10-1 Real Exponents and Exponential Functions

What YOU'LL LEARN

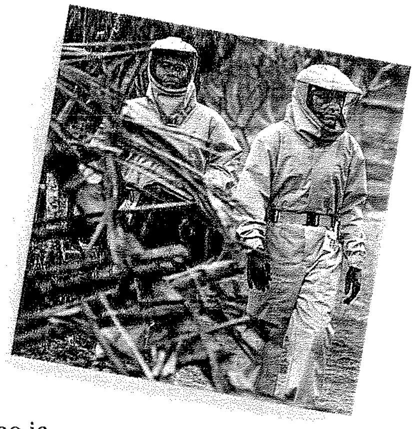
- To simplify expressions and solve equations and inequalities involving real exponents.

Why IT'S IMPORTANT

You can use exponential functions to solve problems involving disease control and animal behavior.

Real World APPLICATION

Disease Control



In a recent movie, a dreadful disease is predicted to spread across the United States in just a matter of days. How could a disease spread so quickly?

Suppose that every 8 hours, a sick person infects 2 people before the disease is diagnosed and he or she is quarantined. Consider each 8 hours as one time period. At the beginning, one individual is infected. During the first time period, this person infects 2 people. During the second time period, the first person is quarantined, but the 2 people he or she infected each infect 2 more people. During the third time period, the 2 people are quarantined, and the 4 people they infected are each responsible for infecting 2 more people. During the third time period, 4×2 or 8 people are infected.

This pattern can be summarized in the table at the right. The number of people infected during a time period y can be expressed as a function of time where x is the number of 8-hour periods. This function, $y = 2^x$, is an **exponential function**.

Time Period	Number Infected	Pattern
0	1	2^0
1	$1 \times 2 = 2$	2^1
2	$2 \times 2 = 4$	2^2
3	$4 \times 2 = 8$	2^3
4	$8 \times 2 = 16$	2^4
⋮	⋮	⋮
x	y	2^x

During the end of the 21st time period (the end of the first week), 2^{21} or 2,097,152 new people will be infected!

CAREER CHOICES

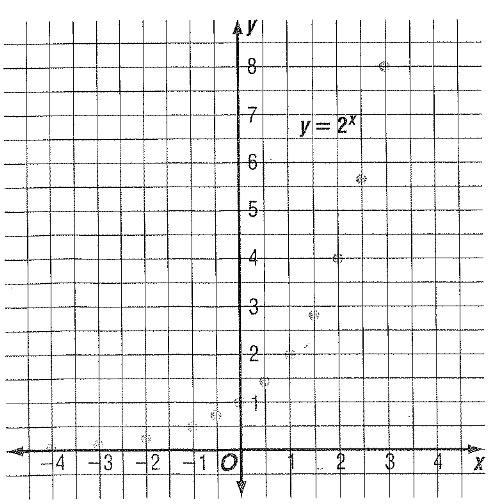
A **physician** diagnoses illnesses and prescribes and administers treatment for their patients. They counsel patients about preventive health care.

An undergraduate degree, completion of a 4-year medical school degree, and graduate medical education are required to be a licensed physician.

For more information, contact:

American Medical Association
515 N. State St.
Chicago, IL 60610

Let's take a close look at the graph of $y = 2^x$. Make a table of values to help draw the curve. *Note that negative values of x have no meaning in the example above.*



x	2^x or y	y
-4	$2^{-4} = \frac{1}{16}$	0.06
-3	$2^{-3} = \frac{1}{8}$	0.13
-2	$2^{-2} = \frac{1}{4}$	0.25
-1	$2^{-1} = \frac{1}{2}$	0.50
$-\frac{1}{2}$	$2^{-\frac{1}{2}} = \frac{1}{2}\sqrt{2}$	0.71
0	$2^0 = 1$	1

x	2^x or y	y
$\frac{1}{2}$	$2^{\frac{1}{2}} = \sqrt{2}$	1.41
1	$2^1 = 2$	2
$\frac{3}{2}$	$2^{\frac{3}{2}} = 2\sqrt{2}$	2.83
2	$2^2 = 4$	4
$\frac{5}{2}$	$2^{\frac{5}{2}} = 4\sqrt{2}$	5.66
3	$2^3 = 8$	8

LOOK BACK

You can refer to Lesson 6-1A for information on solving equations by finding the x-intercepts.

There are two methods that can be used with a graphing calculator to find a solution to exponential equations.

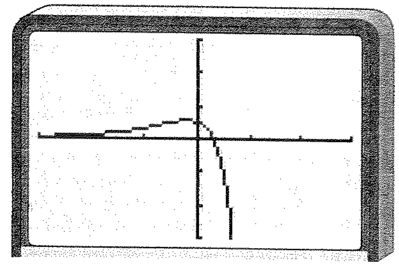
Method 1: Graph each side of the equation as a separate function and estimate the point at which they intersect.

Method 2: Rewrite the equation so that one side equals zero. Graph the related function and find the x-intercept.

Example 3 **Find the solution of $3^x = 2^{5x-1}$ to the nearest hundredth. Use the viewing window [−3, 3] by [−3, 3].**

Use the second method to solve the equation. Rewrite the equation as $3^x - 2^{5x-1} = 0$. Graph the related function $y = 3^x - 2^{5x-1}$.

Enter: [Y=] 3 [^] [X,T,θ,n] [−] 2 [^] [(] 5 [X,T,θ,n] [−] 1 [)] [GRAPH]

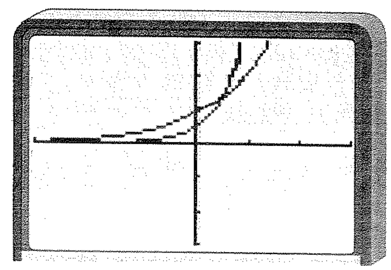


Now use ZOOM-IN or ROOT to approximate the x-intercept.

The solution is about 0.29.

Check this solution using the first method.

Enter: [Y=] 3 [^] [X,T,θ,n] [ENTER] [Y=] 2 [^] [(] 5 [X,T,θ,n] [−] 1 [)] [GRAPH]



Use ZOOM-IN or INTERSECT to find the approximate value of x at the point of intersection.

The solution of 0.29 is correct.

EXERCISES

Use a graphing calculator to obtain a complete graph of each function. Then sketch the graph on a sheet of paper.

1. $y = 10^x$
2. $y = 3.5^x$
3. $y = 0.1^x$
4. $y = 0.05^x$
5. $y = \log_4 x$
6. $y = \log_{0.3} x$

Solve each equation graphically. Round solutions to the nearest hundredth.

7. $4^x = 8$
8. $3.2^x = 52.5$
9. $2.1^{x-5} = 9.7$
10. $0.65^{x+3} = 3^{2x-1}$
11. $2^x = x^2$
12. $1.5^x = 2500$
13. $\log_{10} x = 0.23$
14. $\log_2 (x + 2) = \log_{0.5} 2$
15. $\log_9 (x + 4) = \log_2 x$
16. $3^{4x-7} = 4^{2x+3}$

Since 2^x has not been defined when x is irrational, there are "holes" in the graph of $y = 2^x$. We can expand the domain of $y = 2^x$ to include irrational numbers. The domain will then include all real numbers. Draw the graph of $y = 2^x$ with no "holes" in the graph. The set of points on the graph is complete, or *continuous*, and the graph is a smooth curve. You can use the graph of $y = 2^x$ to estimate the value of 2^x when x is any real number.

Use the graph to estimate the value of $2^{\sqrt{3}}$, which is the value of y when $x = \sqrt{3}$.

$1.7 < \sqrt{3} < 1.8$ since $\sqrt{3} \approx 1.732$.

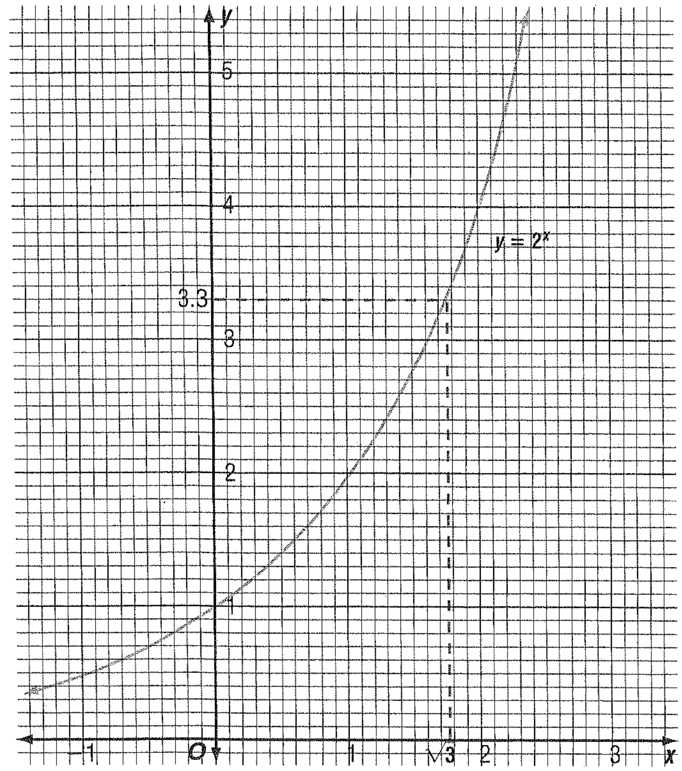
From the graph, the value of y is approximately 3.3.

