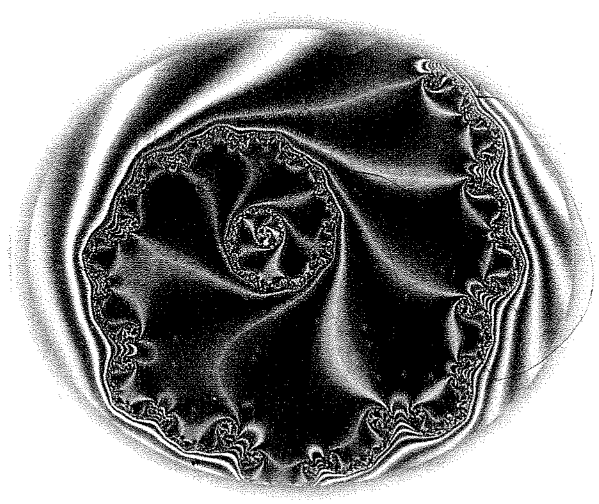
Example 3 **Real World APPLICATION** **Electricity**

Refer to the application at the beginning of the lesson. What would the impedance Z be for a current of $15 + 3j$ amps in a 110-volt circuit?

We found that $Z = \dfrac{110}{15+3j}$.

$$Z = \frac{110}{15+3j}\cdot\frac{15-3j}{15-3j} \quad \textit{Rationalize the denominator.}$$
$$= \frac{110(15-3j)}{234} \quad (15+3j)(15-3j) = 15^2 + 3^2 \text{ or } 234$$
$$= \frac{1650-330j}{234}$$
$$\approx 7.05 - 1.41j$$

The impedance would be approximately $7.05 - 1.41j$ ohms.



The picture at the left, called a **fractal,** was created with the aid of a computer. Benoit Mandelbrot coined the word *fractal* as a label for computer-generated, irregular and fragmented, self-similar shapes. Fractal objects are created using functions that are **iterated,** that is, repeated over and over. The function is evaluated for some initial value of x, and then the function is evaluated again using that result. Repeating this process and plotting the points produces interesting and sometimes beautiful pictures. Some of the most exciting fractals are created using this process with a number like $4 + 3i$ as the initial value.

Example 4 **INTEGRATION Fractal Geometry**

Suppose the function $f(x) = \frac{1}{x^2 + 1}$ is to be iterated to produce a fractal. Find the first two points of the iteration if the initial value is $(1 + i)$.

1st iteration

Replace x with $(1 + i)$.

$$f(1 + i) = \frac{1}{(1 + i)^2 + 1} = \frac{1}{1 + 2i - 1 + 1} = \frac{1}{1 + 2i} = \frac{1}{1 + 2i} \cdot \frac{1 - 2i}{1 - 2i} = \frac{1 - 2i}{5} \text{ or } \frac{1}{5} - \frac{2i}{5}$$

2nd iteration

Replace x with $\frac{1 - 2i}{5}$.

$$f\left(\frac{1 - 2i}{5}\right) = \frac{1}{\left(\frac{1 - 2i}{5}\right)^2 + 1} = \frac{1}{\frac{(1 - 4i - 4) + 25}{25}} = \frac{25}{22 - 4i} = \frac{25}{2(11 - 2i)} \cdot \frac{11 + 2i}{11 + 2i} = \frac{275 + 50i}{2(121 + 4)} = \frac{275 + 50i}{250} \text{ or } \frac{11}{10} + \frac{i}{5}$$

The first two points of the iteration are $\frac{1}{5} - \frac{2i}{5}$ and $\frac{11}{10} + \frac{i}{5}$.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **Describe** how to rationalize the denominator of $\frac{1}{a + bi}$.
2. **Explain** why the product of a complex number and its conjugate is always a real number.
3. **Evaluate** each expression if $z = 1 + 2i$.
   a. z^2
   b. $\frac{1}{z}$
   c. What would you think the product $z^2 \cdot \frac{1}{z}$ would be without substituting $1 + 2i$ for z? Use substitution to verify your answer.
4. Does every complex number have a multiplicative inverse and an additive inverse?

