## Focus On
# PREREQUISITE SKILLS

To be successful in this chapter, you'll need to understand these concepts and be able to apply them. Refer to the lesson in parentheses if you need more review before beginning the chapter.

**Multiply polynomials.** (Lesson 5–2)

**Simplify.**

1. $(x + 6)^2$
2. $(9 - 2x)(4 + 3x)$
3. $(5x - 1)^2$
4. $(x - 12)(7x + 4)$

**Factor polynomials.** (Lesson 5–4)

**Factor completely. If the polynomial is not factorable, write *prime*.**

5. $x^2 + 10x + 25$
6. $x^2 - 5x - 14$
7. $x^2 + x + 2$
8. $x^2 - 18x + 81$

**Simplify square roots containing negative radicands.** (Lesson 5–9)

**Simplify.**

9. $\sqrt{-16}$
10. $\sqrt{-75}$
11. $\sqrt{-180}$
12. $\sqrt{-17}$

**Solve a system of three equations in three variables.** (Lesson 3–7)

**Solve each system of equations.**

13. $x - 2y + z = 7$
    $3x + y - z = 2$
    $2x + 3y + 2z = 7$
14. $x + y + z = 6$
    $2x - 3y + 4z = 3$
    $4x - 8y + 4z = 12$
15. $2x + 6y + 8z = 5$
    $-2x + 9y - 12z = -1$
    $4x + 6y - 4z = 3$

## Focus On
# READING SKILLS

In this chapter, you will learn about ***quadratic functions*** and ***inequalities***. A quadratic equation or function is one in which the highest power of the variable is two. The graphs of quadratic equations are curves. You will also learn about the ***quadratic formula,*** which is a formula for finding the values that satisfy a quadratic equation. The quadratic formula contains a radical expression called the ***discriminant***. The discriminant helps you to discriminate, or distinguish among, the types of solutions to the equation.

# 6–1A Graphing Technology
# Quadratic Functions

*A Preview of Lesson 6–1*

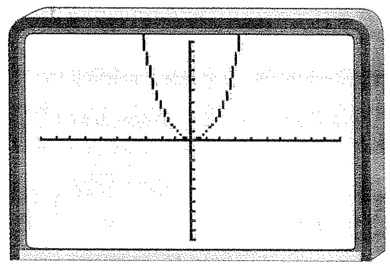
The graphing calculator is a powerful tool for studying graphs of functions. In this lesson, we will study graphs of functions of the form $y = ax^2 + bx + c$, where a $\neq$ 0. These are called **quadratic functions,** and their graphs are called **parabolas.**

**Example 1** **Graph each function in the standard viewing window.**

**a.** $y = x^2$

This function is of the form $y = ax^2 + bx + c$, where $a = 1$, $b = 0$, and $c = 0$.

**Enter:** [Y=] [X,T,θ,*n*] [$x^2$]
[ZOOM] 6

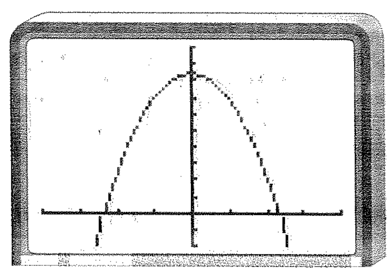


The graph of the function is shaped like a cup. This is the general shape of a parabola. *Notice that $a > 0$.*

**b.** $y = -0.3x^2 + 42$

The function y $= -0.3x^2 + 42$ is of the form $y = ax^2 + bx + c$, where $a = -0.3$, $b = 0$, and $c = 42$.

**Enter:** [Y=] [(–)] .3 [X,T,θ,*n*] [$x^2$] [+] 42 [ZOOM] 6



Nothing appears on the graphics screen. We must change the window to be able to view the complete graph. Press [TRACE] and notice that the coordinates of the vertex, (0, 42), are shown at the bottom of the screen. Since we cannot see the graph in the standard viewing window, change Ymax to 50. Then press [GRAPH].