Use a calculator to check this answer.

Enter: 2 [y^x] 3 [$\sqrt{x}$] [=] 3.321997085

The calculator verifies the estimation from the graph.

$2^{\sqrt{3}} \approx 3.3$



All properties of rational exponents apply to real exponents.

Example 1 **Simplify each expression.**

a. $7^{\sqrt{2}} \cdot 7^{\sqrt{3}}$

$7^{\sqrt{2}} \cdot 7^{\sqrt{3}} = 7^{\sqrt{2}+\sqrt{3}}$ *Product of powers property*

You can use your calculator to verify the result.

b. $\left(8^{\sqrt{3}}\right)^{\sqrt{5}}$

$\left(8^{\sqrt{3}}\right)^{\sqrt{5}} = 8^{\sqrt{3}\cdot\sqrt{5}}$ *Power of a power property*

$= 8^{\sqrt{15}}$

In general, an exponential function can be written in the form $y = ab^x$.

Definition of Exponential Function	**An equation of the form $y = a \cdot b^x$, where $a \neq 0$, $b > 0$, and $b \neq 1$, is called an exponential function with base b.**

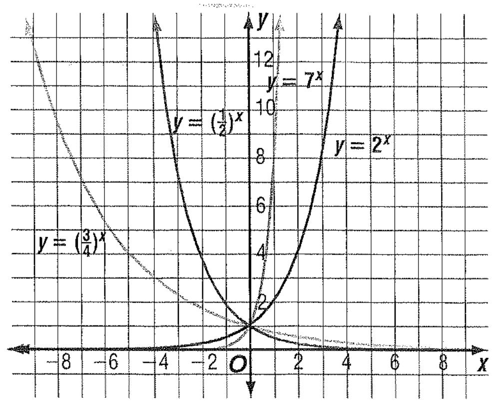
A family of exponential functions is graphed at the right. Notice that $a = 1$ in each case.

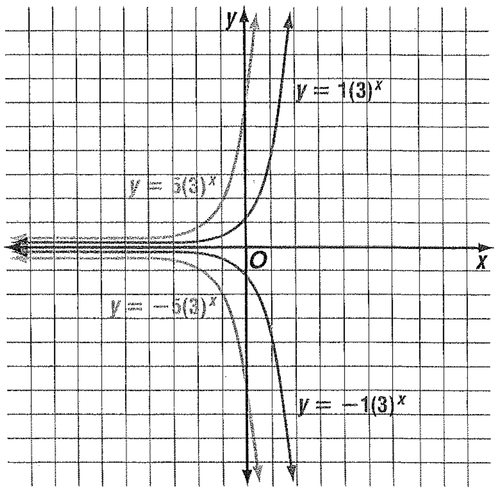
$y = 7^x$ *When $b > 1$, the value of y increases as the value of x increases.*

$y = 2^x$

$y = \left(\frac{3}{4}\right)^x$ *When $0 < b < 1$, the value of y increases as the value of x decreases.*

$y = \left(\frac{1}{2}\right)^x$





Another family of exponential functions is graphed at the left. Study this family of graphs.

$y = 1(3)^x$

$y = 5(3)^x$ *Notice that the y-intercept of each graph equals the value of a.*

$y = -1(3)^x$

$y = -5(3)^x$

The graph of $y = a \cdot b^x$ is the reflection of $y = -a \cdot b^x$ across the x-axis.

Exponential functions are frequently used to model population growth.

Example 2

Real World APPLICATION

Animal Control

During the nineteenth century, rabbits were brought to Australia. Since the rabbits had no natural enemies on that continent, their population increased rapidly. Suppose there were 65,000 rabbits in Australia in 1865 and 2,500,000 rabbits in 1867.

a. Write an exponential equation that could be used to model the rabbit population in Australia. Write the equation in terms of the number of years elapsed since 1865.

b. Estimate the Australian rabbit population in 1872.

a. For the year 1865, the time t equals 0, and the initial population p equals 65,000. Substitute these values in the standard exponential equation to find the value of a.

$p = ab^t$

$65{,}000 = ab^0$ *Replace p with 65,000 and t with 0.*

$65{,}000 = a(1)$ *$b^0 = 1$*

$65{,}000 = a$

For the year 1867, the time t equals 2, and the population p equals 2,500,000. Substitute these values in the standard exponential equation to find the value of b.

$p = ab^t$

$2{,}500{,}000 = 65{,}000b^2$ *Replace p with 2,500,000, a with 65,000, and t with 2.*

$38.46 \approx b^2$ *Division property of equality*

$6.20 \approx b$

The equation that models the rabbit population is $p = 65{,}000(6.20)^t$.

b. For the year 1872, the time t equals 7.

$$p = 65{,}000(6.20)^t$$
$$= 65{,}000(6.20)^7$$
$$\approx 22{,}890{,}495{,}000$$

According to the equation, the rabbit population was about 22,890,495,000 in 1872.

The following property is very useful when solving equations involving exponential functions.

Property of Equality for Exponential Functions	Suppose b is a positive number other than 1. Then $b^{x_1} = b^{x_2}$ if and only if $x_1 = x_2$.

This property also holds for inequalities.

Example 3

a. Solve $64 = 2^{3n+1}$.

$$64 = 2^{3n+1}$$
$$2^6 = 2^{3n+1}$$
$$6 = 3n + 1$$
$$\frac{5}{3} = n$$

The solution is $\frac{5}{3}$.

Check: $64 = 2^{3n+1}$

$$64 \stackrel{?}{=} 2^{3\left(\frac{5}{3}\right)+1}$$
$$64 \stackrel{?}{=} 2^{5+1}$$
$$64 \stackrel{?}{=} 2^6$$
$$64 = 64 \ \checkmark$$

b. Solve $6^{2n-1} > \frac{1}{216}$.

$$6^{2n-1} > \frac{1}{216}$$
$$6^{2n-1} > 6^{-3}$$
$$2n-1 > -3$$
$$n > -1$$

The solution is $n > -1$.

Check: Try $n = 0$.

$$6^{2n-1} > \frac{1}{216}$$
$$6^{2(0)-1} \stackrel{?}{>} \frac{1}{216}$$
$$\frac{1}{6} > \frac{1}{216} \ \checkmark$$

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **You Decide** Todd says that $y = x^2$ is an exponential function. Juan disagrees. Who is correct? Explain.
2. **Describe** the domains and ranges of functions of the form $y = a \cdot b^x$ when:
   a. $a = 1, b > 0$. b. $a > 1, b > 0$. c. $a < 1, b > 0$.
3. **Compare** the graphs of $y = 2^{-x}$, $y = \left(\frac{1}{2}\right)^x$, and $y = (0.5)^x$.
4. **Name** the y-intercepts for the graph of $y = b^x$, where b is any real number.
5. **Explain** why 1 is excluded as a value for a in the property of equality for exponential functions.
6. **Explain** how you could use a calculator to determine which of the following values is the best approximation for the value of x in the equation $3 = 2.3^x$. Give the best approximation.
   a. 1.2 b. 1.3 c. 1.4 d. 1.5

Guided Practice

Use the graph of $y = 2^x$ on page 597 or a calculator to approximate each expression to the nearest tenth.

7. $2^{\sqrt{5}}$

8. $4^{0.6}$

Use the rule of exponents to simplify each expression.

9. $5^{\sqrt{2}} \cdot 5^{3\sqrt{2}}$

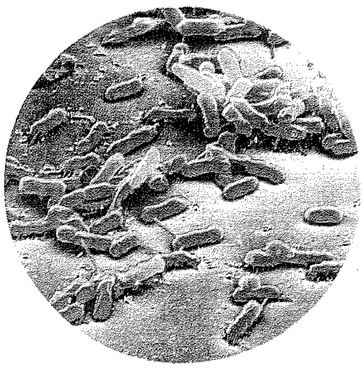
10. $\left(3^{\sqrt{5}}\right)^{\sqrt{5}}$

11. $27^{\sqrt{5}} \div 3^{\sqrt{5}}$

Find the value of a if the graph of an exponential function of the form $y = a \cdot 2^x$ passes through the given point.