Guided Practice

Find the conjugate of each complex number.

5. $7i$
6. $3 + 5i$

Find the product of each complex number and its conjugate.

7. $-10i$
8. $12 + 5i$

Simplify.

9. $\frac{7}{-2i}$
10. $\frac{9 + 3i}{2i}$
11. $\frac{5}{2 + i}$
12. $\frac{5 + i}{1 + 2i}$
13. $\frac{3 - 2i}{1 - i}$
14. $\frac{7}{\sqrt{2} - 3i}$

15. Show that $7 + 3i$ and $\frac{7 - 3i}{58}$ are multiplicative inverses of each other.

EXERCISES

Practice

Find the conjugate of each complex number.

16. $10i$
17. $12 + i$
18. $-15i$
19. $10 - 4i$
20. $7 + i\sqrt{5}$
21. $6 + \sqrt{-7}$

Find the product of each complex number and its conjugate.

22. $-2i$
23. $5 - 2i$
24. $4 + 6i$
25. $1 + i$
26. $3 + 5i\sqrt{2}$
27. $8 - 2i$

Simplify.

28. $\frac{2 + 8i}{3i}$
29. $\frac{3 + 7i}{2i}$
30. $\frac{11 + i}{2 - i}$
31. $\frac{3i}{2 + i}$
32. $\frac{-3i}{5 + 4i}$
33. $\frac{3}{6 + 4i}$
34. $\frac{3 + 5i}{1 + i}$
35. $\frac{2 + i}{3 - i}$
36. $\frac{1 - i}{4 - 5i}$
37. $\frac{2 + 3i}{3 - 2i}$
38. $\frac{5 - 6i}{-3i}$
39. $\frac{3 - 9i}{4 + 2i}$
40. $\frac{8}{\sqrt{2} + i}$
41. $\frac{1}{3 - i\sqrt{2}}$
42. $\frac{4}{\sqrt{3} + 2i}$

Find the multiplicative inverse of each complex number.

43. $6 - 5i$
44. $\frac{-i}{3 + 5i}$
45. $x + yi$

INTEGRATION
Fractal Geometry

Find the 1st and 2nd iteration of each function for the given initial value.

46. $f(x) = x^2 + 2$ for $x = 1 + i$
47. $f(x) = 3x^2 + 2$ for $x = 1 - i$
48. $f(x) = x^2 - x$ for $x = i + 3$

Simplify.

49. $\frac{3 - i\sqrt{5}}{3 + i\sqrt{5}}$
50. $\frac{1 + i\sqrt{3}}{1 - i\sqrt{3}}$
51. $\frac{1 - i}{(1 + i)^2}$
52. $\left(\frac{\sqrt{3}}{2 + 3i}\right)^2$
53. $\frac{(2 + 3i)^2}{(3 + i)^2}$
54. $\frac{(4 + 3i)^2}{(3 - 4i)^2}$

Critical Thinking

55. If $z = a + bi$ and $w = c + di$ are complex numbers, then their conjugates are denoted by $z = a - bi$ and $w = c - di$, respectively. Determine if the conjugate of the product zw is equal to the product of the conjugates. That is, would $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$? Why or why not?

56. Show that $\frac{-1 + i\sqrt{3}}{2}$ is a cube root of 1.

Applications and Problem Solving

Real World

CAREER CHOICES

Engineers apply the principles of science and mathematics to solve practical technical problems. **Electrical engineers** design, develop, test, and supervise the manufacturing of electrical and electronic equipment. They comprise more than one fourth of all engineers. A bachelor's degree in engineering is the minimum requirement.