To see both intercepts of the graph, we have to change the window again. Try changing Xmin to −20 and Xmax to 20. *You may want to use scale factors of 5 on both axes.*

The graph of $y = -0.3x^2 + 42$ is a parabola that opens downward. *Notice that $a < 0$.*

The graph of the quadratic function $y = ax^2 + bx + c$ represents all the values of $x$ and $y$ that satisfy the equation. When we solve the quadratic equation $ax^2 + bx + c = 0$, we are interested only in values of $x$ that make the expression $ax^2 + bx + c$ equal to 0. These values are represented by the points at which the graph of the function crosses the $x$-axis, since the $y$ values of these points are 0. The $x$-intercepts of the quadratic function are called the **solutions** or **roots** of the quadratic equation.

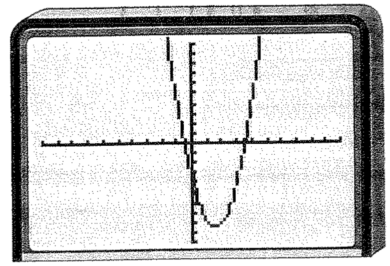
There are three possible outcomes when solving a quadratic equation. The equation will have two real solutions, one real solution, or no real solutions.

**Example 2** **Use a graphing calculator to solve $2x^2 - 6x - 4 = 0$ to the nearest hundredth.**

Begin by graphing the related function $y = 2x^2 - 6x - 4$ in the standard viewing window.

**Enter:** [Y=] 2 [X,T,θ,n] [$x^2$] [–] 6 [X,T,θ,n] [–] 4 [ZOOM] 6

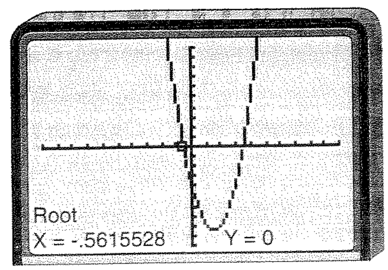
TRACE to one of the $x$-intercepts and press [ZOOM] 2 [ENTER]. The more times you repeat this process, the more accurate your answer will be.



The solutions of this equation are −0.56 and 3.56, to the nearest hundredth.

You can also use the ROOT feature of the calculator to find the solutions automatically. This involves defining an interval that includes the root. A lower bound is a point on the graph just to the left of the solution and an upper bound is a point on the graph just to the right of the solution.

**Enter:** [2nd] [CALC] 2




Using the arrow keys, locate a lower bound for the first solution and press [ENTER]. Similarly, locate an upper bound for the root and press [ENTER]. Finally, after the Guess? prompt, press [ENTER].

Repeat this process for the second root.

It should be noted that solving equations graphically provides only approximate solutions. While approximate solutions are adequate for many applications, if an exact answer is required, an algebraic technique usually must be used.

## EXERCISES

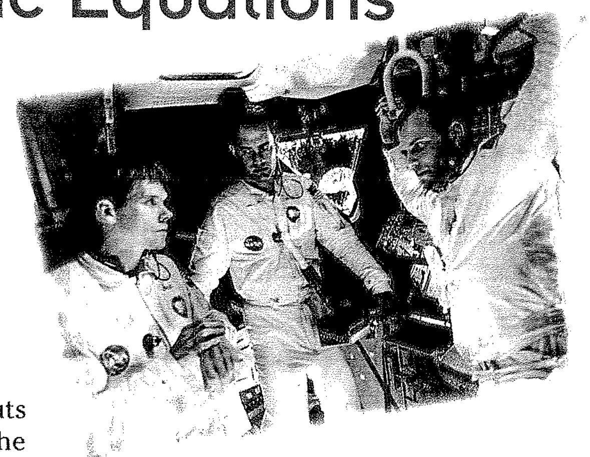
**Determine a viewing window that gives a complete graph of each function.**

1. $y = 4x^2 + 11$
2. $y = 7.5x^2 + 9.5$
3. $y = 6x^2 + 250x + 725$
4. $y = x^2 + 4x - 15$
5. $y = -2x^2 - x - 15$
6. $y = x^2 + 30x + 225$

**Solve each quadratic equation to the nearest hundredth by using a graphing calculator.**

7. $4x^2 + 11 = 0$
8. $7.5x^2 + 9.5 = 0$
9. $6x^2 + 250x + 725 = 0$
10. $x^2 + 4x - 15 = 0$

# 6-1 Solving Quadratic Equations by Graphing



**What YOU'LL LEARN**

- To write functions in quadratic form,
- to graph quadratic functions, and
- to solve quadratic equations by graphing.