12. $A(2, 12)$

13. $B(3, -16)$



Escherichia coli

Solve each equation or inequality. Check your solution.

14. $3^n = 81$

15. $2^{2n} \leq \frac{1}{16}$

16. $\left(\frac{1}{7}\right)^{b-3} = 343$

17. **Biology** Mitosis is a process of cell duplication in which one cell divides into two. The *Escherichia coli* is one of the fastest growing bacteria. It can reproduce itself in 15 minutes. If you begin with one *Escherichia coli* cell, how many cells will there be in one hour?

EXERCISES

Practice

Use the graph of $y = 2^x$ on page 597 or a calculator to evaluate each expression to the nearest tenth.

18. $2^{1.7}$

19. $2^{-0.5}$

20. $2^{\sqrt{2}}$

21. $2^{-1.1}$

22. $16^{0.4}$

23. $8^{-0.3}$

Simplify each expression.

24. $\left(2^{\sqrt{2}}\right)^{\sqrt{8}}$

25. $4^{\sqrt{2}} \cdot 4^{2\sqrt{2}}$

26. $7^{3\sqrt{2}} \div 7^{\sqrt{2}}$

27. $\left(y^{\sqrt{3}}\right)^{\sqrt{12}}$

28. $5^{\sqrt{3}} \cdot 5^{\sqrt{27}}$

29. $64^{\sqrt{7}} \div 2^{\sqrt{7}}$

30. $2\left(3^{\sqrt{2}}\right)\left(3^{-\sqrt{2}}\right)$

31. $\left(a^{\sqrt{5}}\right)^{\sqrt{20}}$

32. $\left(m^{\sqrt{3}} + n^{\sqrt{2}}\right)^2$

Find the value of a if the graph of an exponential function of the form $y = a \cdot 3^x$ passes through the given point.

33. $C(2, 36)$

34. $D(-1, 15)$

35. $E(4, -81)$

36. $F(-2, -2)$

37. $G(5, 27)$

38. $H\left(-3, \frac{1}{9}\right)$

Solve each equation or inequality. Check your solution.

39. $3^{4x} = 3^{3-x}$

40. $5^{n-3} \geq \frac{1}{25}$

41. $\frac{1}{32} = 2^{1-m}$

42. $9^{2p} = 27^{p-1}$

43. $16^n > 8^{n+1}$

44. $\left(\frac{1}{9}\right)^m = 81^{m+4}$

45. $2^x \cdot 4^{x+5} = 4^{2x-1}$

46. $2^{5x} \cdot 16^{1-x} = 4^{x-3}$

47. $25^x = 5^{x^2-15}$

48. On the same coordinate plane, graph $y = 4^x$, $y = -(4)^x$, and $y = \left(\frac{1}{4}\right)^x$. Compare the graphs.

Graphing Calculator

Use a graphing calculator to graph each of the following.

49. $y = 2.1^x$

50. $y = 0.5(2.1)^x$

51. $y = -0.2(2.1)^x$

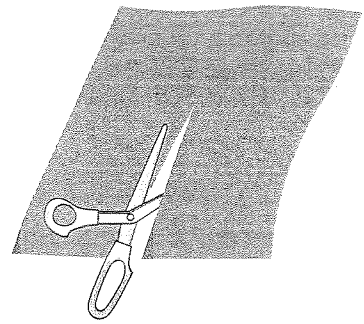
Critical Thinking

52. Using a calculator, approximate to the nearest tenth the value of x when $y = 2$ in the expression $y = 2.3^x$.

Applications and Problem Solving

Real World

53. **Look for a Pattern** A large piece of paper is cut in half, and one of the resulting pieces is placed on top of the other. Then the pieces in the stack are cut in half and placed on top of each other. Suppose this procedure is repeated several times.



a. How many pieces will be in the stack after the first cut? after the second cut? after the third cut? after the fourth cut?

b. Use the pattern in step a to write an equation for the number of pieces in the stack after x cuts.

c. The thickness of ordinary paper is about 0.003 inch. Write an equation for the thickness of the stack of paper after x cuts.

d. How thick will the stack of paper be after 30 cuts?

54. **Animal Behavior** Studies show an animal will defend an area in square yards that is directly proportional to the 1.31 power of the animal's weight in pounds.

a. If a 45-pound beaver will defend 170 square yards, write an equation for the area a defended by a beaver weighing w pounds.

b. Thousands of years ago, some beavers grew to be 11 feet long and weighed 430 pounds. Use your equation in step a to determine the area defended by these animals.

55. **Atmospheric Pressure** Atmospheric pressure decreases at higher altitudes. In the equation $P = 14.7(10)^{-0.02h}$, P represents the atmospheric pressure in pounds per square inch, and h represents the altitude above sea level in miles.

a. The elevation of Boston, Massachusetts, is about sea level. Find the atmospheric pressure in Boston.

b. The elevation of Denver, Colorado, is about 1 mile. Find the atmospheric pressure in Denver.

c. The highest mountain in the world is Mt. Everest in Nepal, which is about 5.5 miles above sea level. Find the atmospheric pressure at the top of this mountain.

d. About 6 million years ago, the Mediterranean Sea dried up. The bottom of the valley formed by this dry sea was about 1.9 miles below sea level. Find the atmospheric pressure at the bottom of this valley.

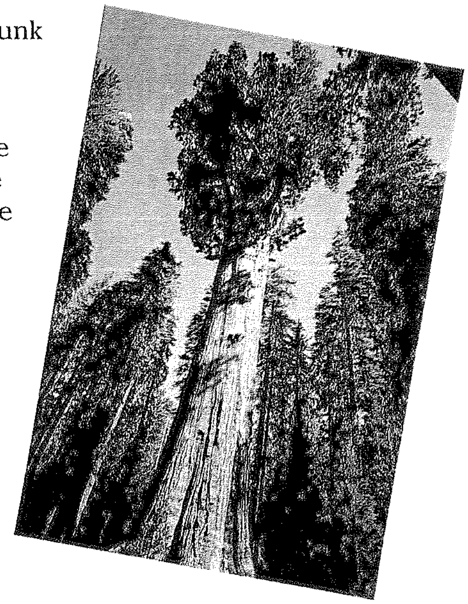
e. Graph the equation for the atmospheric pressure. Explain the meaning of the points on the graph that have negative values for h.

56. **City Planning** The school board in Orlando, Florida, needs to project the population growth for the remainder of the century to plan for the construction of new schools. One of the formulas the board can use is $P = 164{,}693(2.7)^{0.007t}$, where t represents the number of years since 1990 and 164,693 was the population of Orlando according to the 1990 census. Use a calculator to determine how large the population will be in the year 2000.

57. **Forestry** The diameter of the base of a tree trunk in centimeters varies directly with the $\frac{3}{2}$ power of its height in meters.

   a. A young sequoia tree is 6 meters tall, and the diameter of its base is 19.1 centimeters. Use this information to write an inequality for the diameter d of the base of a sequoia tree if its height is at least h meters high.

   b. One of the oldest living things on Earth is the General Sherman Tree in Sequoia National Park in California. This sequoia is between 2200 and 2500 years old. If it is at least 80 meters high, find the diameter at its base.



Mixed Review

58. Solve $\frac{6}{a-7} = \frac{a-49}{a^2-7a} + \frac{1}{a}$. (Lesson 9–5)

59. State the number of positive real zeros, negative real zeros, and imaginary zeros for $f(x) = -x^4 - x^2 - x - 1$. (Lesson 8–4)

60. **SAT Practice** **Grid-in** A ball bounces up $\frac{3}{4}$ of the distance it falls when dropped, and on each bounce thereafter, it bounces $\frac{3}{4}$ of the previous height. If it is dropped from a height of 64 feet, how many feet will it have traveled when it hits the ground the fourth time?