For more information, contact:

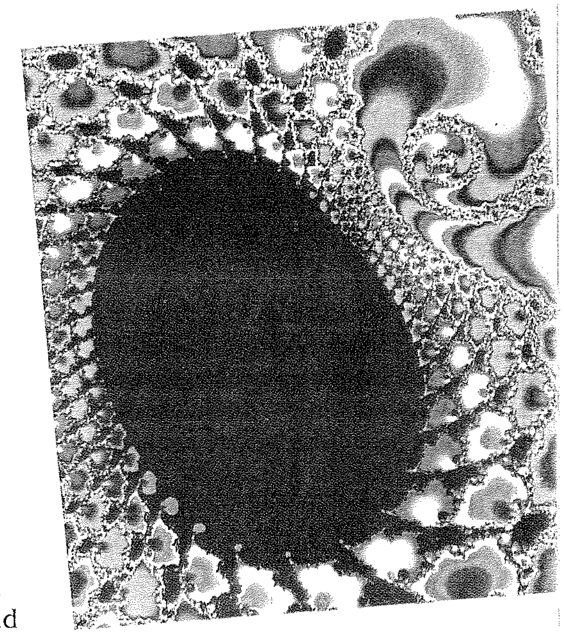
Institute of Electrical and Electronic Engineers
1828 L St. NW, Suite 1202
Washington, DC 20036

57. **Electrical Engineering** Refer to the application at the beginning of the lesson. Copy and complete the table.

	a.	b.	c.	d.
E (volts)	$60 + 112j$	$85 + 110j$	$-50 + 100j$	$-70 + 240j$
Z (ohms)	$10 + 6j$	$3 - 4j$		
I (amps)			$-6 + 2j$	$-5 + 4j$

58. **Electrical Circuitry** In a two-battery flashlight, the positive terminal of the first battery touches the negative terminal of the second. The positive terminal of the second battery touches the center terminal of the light bulb. A metal strip connects the bulb to the switch, which is connected to the negative terminal of the first battery. When you turn the flashlight on, the switch completes the circuit, and the bulb lights up.
   a. Suppose each of the batteries is 2.5 volts. What is the total impedance of the circuit in the flashlight if the current is $(1 + 2j\sqrt{2})$?
   b. What is the total current if two 1.5 volt batteries are used and the impedance is $(2 + 3j)$ ohms?

59. **Fractal Geometry** Suppose the function $f(x) = x^2 - 1$ is to be iterated to produce a fractal. Find the first four points of the iteration if the initial value of x is $(1 + i)$. Write the points as ordered pairs.



Mixed Review

Solve each equation.

60. $3a^2 + 24 = 0$ (Lesson 5–9)

61. $2x + 7 = -x\sqrt{2}$ (Lesson 5–8)

62. Factor $(x + y)^2 - \frac{1}{4}$. (Lesson 5–4)

63. **ACT Practice** The girls' soccer team has three top scorers: Beth, Nina, and Sonia. Between them, they scored 30 goals. Beth scored three times as many goals as Nina. The combined goals of Beth and Sonia were twice that of Nina and Sonia. How many goals did Beth score?

   A 5 **B** 6 **C** 10 **D** 15 **E** 18

64. Find the inverse of $\begin{bmatrix} 3 & 1 \\ -4 & 1 \end{bmatrix}$. (Lesson 4–5)

Solve each system algebraically or by graphing.

65. $6x - 2y - 3z = -10$
    $-6x + y + 9z = 3$
    $8x - 3y = -16$ (Lesson 3–7)

66. $x > 1$
    $y < -1$
    $y < x$ (Lesson 3–4)

67. Is the relation $\{(0, 0), (1, 0)\}$ a function? Explain your answer. (Lesson 2–1)

68. Solve $3(2m - 3) \geq 9$. (Lesson 1–6)

69. Evaluate $[(-8 + 3) \times 4 - 2] \div 6$. (Lesson 1–1)

For Extra Practice, see page 888.

CLOSING THE

In•ves•ti•ga•tion

3-2-1-Blast-Off!

Refer to the Investigation on pages 180–181.

When a new launch system is needed and a contract is given to a research company, the company employs engineers, scientists, physicists, drafters, and others to work together toward the final plans for the launcher. Accuracy is examined as well as the cost factor. Many of the skills you have used in this Investigation are used by professionals to design the right launcher for the job.

Analyze

You have conducted experiments and organized your data in various ways. It is now time to analyze your findings and state your conclusions.

PORTFOLIO ASSESSMENT

You may want to keep your work on this Investigation in your portfolio.

1. Look over the data and organize a table of all the data from the shoot-off including all the distances of each team.
2. Make a list of statistical measures and accuracy measures calculated from each team's data. These should include two sets of measures: the median, quartiles, least and greatest values, and interquartile range; the error range, tolerance, relative error, average error, and standard deviation. Did the data collected from the two sets of measures lead you to the same conclusion about the launcher's accuracy? Explain.
3. Analyze each of the launchers that were used in the shoot-off and rank them from most accurate to least accurate. Use mathematical measures to justify your rankings. Explain what measures you used and how you used them to evaluate the accuracy of the launchers.

Write

Your client has asked you for a report on both the design and calibration of your launcher and an analysis of the results of the shoot-off.

4. Summarize the procedures you used to design the launcher. Describe the steps and procedures used to design your launcher, from the receipt of the instructions to the demonstration shoot-off.
5. Draw a detailed blueprint of the launch system that specifies its parts and the construction of the system.
6. Include in this report an operation manual that describes how the launcher operates. Include a description of the calibration system used for accuracy. A list or table that compares the launch system settings with actual shot distances should also be included. Describe the mathematical relationship between the settings and target distances.
7. Summarize the accuracy of the launch system. Include test data recorded from shots taken at each distance: 50 cm, 100 cm, 150 cm, 200 cm, and 250 cm. A graph depicting the location of the test shots and the target distance should also be included.

interNET CONNECTION

Data Collection and Comparison To share and compare your data with other students, visit: **www.algebra2.glencoe.com**

CHAPTER 5 HIGHLIGHTS

INET CONNECTION

Chapter Review For additional lesson-by-lesson review, visit: **www.algebra2.glencoe.com**

VOCABULARY

After completing this chapter, you should be able to define each term, property, or phrase and give an example or two of each.

Algebra

binomial (p. 261)
coefficient (p. 255)
complex conjugates (p. 317)
complex number (p. 311)
conjugates (p. 292)
constant (p. 255)
degree (p. 255)
equal complex numbers (p. 312)
extraneous solution (p. 305)
factors (p. 274)
FOIL method (p. 263)
imaginary unit (p. 310)
iterate (p. 318)
like radical expressions (p. 291)
like terms (p. 261)
monomial (p. 255)
nth root (p. 282)
polynomial (p. 261)
power (p. 255)
prime number (p. 274)
principal root (p. 282)
pure imaginary number (p. 310)
radical equations (p. 305)
rational exponent (p. 298)
rationalizing the denominator (p. 290)
scientific notation (p. 254)
simplify (p. 255)
square root (p. 281)
square root inequalities (p. 306)
synthetic division (p. 269)
term (p. 261)
trinomial (p. 261)

Geometry

fractal (p. 318)

Problem Solving

identify subgoals (p. 296)

UNDERSTANDING AND USING THE VOCABULARY

Choose a word or term that best completes each statement or phrase.

1. A number is expressed in ________ when it is in the form of $a \cdot 10^n$, where $1 \leq a < 10$ and n is an integer.
2. Monomials that contain no variables are known as ________.
3. ________ is the process used to eliminate radicals from the denominator or fractions from a radicand.
4. A shortcut method known as ________ is used to divide polynomials by binomials.
5. Real numbers that cannot be written as terminating or repeating decimals are ________.
6. The ________ is used to multiply two binomials.
7. In an algebraic term that is the product of a number and a variable, the number is the ________ of the variable.
8. A ________ is an expression that is a number, a variable, or the product of a number and one or more variables.
9. A solution of a transformed equation that is not a solution of the original equation is an ________.
10. ________ are imaginary numbers of the form $a + bi$ and $a - bi$.
11. For any number a and b, if $a^2 = b$, then a is the ________ of b.
12. A polynomial comprised of three unlike terms is known as a ________.
13. The ________ is the degree of the monomial of the greatest degree.
14. When there is more than one root, the ________ is the nonnegative root.
15. ________ are computer-generated, irregular and fragmented, self-similar shapes created using functions that are iterated—that is, repeated over and over.