61. **Cartography** Edison is located at (9, 3) on the road map. Kettering is located at (12, 5) on the same map. Each side of a grid on the map represents 10 miles. Use the distance formula to approximate the distance between Edison and Kettering. (Lesson 7–1)

62. Solve $x^2 + 14x - 12 = 0$ by completing the square. (Lesson 6–3)

63. Find the multiplicative inverse of $3 - 4i$. (Lesson 5–10)

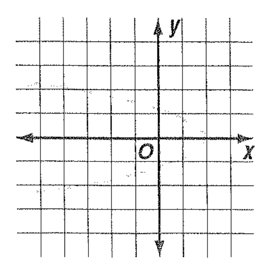
64. Simplify $\frac{(3+\sqrt{5})}{(1+\sqrt{2})}$. (Lesson 5–6)

65. If $A = \begin{bmatrix} -2 & 3 \\ 1 & 10 \\ 0 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 3 \\ 1 & 4 \end{bmatrix}$, find AB. (Lesson 4–[illegible])

66. Solve the system of equations. (Lesson 3–7)

   $r + s + t = 15$
   $r + t = 12$
   $s + t = 10$

67. Use the vertical line test to determine whether the relation graphed at the right is a function. (Lesson 2–1)



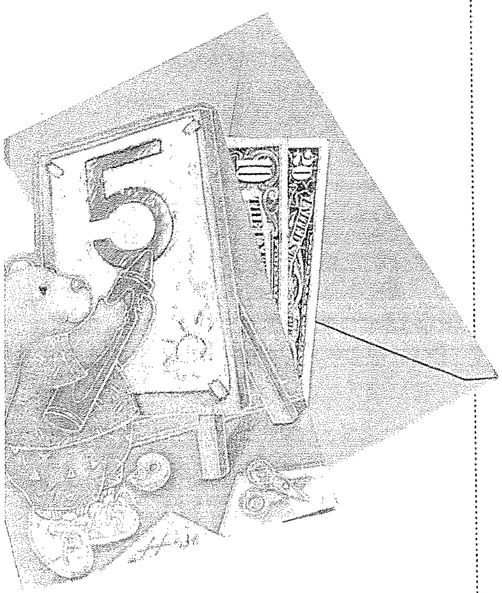
68. State the property illustrated by the following equation. $4 + (a + r) = (4 + a) + r$ (Lesson [illegible])

For Extra Practice see page 899

10–1B Graphing Technology
Curve Fitting with Real-World Data

An Extension of Lesson 10–1

As we have seen in earlier chapters, we are often confronted with data for which we need to find an equation that best describes the information.



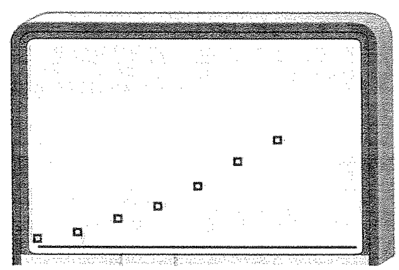
Example 1 **In 1985, Kayla received \$30.00 from her grandparents for her fifth birthday. Her mother deposited it into a bank account for her. Both Kayla and her mother forgot about the money and made no further deposits or withdrawals. The table to the right shows the account balance for several years.**

Elapsed Time (years)	Balance
0	\$30.00
5	\$41.10
10	\$56.31
15	\$77.16
20	\$105.71
25	\$144.83
30	\$198.43

a. Use a graphing calculator to enter the data and draw a scatter plot that shows how the account balance is related to time.

b. If Kayla discovers the account with the birthday money on her 50th birthday, how much will she have in the account?

a. Enter the elapsed time data into the L1 list on the STAT menu and enter the balance data into the L2 list. Be sure to clear the Y= list. Press [2nd] [STAT PLOT] and select plot 1. Make sure that plot 1 is on, scatter plot is chosen, the x list is L1, and the y list is L2. Use the viewing window [0, 45] with a scale factor of 5 by [0, 300] with a scale factor of 20. Press [GRAPH].



LOOK BACK

You can refer to Lessons 2-5B and 8-3B for information on entering data and graphing scatter plots on the graphing calculator.

We see from the data that the equation that best fits the data must be a curve. This means the equation is probably polynomial or exponential. Let's try an exponential model. To determine the exponential equation that best fits the data, use the exponential regression feature of the calculator.

Enter: [STAT] [▶] 0 [2nd] L1 [,] [2nd] L2 [ENTER]

The equation is $y = 29.99908551(1.065001351)^x$.

The calculator also reports an r-value of 0.999999998. Recall that this number is a correlation coefficient that indicates how well the equation fits the data. A perfect fit would be $r = 1$. Therefore, we can conclude that this equation is indeed a good fit for the data.

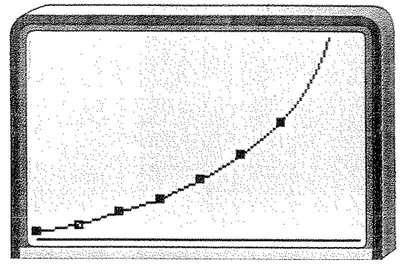
(continued on the next page)

To check this equation visually, overlap the graph of the equation with the scatter plot.

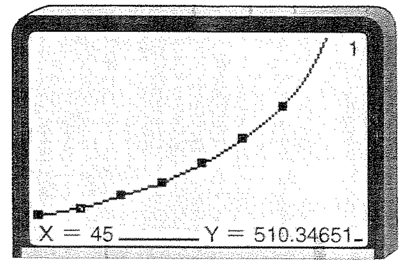
Enter: [Y=] [VARS] 5

[▶] [▶] 7

[GRAPH]

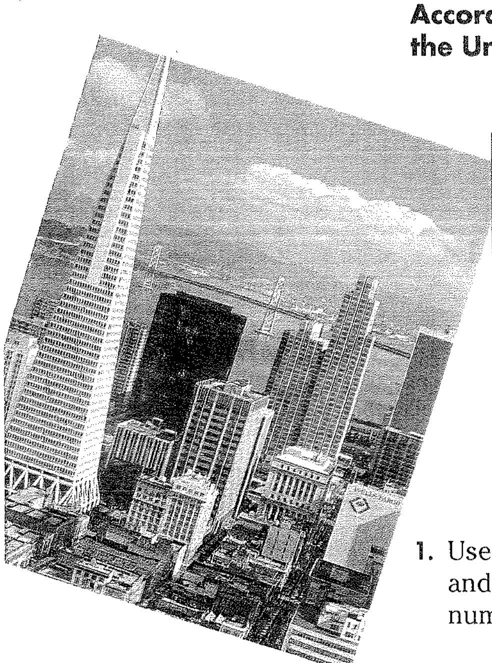


b. On Kayla's 50th birthday, the money will have been in the account for 50 − 5 or 45 years. From the graphics screen, enter [2nd] [CALC] 1 45 [ENTER]. (Be sure your viewing window is large enough to include $x = 45$.) The calculator returns a y-value of 510.34651. Kayla will have $510.35 in the account when she is 50 years old.



EXERCISES

According to the World Almanac, the population per square mile in the United States has changed dramatically over a period of years.



Year	People per Square Mile	Year	People per Square Mile
1790	4.5	1900	21.5
1800	6.1	1910	26.0
1810	4.3	1920	29.9
1820	5.5	1930	34.7
1830	7.4	1940	37.2
1840	9.8	1950	42.6
1850	7.9	1960	50.6
1860	10.6	1970	57.5
1870	10.9	1980	64.0
1880	14.2	1990	70.3
1890	17.8		

1. Use a graphing calculator to draw a scatter plot of the data. Then calculate and graph the curve of best fit that shows how the year is related to the number of people per square mile. Use ExpReg for this example.

2. Write the equation of best fit. Write a sentence that describes the fit of the graph to the data.

3. Based on the graph, estimate the population density for 2000. Check this using the CALC value.

4. Do you think there are any other types of equations that would be good models for this data? Why or why not?

5. **History** What event occurred between 1800 and 1810 that would account for the sudden big decrease in population per square mile?

10-2 Logarithms and Logarithmic Functions

What YOU'LL LEARN

- To write exponential equations in logarithmic form and vice versa,
- to evaluate logarithmic expressions, and
- to solve equations and inequalities involving logarithmic functions.

Why IT'S IMPORTANT

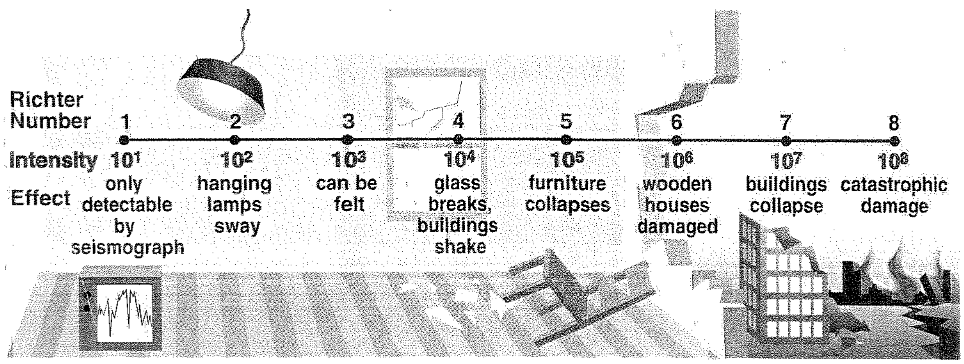
You can use logarithmic functions to solve problems involving chemistry and geology.