SKILLS AND CONCEPTS

OBJECTIVES AND EXAMPLES

Upon completing this chapter, you should be able to:

- multiply and divide monomials (Lesson 5–1)

$(3x^4y^6)(-8x^3y)$

$= (3)(-8)x^{4+3}y^{6+1}$ *Multiplying powers*

$= -24x^7y^7$

REVIEW EXERCISES

Use these exercises to review and prepare for the chapter test.

Simplify. Assume that no variable equals 0.

16. $m^3 \cdot m^5$

17. $f^{-7} \cdot f^4$

18. $(3x^2)^3$

19. $(2y)(4xy^3)$

20. $\left(\frac{3}{5}c^2f\right)\left(\frac{4}{3}cd\right)^2$

21. $\frac{1}{x^0 + y^0} - \frac{x^0 + y^0}{1}$

22. $3(ab)^3(4ac^2) + c(4ab)(5a^3b^2c)$

- represent numbers in scientific notation (Lesson 5–1)

$31{,}000 = 3.1 \times 10{,}000$

$= 3.1 \times 10^4$

$0.007 = 7 \times 0.001$

$= 7 \times 10^{-3}$

Evaluate. Express each answer in both scientific and decimal form.

23. $(2000)(85{,}000)$

24. $(0.0014)^2$

25. $5{,}400{,}000 \div 6000$

- add, subtract, and multiply polynomials (Lesson 5–2)

$(5x^2 + 4x) - (3x^2 + 6x - 7)$

$= 5x^2 + 4x - 3x^2 - 6x + 7$

$= (5x^2 - 3x^2) + (4x - 6x) + 7$

$= 2x^2 - 2x + 7$

$(9k + 4)(7k - 6)$

$= (9k)(7k) + (9k)(-6) + (4)(7k) + (4)(-6)$

$= 63k^2 - 54k + 28k - 24$

$= 63k^2 - 26k - 24$

Simplify.

26. $(4c - 5) - (c + 11) + (-6c + 17)$

27. $(11x^2 + 13x - 15) - (7x^2 - 9x + 19)$

28. $-6m^2(3mn + 13m - 5n)$

29. $(d - 5)(d + 3)$

30. $x^{-8}y^{10}(x^{11}y^{-9} + x^{10}y^{-6})$

31. $(2a^2 + 6)^2$

32. $-5f^{12}(4f^3g + 2f)$

33. $(2b - 3c)^3$

- divide polynomials by binomials using synthetic division (Lesson 5–3)

Find $(4x^4 - x^3 - 19x^2 + 11x - 2) \div (x - 2)$.

$4x^4 - x^3 - 19x^2 + 11x - 2$

↓ ↓ ↓ ↓ ↓

2	4	−1	−19	11	−2
		8	14	−10	2
	4	7	−5	1	0

The quotient is $4x^3 + 7x^2 - 5x + 1$.

Find each quotient.

34. $(2x^4 - 6x^3 + x^2 - 3x - 3) \div (x - 3)$

35. $(10x^4 + 5x^3 + 4x^2 - 9) \div (x + 1)$

36. $(x^2 - 5x + 4) \div (x - 1)$

37. $(5x^4 + 18x^3 + 10x^2 + 3x) \div (x^2 + 3x)$

OBJECTIVES AND EXAMPLES

- factor polynomials (Lesson 5–4)

$4x^3 - 6x^2 + 10x - 15$
$= (4x^3 - 6x^2) + (10x - 15)$
$= 2x^2(2x - 3) + 5(2x - 3)$
$= (2x^2 + 5)(2x - 3)$

REVIEW EXERCISES

Factor completely. If the polynomial is not factorable, write *prime*.

38. $200x^2 - 50$
39. $10a^3 - 20a^2 - 2a + 4$
40. $5w^3 - 20w^2 + 3w - 12$
41. $s^3 + 512$
42. $x^2 - 7x + 5$

- simplify radicals having various indices (Lesson 5–5)

$\pm\sqrt{81x^4} = \pm\sqrt{(9x^2)^2}$ or $\pm 9x^2$

$\sqrt[7]{2187x^{14}y^{35}} = \sqrt[7]{(3x^2y^5)^7}$ or $3x^2y^5$

Simplify.

43. $\pm\sqrt{256}$
44. $\sqrt[3]{-216}$
45. $\sqrt{-(-8)^2}$
46. $\sqrt[5]{c^5d^{15}}$
47. $\pm\sqrt{(x^4 - 3)^2}$
48. $\sqrt[3]{(512 + x^2)^3}$
49. $\sqrt[4]{16m^8}$
50. $\sqrt{a^2 - 10a + 25}$

- add, subtract, multiply, and divide radical expressions (Lesson 5–6)

$6\sqrt[5]{32m^3} \cdot 5\sqrt[5]{1024m^2}$
$= 6 \cdot 5\sqrt[5]{(32m^3 \cdot 1024m^2)}$
$= 30\sqrt[5]{2^5 4^5 m^5}$
$= 30\sqrt[5]{2^5} \cdot \sqrt[5]{4^5} \cdot \sqrt[5]{m^5}$
$= 30 \cdot 2 \cdot 4 \cdot m$ or $240m$

Simplify.

51. $\sqrt[4]{64}$
52. $\sqrt{5} + \sqrt{20}$
53. $5\sqrt{12} - 3\sqrt{75}$
54. $6\sqrt[5]{11} - 8\sqrt[5]{11}$
55. $\left(\sqrt{8} + \sqrt{12}\right)^2$
56. $\sqrt{8} \cdot \sqrt{15} \cdot \sqrt{21}$
57. $\dfrac{1}{3 + \sqrt{5}}$
58. $\dfrac{\sqrt{10}}{4 + \sqrt{2}}$

- evaluate expressions in either exponential or radical form (Lesson 5–7)

$32^{\frac{4}{5}} \cdot 32^{\frac{2}{5}} = 32^{\left(\frac{4}{5}+\frac{2}{5}\right)}$
$= 32^{\frac{6}{5}}$
$= (2^5)^{\frac{6}{5}}$
$= 2^6$ or 64

$\dfrac{3x}{y^{-\frac{3}{2}} \cdot \sqrt[3]{z}} = \dfrac{3xy^{\frac{3}{2}}}{z^{\frac{1}{3}}} \cdot \dfrac{z^{\frac{2}{3}}}{z^{\frac{2}{3}}}$
$= \dfrac{3xy^{\frac{3}{2}}z^{\frac{2}{3}}}{z}$

Evaluate.

59. $27^{-\frac{2}{3}}$
60. $9^{\frac{1}{3}} \cdot 9^{\frac{5}{3}}$
61. $\left(\frac{8}{27}\right)^{-\frac{2}{3}}$

Simplify.

62. $\dfrac{1}{y^{\frac{2}{5}}}$
63. $\dfrac{xy}{\sqrt[3]{z}}$
64. $\dfrac{3x + 4x^2}{x^{-\frac{2}{3}}}$

- solve equations containing radicals (Lesson 5–8)

$\sqrt{3x - 8} + 1 = 3$
$\sqrt{3x - 8} = 2$
$(\sqrt{3x - 8})^2 = 2^2$
$3x - 8 = 4$
$x = 4$

Solve each equation. Be sure to check for extraneous solutions.