Real World APPLICATION

Geology

Logarithms are exponents. They were once used to simplify calculations, but the advent of calculators and computers caused calculation with logarithms to be used less and less.

An example of logarithms at work is the Richter scale. The Richter scale is used to measure the strength of an earthquake. It is a logarithmic scale based on the powers of ten. The table below gives the effects of earthquakes of various intensities.



The 1906 San Francisco earthquake measured 8.3 on the Richter scale. The Loma Prieta earthquake that interrupted the 1989 World Series in San Francisco measured 7.1. *We will compare the magnitudes of these two famous earthquakes in Example 2.*

The tables below show two related exponential equations. You will recognize the equation in the table on the left as an exponential function.

Given the exponent, x, compute the power of 2 as y.

x	$2^x = y$	y
-1	$2^{-1} = y$	?
2	$2^2 = y$	?
3	$2^3 = y$	?
6	$2^6 = y$	?

Given x as the power of 2, compute the exponent, y.

y	$2^y = x$	x
?	$2^y = \frac{1}{2}$	$\frac{1}{2}$
?	$2^y = 4$	4
?	$2^y = 8$	8
?	$2^y = 64$	64

In the relation shown in the table on the right, $2^y = x$, the exponent y is called the **logarithm**, base 2, of x. This relation is written $\log_2 x = y$ and is read "the log base 2 of x is equal to y." The logarithm corresponds to the exponent. Study the diagram below.

Exponential Equation	**Logarithmic Equation**
$n = b^p$	$p = \log_b n$

exponent or logarithm

base

number

Definition of Logarithm

Suppose $b > 0$ and $b \neq 1$. For $n > 0$, there is a number p such that $\log_b n = p$ if and only if $b^p = n$.

The chart below shows some equivalent exponential and logarithmic equations.

Exponential Equation	Logarithmic Equation
$5^2 = 25$	$\log_5 25 = 2$
$10^5 = 100{,}000$	$\log_{10} 100{,}000 = 5$
$8^0 = 1$	$\log_8 1 = 0$
$2^{-4} = \frac{1}{16}$	$\log_2 \frac{1}{16} = -4$
$9^{\frac{1}{2}} = 3$	$\log_9 3 = \frac{1}{2}$

You can find the value of a variable in a logarithmic equation $\log_b x = y$ when values for two of the variables are known.

Example 1 **Solve each equation.**

a. $\log_9 x = \frac{3}{2}$

$$\log_9 x = \frac{3}{2}$$

$$9^{\frac{3}{2}} = x \quad \textit{Definition of logarithm}$$

$$(3^2)^{\frac{3}{2}} = x$$

$$3^3 = x \quad \textit{Power of a power}$$

$$27 = x$$

b. $\log_4 256 = y$

$$\log_4 256 = y$$

$$4^y = 256 \quad \textit{Definition of logarithm}$$

$$(2^2)^y = 2^8$$

$$2^{2y} = 2^8 \quad \textit{Power of a power}$$

$$2y = 8 \quad \textit{Property of equality for exponential functions}$$

$$y = 4$$

Example 2 **Refer to the application at the beginning of the lesson. Compare the magnitude of the 1906 quake to the 1989 quake.**

Real World APPLICATION

Geology

Let x represent the measure of the 1906 quake and let y represent the measure of the 1989 quake.

Quake	Richter Value as Exponents	Richter Value as Logarithms
1906	$10^{8.3} = x$	$8.3 = \log_{10} x$
1989	$10^{7.1} = y$	$7.1 = \log_{10} y$

Use the Richter values as exponents to find the ratio of the magnitude of the 1906 quake to the 1989 quake.

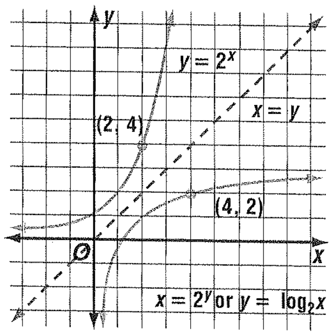
$$\frac{x}{y} = \frac{10^{8.3}}{10^{7.1}}$$

$$= 10^{1.2} \quad \textit{Division of powers}$$

$$\approx 15.8$$

Therefore, the intensity of the 1906 earthquake was approximately 16 times greater than that of the 1989 quake.

Let's look at the graphs of an exponential function and its corresponding logarithmic function. In fact, you can use a table of values for $y = 2^x$ to make a table of values for $x = 2^y$.



$y = 2^x$

x	y
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

$x = 2^y$ or $y = \log_2 x$

x	y
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

The x and y values are reversed.

For every point (a, b) on the graph of $y = 2^x$, there is a point on the graph of $y = \log_2 x$ with coordinates (b, a).

Notice that the graphs are reflections of each other over the line $y = x$. The fact that x and y switch places is apparent in the domains and ranges.

	For $y = 2^x$	For $x = 2^y$ or $y = \log_2 x$
Domain	all real numbers	positive real numbers
Range	positive real numbers	all real numbers

The relations are inverses of each other. Using the vertical line test, you can see that no vertical line can intersect the graph of $y = \log_2 x$ in more than one place, so $y = \log_2 x$ is a function, called a **logarithmic function**.

Definition of Logarithmic Function	**An equation of the form $y = \log_b x$, where $b > 0$ and $b \neq 1$, is called a logarithmic function.**

LOOK BACK

You can refer to Lesson 8-7 for information on composition of functions.

Since the exponential function $y = b^x$ and the logarithmic function $y = \log_b x$ are inverses of each other, their composites are the identity function. Let $f(x) = \log_b x$ and $g(x) = b^x$. For $f(x)$ and $g(x)$ to be inverses, it must be true that $f(g(x)) = x$ and $g(f(x)) = x$.

$$f(g(x)) = x \qquad g(f(x)) = x$$
$$f(b^x) = x \qquad g(\log_b x) = x$$
$$\log_b b^x = x \qquad b^{\log_b x} = x$$

Example 3 **Evaluate each expression.**

a. $\log_5 5^3$

$\log_5 5^3 = 3$ *$\log_b b^x = x$*

b. $6^{\log_6(2x+5)}$

$6^{\log_6(2x+5)} = 2x + 5$ *$b^{\log_b x} = x$*

A property similar to the property for exponential functions applies to the logarithmic functions.

Property of Equality for Logarithmic Functions

Suppose $b > 0$ and $b \neq 1$. Then $\log_b x_1 = \log_b x_2$ if and only if $x_1 = x_2$.

This property also holds for inequalities.

Example 4 Solve each equation or inequality.

a. $\log_{10}(t^2 - 6) = \log_{10} t$

$$\log_{10}(t^2 - 6) = \log_{10} t$$
$$t^2 - 6 = t$$
$$t^2 - t - 6 = 0$$
$$(t - 3)(t + 2) = 0$$
$$t = 3 \quad \text{or} \quad t = -2$$

Check: Eliminate -2, because $\log_b x$ is defined only if $x > 0$. Thus, the solution is 3.

b. $\log_3(3x - 5) \geq \log_3(x + 7)$

$$\log_3(3x - 5) \geq \log_3(x + 7)$$
$$3x - 5 \geq x + 7$$
$$2x \geq 12$$
$$x \geq 6$$

Check: Try $x = 8$.

$$\log_3(3x - 5) \geq \log_3(x + 7)$$
$$\log_3[3(8) - 5] \geq \log_3(8 + 7)$$
$$\log_3 19 \geq \log_3 15$$

Since $\log_b x$ increases as x increases and $\log_b x > 0$, $\log_3 19 \geq \log_3 15$.

Test other values to verify the reasonableness of your solution.

CHECK FOR UNDERSTANDING

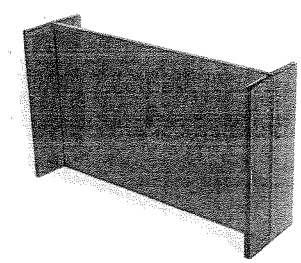
Communicating Mathematics

Study the lesson. Then complete the following.

1. **Give** another name for the exponent y in the equation $2^y = x$.
2. **Write** an example of a logarithmic function.
3. **Describe** the domain of $y = 2^x$.
4. **Describe** the range of $y = \log_2 x$.

MODELING MATHEMATICS

5. **Draw** a graph of $y = 3^x$. Place a geomirror along the line represented by the equation $y = x$. Use the reflection seen in the geomirror to draw the graph of $y = \log_3 x$.