65. $y^{\frac{1}{3}} - 7 = 0$
66. $(x - 2)^{\frac{3}{2}} = -8$
67. $6 + 2x\sqrt{3} = 0$
68. $\sqrt{3t - 5} - 3 = 4$
69. $\sqrt{1 + 8v} - 2 = v$
70. $\sqrt[4]{2x - 1} = 2$
71. $\sqrt{y + 5} = \sqrt{2y - 3}$
72. $\sqrt{y + 1} + \sqrt{y - 3} = 5$

OBJECTIVES AND EXAMPLES

- add, subtract, and multiply complex numbers (Lesson 5–9)

$(15 - 2i) + (5i - 11) = (15 - 11) + (-2i + 5i)$
$= 4 + 3i$

$(2 + 3i)(4i - 11) = (2)(4i) + (2)(-11) + (3i)(4i) + (3i)(-11)$
$= 8i - 22 + 12i^2 - 33i$
$= 8i - 22 + 12(-1) - 33i$
$= 8i - 33i - 22 - 12$
$= -34 - 25i$

REVIEW EXERCISES

Simplify.

73. $\sqrt{-256}$
74. $\sqrt[6]{-64m^{12}}$
75. $(13i - 2)5i$
76. $(7 - 4i) - (-3 + 6i)$
77. $-6\sqrt{-a} \cdot 2\sqrt{-b}$
78. i^6
79. i^{85}
80. $(3 + 4i)(5 - 2i)$
81. $(\sqrt{6} + i)(\sqrt{6} - i)$

- simplify rational expressions containing complex numbers in the denominator (Lesson 5–10)

$\frac{7i}{2 + 4i} = \frac{7i}{2 + 4i} \cdot \frac{2 - 4i}{2 - 4i}$
$= \frac{7i(2 - 4i)}{4 + 4^2}$
$= \frac{14i - 28i^2}{20}$
$= \frac{14i + 28}{20}$ or $\frac{7}{5} + \frac{7}{10}i$

Simplify.

82. $\frac{1 + i}{1 - i}$
83. $\frac{4 - 3i}{4 + 3i}$
84. $\frac{4}{4 + 5i}$
85. $\frac{1 + i\sqrt{2}}{1 - i\sqrt{2}}$

APPLICATIONS AND PROBLEM SOLVING

86. **Aerospace Engineering** Scientists expect that on future space stations, artificial gravity will be created by rotating all or part of the space station. The formula $N = \frac{1}{2\pi}\sqrt{\frac{a}{r}}$ gives the number of rotations N required per second to maintain an acceleration of gravity of a meters per second squared (m/s^2) on a satellite with a radius of r meters. The acceleration of gravity on Earth is $9.8\ m/s^2$. How many rotations per minute will produce an artificial gravity that is equal to half of the gravity on Earth in a space station 25 meters wide? (Lesson 5–5)

87. **Law Enforcement** A police investigator measured the skid marks left by a car to be approximately 120 feet. The driver of the car claims that she was not exceeding the 40-mph speed limit. Is she telling the truth? (Recall that $s = 2\sqrt{5\ell}$, where s represents speed, measured in miles per hour, and ℓ represents the length of the skidmarks, measured in feet.) How fast was she driving? (Lesson 5–6)

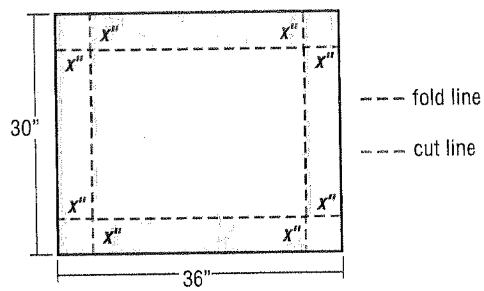
88. **Archaeology** Since carbon-14 is present in all living organisms and decays at a predictable rate after death, archaeologists use the amount of carbon-14 left in a fossil to extimate the age of the fossil. This is commonly called *carbon dating*. The approximate number of milligrams A of carbon-14 left in a fossil after 5000 years can be found using the formula $A = A_0(2.7)^{-\frac{3}{5}}$, where A_0 is the initial amount of carbon-14 in the organism. Find the amount of carbon-14 left in an organism that contained 500 milligrams of carbon-14. (Lesson 5–7)

A practice test for Chapter 5 is provided on page 916.