Guided Practice

Write each equation in logarithmic form.

6. $3^3 = 27$
7. $2^5 = 32$
8. $4^{-2} = \frac{1}{16}$

Write each equation in exponential form.

9. $\log_5 125 = 3$
10. $\log_8 4 = \frac{2}{3}$
11. $\log_{10} 0.001 = -3$

Evaluate each expression.

12. $\log_3 \frac{1}{27}$
13. $\log_{16} 4$
14. $5^{\log_5 25}$

Solve each equation or inequality. Check your solution.

15. $\log_7 y = -2$
16. $\log_b 64 = 3$
17. $\log_{\frac{1}{3}} 27 = x$
18. $\log_5(2x - 3) > \log_5(x + 2)$
19. $\log_{10}(x^2 + 36) = \log_{10} 100$
20. $\log_3 3^{(2x - 1)} = 7$
21. **Geology** How much stronger is an earthquake with a Richter scale rating of 7 than an aftershock with a rating of 4?

EXERCISES

Practice

Evaluate each expression.

22. $\log_{10} 1000$
23. $\log_5 25$
24. $\log_{14} 196$
25. $\log_3 \frac{1}{81}$
26. $\log_2 \frac{1}{128}$
27. $\log_{36} 6$
28. $\log_8 8^4$
29. $3^{\log_3 243}$
30. $7^{\log_7 (x+3)}$

Solve each equation or inequality. Check your solution.

31. $\log_2 x = 5$
32. $\log_3 27 = y$
33. $\log_b 9 = 2$
34. $\log_5 \sqrt{5} = y$
35. $\log_{25} x = \frac{3}{2}$
36. $\log_b 0.01 = -2$
37. $\log_{\frac{1}{10}} x = -3$
38. $\log_{3x} 125 = 3$
39. $\log_{x+2} 16 = 2$
40. $\log_8 (3x - 1) = \log_8 (2x^2)$
41. $\log_2 (4x + 10) - \log_2 (x + 1) < 3$
42. $\log_{10} (x^2 + 16) = \log_{10} 80$
43. $4^{\log_4 (x-1)} = -0.5$
44. $\log_2 2^{(3x+2)} = 14$
45. $3^{\log_3 10} = x$
46. $\log_{10} (\log_8 8) = x$
47. $\log_4 (\log_2 16) = y$

Graph each pair of equations on the same axes.

48. $y = \log_5 x$ and $y = 5^x$
49. $y = \log_{\frac{1}{3}} x$ and $y = \left(\frac{1}{3}\right)^x$
50. Study the graphs in Exercises 48 and 49. What is the relationship of each pair of graphs?
51. Graph $y = \log_{10} x$, $y = \log_5 x$, $y = \log_{\frac{1}{3}} x$ and $y = \log_{\frac{1}{2}} x$ on the same set of axes. Assume that the parent graph is the graph of $y = \log_{10} x$. Describe this family of graphs in terms of the parent graph.

Show that each statement is true.

52. $\log_4 4 + \log_4 16 = \log_4 64$
53. $\log_4 16 = 2 \log_4 4$
54. $\log_2 8 \cdot \log_8 2 = 1$
55. $\log_{10} [\log_3 (\log_4 64)] = 0$

Critical Thinking

56. If $x = \log_{10} 2460$, the value of x is between two consecutive integers. Name these integers and explain how you determined the values.

Applications and Problem Solving

Real World

57. **Geology** The Seattle quake of April 29, 1965, measured 7.0 on the Richter scale. The San Francisco quake of 1989 measured 7.1. How many times more severe was the San Francisco quake than the Seattle quake?

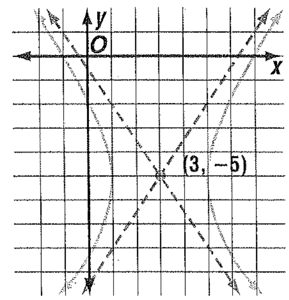
58. **Chemistry** The pH of a solution is a measure of its acidity and is written as a logarithm to the base 10. A low pH indicates an acidic solution, and a high pH indicates a basic solution. Neutral water has a pH of 7. Acid rain has a pH of 4.2. How many more times acidic is the acid rain than neutral water?

Mixed Review

59. Simplify $11^{\sqrt{5}} \cdot 11^{\sqrt{45}}$. (Lesson 10–1)
60. **Physics** The volume of any gas varies inversely with its pressure as long as the temperature remains constant. If a helium-filled balloon has a volume of 3.4 cubic decimeters at a pressure of 120 kilopascals, what is its volume at 101.3 kilopascals? (Lesson 9–2)

61. If $f = \{(2, 1), (-1, 6), (3, 2)\}$ and $g = \{(2, 2), (6, -1), (1, 5)\}$, express $f \circ g$ and $g \circ f$, if they exist, as sets of ordered pairs. (Lesson 8–7)

62. Write the equation of the hyperbola in the graph at the right. (Lesson 7–5)



63. Write a quadratic equation that has roots 6 and −6. (Lesson 6–5)

64. Solve $x^2 + 6x = -9$ by factoring. (Lesson 6–2)

65. **Physics** Find the time t (in seconds) that it takes for a free-falling object to fall a distance s of 200 feet. Use the formula $t = \frac{1}{4}\sqrt{s}$. (Lesson 5–6)

66. **ACT Practice** If $\frac{t^2 - 16}{4 + t} = 12$, then $t = ?$

A −8 B 4 C 8 D 16 E 24

67. Solve the system of inequalities by graphing. (Lesson 3–4)

$y - x \leq 3$

$y \geq x - 2$

68. Find the y-intercept and the x-intercept of the graph of $3x + 5y = 30$. (Lesson 2–3)

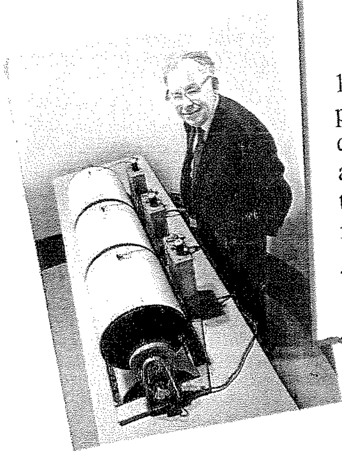
For Extra Practice, see page 899.

69. Solve $8x + 5 < 7x - 3$. (Lesson 1–6)

Mathematics and SOCIETY

Earthquake!

The excerpt below appeared in an article in *Science News* on October 15, 1994.



When Charles Richter invented the concept of seismic magnitude, he made it easy to compare earthquakes. Anyone who can count to 10 will recognize that a magnitude 7.0 shock packs a bigger punch than a 6.0 quake. But the question "How much bigger?" is not so easily answered. In the original definition of magnitude, a 1-point increase meant that peak waves recorded by a Wood-Anderson seismometer jumped by a factor of 10. . . . Seismologists themselves compare earthquakes using seismic moments. . . . But moments are expressed in unwieldy numbers, such as 2×10^{27} newton meters—clearly not an appealing figure for the public. Pat Jorgenson, a USGS spokeswoman in Menlo Park, Calif., says she would prefer to discuss quakes in terms of something people can comprehend. . . . In that vein, a magnitude 1.0 earthquake would equal roughly 6 ounces of TNT. For a magnitude 5.0, think of 1000 tons of TNT. . . . The largest recorded earthquake, of moment magnitude 9.5, in Chile in 1960, equaled about 3 billion tons of TNT. ■

1. Between one-unit intervals on the Richter scale, the magnitude of earthquake strength increases by a factor of 10. What is the mathematical name for this type of scale?
2. The largest earthquakes can be several billion times stronger than the smallest. What problems might arise if you tried to design a simpler, more understandable measuring scale?
3. Choose or invent your own unit of measurement to measure quakes.

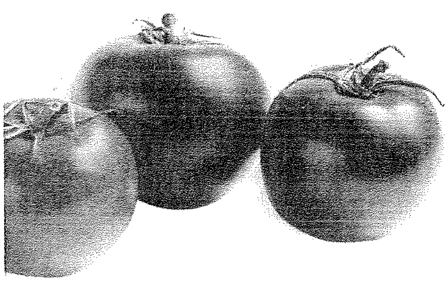
10-3 Properties of Logarithms

What YOU'LL LEARN

- To simplify and evaluate expressions using properties of logarithms, and
- to solve equations involving logarithms.

Why IT'S IMPORTANT

You can use logarithms to solve problems involving biology and medicine.



CONNECTION

Chemistry

The levels of the acidity of the foods we eat is a concern to some health-conscious consumers. Most of the foods that we consume tend to be more acidic than basic. The pH scale measures acidity; a low pH indicates an acidic solution, and a high pH indicates a basic solution. It is another example of a logarithmic scale based on powers of ten.

pH Levels of Product	Common Products pH Level
Lemon juice	2.1
Sauerkraut	3.5
Tomatoes	4.2
Black coffee	5.0
Milk	6.4
Pure water	7.0
Eggs	7.8
Milk of magnesia	10.0

Black coffee has a pH of 5, while neutral water has a pH of 7. Black coffee is one hundred times more acidic than neutral water, since $10^{7-5} = 10^2$ or 100.

Since logarithms are exponents, the properties of logarithms can be derived from the properties of exponents that you already know. Recall that the product of powers is found by adding exponents.

$$\log_2 (8 \cdot 32) = \log_2 (2^3 \cdot 2^5) = \log_2 (2^{3+5}) = 3 + 5$$

$$\log_2 8 + \log_2 32 = \log_2 2^3 + \log_2 2^5 = 3 + 5$$

So, $\log_2 (8 \cdot 32) = \log_2 8 + \log_2 32$. This example indicates that the logarithm of a product is the sum of the logarithms of the factors.

Product Property of Logarithms	For all positive numbers m, n, and b, where $b \neq 1$, $\log_b mn = \log_b m + \log_b n$.

To prove this property, let $b^x = m$ and $b^y = n$.

Then $\log_b m = x$ and $\log_b n = y$.

$$b^x b^y = mn$$

$$b^{x+y} = mn \quad \text{Multiplying powers}$$

$$\log_b b^{x+y} = \log_b mn \quad \text{Property of equality for logarithmic functions}$$

$$x + y = \log_b mn \quad \text{Definition of inverse functions}$$

$$\log_b m + \log_b n = \log_b mn \quad \text{Replace } x \text{ with } \log_b m \text{ and } y \text{ with } \log_b n.$$

Example 1 **Given $\log_3 5 \approx 1.4650$, find each logarithm.**

a. $\log_3 45$

$$\log_3 45 = \log_3 (3^2 \cdot 5) = \log_3 3^2 + \log_3 5 \approx 2 + 1.4650 \text{ or } 3.4650$$

b. $\log_3 25$

$$\log_3 25 = \log_3 (5 \cdot 5) = \log_3 5 + \log_3 5 \approx 1.4650 + 1.4650 \text{ or } 2.9300$$

To find the quotient of powers, you subtract the exponents.

$$\log_2 \frac{32}{8} = \log_2 \frac{2^5}{2^3} = \log_2 (2^{5-3}) = 5 - 3 = 2$$

$$\log_2 32 - \log_2 8 = \log_2 2^5 - \log_2 2^3 = 5 - 3 = 2$$

So, $\log_2 \frac{32}{8} = \log_2 32 - \log_2 8$. This example indicates that the logarithm of a quotient can be found by subtracting the logarithm of the denominator from the logarithm of the numerator.

Quotient Property of Logarithms	**For all positive numbers m, n, and b, where $b \neq 1$, $\log_b \frac{m}{n} = \log_b m - \log_b n$.**

To prove this property, let $b^x = m$ and $b^y = n$.

Then $\log_b m = x$ and $\log_b n = y$.

$$\frac{b^x}{b^y} = \frac{m}{n}$$

$$b^{x-y} = \frac{m}{n} \quad \textit{Division of powers}$$

$$\log_b b^{x-y} = \log_b \frac{m}{n} \quad \textit{Property of equality for logarithmic functions}$$

$$x - y = \log_b \frac{m}{n} \quad \textit{Definition of inverse functions}$$

$$\log_b m - \log_b n = \log_b \frac{m}{n} \quad \textit{Replace x with } \log_b m \textit{ and y with } \log_b n.$$

Example 2

Given $\log_4 5 \approx 1.1610$ and $\log_4 15 \approx 1.9534$, find each logarithm.

a. $\log_4 \frac{5}{16}$

$$\log_4 \frac{5}{16} = \log_4 \frac{5}{4^2} = \log_4 5 - \log_4 4^2 \approx 1.1610 - 2 \text{ or } -0.8390$$

b. $\log_4 3$

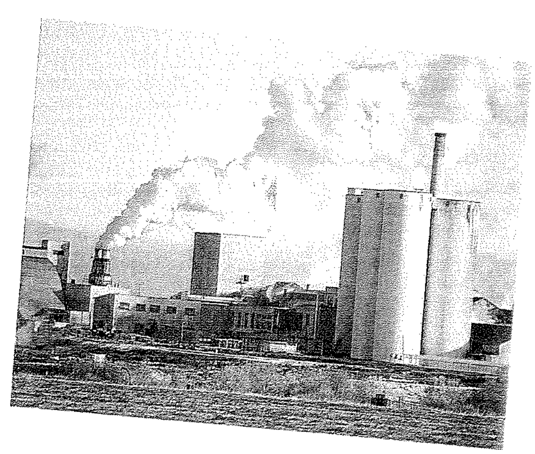
$$\log_4 3 = \log_4 \frac{15}{5} = \log_4 15 - \log_4 5 \approx 1.9534 - 1.1610 \text{ or } 0.7924$$

Example 3

CONNECTION

Chemistry

The pH of a substance is the concentration of hydrogen ions, $[H^+]$, measured in moles of hydrogen per liter of substance. It is given by the formula $\text{pH} = \log_{10} \frac{1}{[H^+]}$. Find the amount of hydrogen in a liter of acid rain that has a pH of 4.2.



Explore Read the problem. You know the formula for finding pH and the pH of the rain. You want to find the amount of hydrogen in a liter of this rain.

Plan Write the equation. Then, solve for $[H+]$.

Solve $\quad pH = \log_{10} \frac{1}{[H+]}$ *Replace pH with 4.2.*

$4.2 = \log_{10} \frac{1}{[H+]}$

$4.2 = \log_{10} 1 - \log_{10} [H+]$ *Quotient property of logarithms*

$4.2 = 0 - \log_{10} [H+]$ *$\log_{10} 1 = 0$*

$4.2 = -\log_{10} [H+]$

$-4.2 = \log_{10} [H+]$

$10^{-4.2} = [H+]$ *Definition of logarithm*

There are $10^{-4.2}$, or about 0.000063, moles of hydrogen in a liter of this rain.

Examine Check the solution.

$4.2 \stackrel{?}{=} \log_{10} \frac{1}{10^{-4.2}}$

$4.2 \stackrel{?}{=} \log_{10} 1 - \log_{10} 10^{-4.2}$

$4.2 \stackrel{?}{=} 0 - (-4.2)$

$4.2 = 4.2$ ✓

The power of a power is found by multiplying the two exponents.

$$\log_3 9^4 = \log_3 (3^2)^4 = \log_3 3^{2 \cdot 4} = 2 \cdot 4$$

$$4 \log_3 9 = (\log_3 9) \cdot 4 = (\log_3 3^2) \cdot 4 = 2 \cdot 4$$

So, $\log_3 9^4 = 4 \log_3 9$. This example indicates that the logarithm of a power is the product of the logarithm and the exponent.

Power Property of Logarithms	**For any real number p and positive numbers m and b, where $b \neq 1$, $\log_b m^p = p \cdot \log_b m$.**

You will prove this property in Exercise 3.

Example 4 illustrates how to use the properties of logarithms to solve equations involving logarithms.

Example 4 **Solve each equation.**

a. $\mathbf{2 \log_3 6 - \frac{1}{4} \log_3 16 = \log_3 x}$

$2 \log_3 6 - \frac{1}{4} \log_3 16 = \log_3 x$

$\log_3 6^2 - \log_3 16^{\frac{1}{4}} = \log_3 x$ *Power property of logarithms*

$\log_3 36 - \log_3 2 = \log_3 x$

$\log_3 \frac{36}{2} = \log_3 x$ *Quotient property of logarithms*

$\log_3 18 = \log_3 x$

$18 = x$ *Property of equality for logarithmic functions*

The solution is 18.

b. $\log_{10} z + \log_{10}(z + 3) = 1$

$$\log_{10} z + \log_{10}(z + 3) = 1$$

$$\log_{10} z(z + 3) = 1 \quad \text{Product property of logarithms}$$

$$z(z + 3) = 10^1 \quad \text{Definition of logarithm}$$

$$z^2 + 3z - 10 = 0$$

$$(z + 5)(z - 2) = 0$$

$$z + 5 = 0 \quad \text{or} \quad z - 2 = 0 \quad \text{Zero product property}$$

$$z = -5 \qquad\qquad z = 2$$

Check:

$$\log_{10} z + \log_{10}(z + 3) = 1$$

$$\log_{10}(-5) + \log_{10}(-5 + 3) \stackrel{?}{=} 1$$

$$\log_{10}(-5) + \log_{10}(-2) \stackrel{?}{=} 1$$

Both $\log_{10}(-5)$ and $\log_{10}(-2)$ are undefined so -5 is not a solution.

$$\log_{10} z + \log_{10}(z + 3) = 1$$

$$\log_{10} 2 + \log_{10}(2 + 3) \stackrel{?}{=} 1$$

$$\log_{10} 2 + \log_{10} 5 \stackrel{?}{=} 1$$

$$\log_{10}(2 \cdot 5) \stackrel{?}{=} 1$$

$$\log_{10} 10 \stackrel{?}{=} 1$$

$$1 = 1 \ \checkmark$$

The only solution is 2.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **Describe** what the pH of a solution indicates and how it is related to logarithms.
2. **Name** the properties that serve as guidelines to derive the properties of logarithms.
3. **Prove** the power property of logarithms.