ALTERNATIVE ASSESSMENT

PERFORMANCE ASSESSMENT TASK

Area and Volume An open box can be made from a piece of cardboard 30 inches wide by 36 inches long by folding congruent square tabs from each corner.



1. Obtain a piece of cardboard the same size as the one shown above. Mark the box as shown with 3-inch squares at each corner. Cut along the cut lines, fold in the tabs, and fold up the sides to make the box. What is the volume of the box?
2. Find the algebraic representation $V(x)$ of the volume of the box when x is the length of one side of the square tab in the corner. Make a table of values if the side length of the squares is 2, 4, 6, 8, 10, 12, and 14 inches. How does the volume of the box change when the size of the square tabs increases in size?
3. Find the algebraic representation $A(x)$ of the area of the bottom of the box. How does the area of the bottom of the box change when the size of the square tabs increases in size?
4. Is the measure of the area of the box ever equal to the measure of the volume of the box? What is the maximum volume of the box? What is the minimum volume of the box?

THINKING CRITICALLY

As you may recall, a set is closed under an operation if the result of performing the operation on any two elements of the set is an element of the set. For example, adding two whole numbers always results in a whole number.

Is the set of irrational numbers closed under any of the four basic operations? Give examples to support your answer.

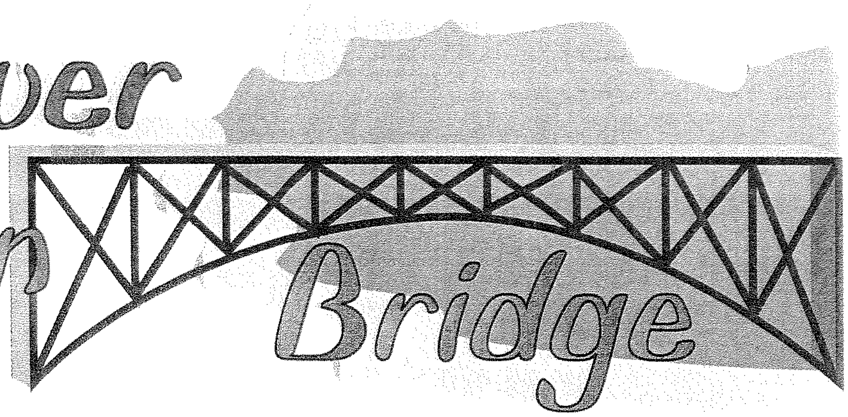
PORTFOLIO

Select one of the assignments from this chapter that you found to be difficult to master. Explain why you found this to be the case. How might this assignment be presented in a different way that may help other students having the same difficulty?

LONG-TERM PROJECT

In•ves•ti•ga•tion

the River Canyon Bridge



MATERIALS NEEDED

- cardboard box
- scissors
- glue
- ruler
- toothpicks
- string
- pipe cleaners
- straws
- wooden craft sticks

You work in the engineering division of a company that builds bridges. Your company has just been awarded a contract to build a six-lane highway bridge across a deep river canyon. The canyon is 550 yards deep and 110 yards wide.

The walls of the canyon are very steep with an almost vertical drop. The river below the proposed bridge is very swift, with strong rapids, especially during the spring when the snow thaws. The river bed consists of soft sand.

Your team of four designers is contemplating the best design for this bridge. A traditional bridge built on stilts would be weak, especially in strong winds. The stilts would also be very expensive and difficult to construct, so you remove that design from consideration.

The company's consultant states that a bridge that includes a parabolic arch is best suited for this situation. Your task is to determine the dimensions of the bridge while ensuring that the arch is shaped like a parabola.

In this Investigation, you will examine several ways in which a bridge could be constructed over the river canyon.