MATH JOURNAL

4. **Assess Yourself** List three properties of exponents and their related properties of logarithms. Do you find the properties of exponents or the properties of logarithms easier to understand? Do you have difficulty understanding or applying any of these properties?

Guided Practice

Express each logarithm as the sum or difference of simpler logarithmic expressions.

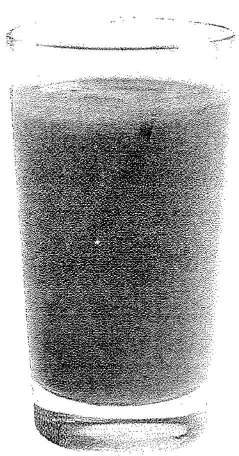
5. $\log_4 x^2y$
6. $\log_3 (xy)^3$
7. $\log_5 \frac{ac}{b}$
8. $\log_2 r^{\frac{1}{3}}t$

Use $\log_2 3 \approx 1.585$ and $\log_2 7 \approx 2.807$ to evaluate each expression.

9. $\log_2 \frac{7}{3}$
10. $\log_2 36$
11. $\log_2 0.75$

Solve each equation.

12. $\log_2 5 + \log_2 x = \log_2 15$
13. $\log_5 16 - \log_5 2t = \log_5 2$
14. $\log_{10} 7 + \log_{10}(n - 2) = \log_{10} 6n$
15. $\log_2 (y + 2) - 1 = \log_2 (y - 2)$
16. **Chemistry** If the pH level of tomato juice is 4.1 and pH level of baking soda is 8.5, how much more acidic is tomato juice than baking soda?



EXERCISES

Practice

Use $\log_3 2 \approx 0.6310$ and $\log_3 7 \approx 1.7712$ to evaluate each expression.

17. $\log_3 4$
18. $\log_3 49$
19. $\log_3 \frac{7}{2}$
20. $\log_3 18$
21. $\log_3 \frac{2}{3}$
22. $\log_3 54$
23. $\log_3 108$
24. $\log_3 \frac{18}{49}$
25. $\log_3 \frac{7}{9}$

Solve each equation.

26. $\log_3 2 + \log_3 7 = \log_3 x$
27. $\log_5 42 - \log_5 6 = \log_5 k$
28. $\log_5 m = \frac{1}{3} \log_5 125$
29. $\log_{10} y = \frac{1}{4} \log_{10} 16 + \frac{1}{2} \log_{10} 49$
30. $\log_9 5 + \log_9 (n + 1) = \log_9 6n$
31. $3 \log_5 x - \log_5 4 = \log_5 16$
32. $2 \log_3 y + \log_3 0.1 = \log_3 5 + \log_3 2$
33. $\log_{10} a + \log_{10} (a + 21) = 2$
34. $\log_6 48 - \log_6 \frac{16}{5} + \log_6 5 = \log_6 5x$
35. $\log_3 64 - \log_3 \frac{8}{3} + \log_3 2 = \log_3 4r$
36. $\log_6 (b^2 + 2) + \log_6 2 = 2$
37. $\log_3 (5z + 5) - \log_3 (z^2 - 1) = 0$

Solve for a.

38. $\log_n a = \log_n (y + 3) - \log_n 3$
39. $\log_b 2a - \log_b x^3 = \log_b x$
40. $\log_x a^2 + 5 \log_x y = \log_x a$
41. $\log_b 4 + 2 \log_b a = 2 \log_b (n + 1)$

Write each expression as one logarithm.

42. $2 \log_b x + \frac{1}{3} \log_b (x + 2) - 4 \log_b (x - 3)$
43. $\log_b (xy^2) + 2 \log_b \frac{x}{y} - 3 \log_b \left(yx^{\frac{2}{3}}\right)$

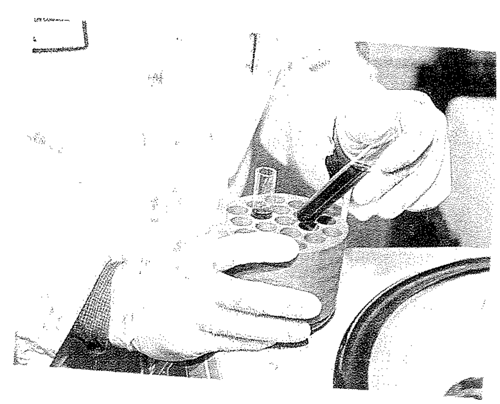
Critical Thinking

44. If $\log_m y = \log_m a - \log_m b - \log_m c$, express y in terms of a, b, and c.

Applications and Problem Solving

Real World

45. **Medicine** The pH of a person's blood can be found by using the Henderson-Hasselbach formula. The formula is $\text{pH} = 6.1 + \log_{10} \frac{B}{C}$, where B represents the concentration of bicarbonate, which is a base, and C represents the concentration of carbonic acid, which is an acid. Most people have a blood pH of about 7.4.
   a. Use a property of logarithms to write the equation without a fraction.
   b. A pH of 7 is neutral, and pH numbers less than 7 represent acidic solutions. pH levels greater than 7 represent basic solutions. Is blood normally an acid, a base, or a neutral?
   c. Use a scientific calculator to find the pH of a person's blood if the concentration of bicarbonate is 25 and concentration of carbonic acid is 2.



46. **Biology** The formula for the energy needed to transport a substance from the outside of a living cell to the inside of that cell is $E = 1.4(\log_{10} C_2 - \log_{10} C_1)$, where E represents the energy in kilocalories per gram molecule, C_1 represents the concentration outside the cell, and C_2 represents the concentration inside the cell.
   a. Use the properties of logarithms to write the value of E as one logarithm.
   b. If the concentration inside the cell is three times the concentration outside the cell, find the energy for a substance to travel from the outside to the inside.

Mixed Review

47. Solve $\log_3 243 = y$. (Lesson 10–2)
48. Find $\frac{x+2}{x+3} \div \frac{x^2+x-12}{x^2-9}$. Write the answer in simplest form. (Lesson 9–3)
49. Find $p(a+1)$ if $p(x) = 4x - x^2 - 4$. (Lesson 8–1)
50. **Geometry** Find the coordinates of the midpoint of $\overline{AB}$ with $A(5, 5)$ and $B(6, -7)$. (Lesson 7–1)
51. State whether $x^2 + 8x + 64$ is a perfect square. (Lesson 6–3)
52. **ACT Practice** What are all the values for x such that $x^2 < 3x + 10$?
    A $x < -2$ B $-2 < x < 5$ C $x > -2$ D $x < 5$ E $x > 5$
53. Factor $x^3 + 2x^2 - 35x$. (Lesson 5–4)
54. Solve the system of equations. (Lesson 3–2)
    $2x + 3y = 2$
    $3x - 4y = -14$

For Extra Practice, see page 899.

WORKING ON THE In•ves•ti•ga•tion

Refer to the Investigation on pages 590–591.

Refer to the data generated from your shooting experiments with the three rubber bands. Examine the tables you created. Explain any patterns you notice from the tables.

1 Create a coordinate plane with the horizontal axis representing the length the rubber band was stretched and the vertical axis representing the distance the rubber band flew for each rubber band. Graph the data for each rubber band. Draw a best-fit line or curve.

2 Describe the shape of each of the graphs. Are any of the graphs functions? If they are functions, describe the type of function you think each one is and justify your answer.

3 Refer to the third column of each table. Compute the difference between the distance each shot flew and the distance the previous shot flew. What pattern, if any, do you notice? What is the relationship between the data in the second column and the data in the third column?

4 Compare the graphs of the three rubber bands. Are the graphs the same? Explain. If there are differences, describe them.

Add the results of your work to your Investigation Folder.