**Why IT'S IMPORTANT**

You can graph quadratic functions to solve problems involving space science and physics.

**Real World APPLICATION**

### Space Science

In a recent movie, actors Tom Hanks, Kevin Bacon, and Bill Paxton play real-life astronauts Jim Lovell, John Swigart, and Fred Haise in the story of the 1970 moon mission that nearly ended in disaster. During the filming of the movie, the actors trained on an airplane called the "Vomit Comet"—a stripped-down KC-135 used to help astronauts get accustomed to weightlessness. After climbing to a designated height, weightlessness begins as the jet arcs over a 36,000-foot peak and then dives toward the ground. For 23 seconds, the occupants are weightless.

The path of the Vomit Comet can be represented by the graph of a **quadratic function.** A quadratic function is a function described by an equation that can be written in the form $f(x) = ax^2 + bx + c$, where $a \neq 0$. In a quadratic function, $ax^2$ is called the **quadratic term**, $bx$ is the **linear term,** and $c$ is the **constant term.**

**Example 1** **Write each function in quadratic form. Identify the quadratic term, the linear term, and the constant term.**

**a.** $f(x) = 2x^2 + 7 - 9x$

In quadratic form, the function is written as $f(x) = 2x^2 - 9x + 7$. The quadratic term is $2x^2$, the linear term is $-9x$, and the constant term is 7.

**b.** $f(x) = (x + 4)^2 - 20$

$f(x) = (x + 4)^2 - 20$
$= x^2 + 8x + 16 - 20$ *Square x + 4.*
$= x^2 + 8x - 4$ *Simplify.*

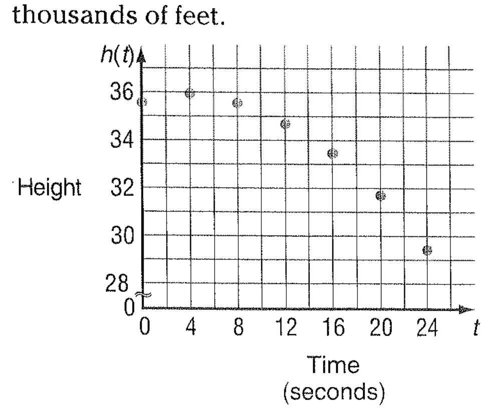
The quadratic term is $x^2$, the linear term is $8x$, and the constant term is $-4$.

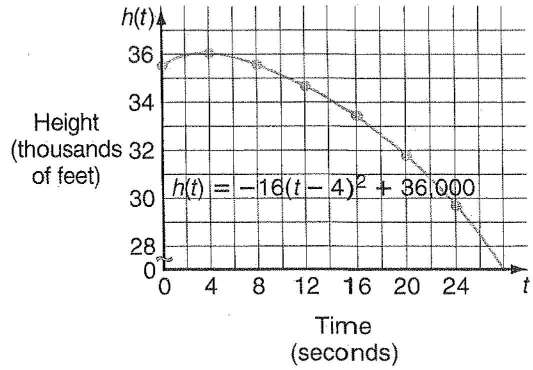
Suppose the flight of the Vomit Comet can be modeled by the quadratic function $h(t) = -16(t - 4)^2 + 36,000$.

To graph this function, we can create a table of values, as shown below.

| $t$ | $-16(t - 4)^2 + 36,000$ | $h(t)$ |
|---|---|---|
| 0 | $-16(-4)^2 + 36,000$ | 35,744 |
| 4 | $-16(0)^2 + 36,000$ | 36,000 |
| 8 | $-16(4)^2 + 36,000$ | 35,744 |
| 12 | $-16(8)^2 + 36,000$ | 34,976 |
| 16 | $-16(12)^2 + 36,000$ | 33,696 |
| 20 | $-16(16)^2 + 36,000$ | 31,904 |
| 24 | $-16(20)^2 + 36,000$ | 29,600 |

Next, we can plot the points $(t, h(t))$, where height is in thousands of feet.






We can now connect the points in a smooth curve. The weightlessness begins when the jet arcs at an elevation of 36,000 feet. Weightlessness continues for 23 seconds. The jet then pulls out of the dive and climbs for another weightless session.

**CAREER CHOICES**

An **astronaut** is a person trained to pilot a spacecraft, operate any of its various systems, or conduct scientific experiments aboard such a craft during space flights.

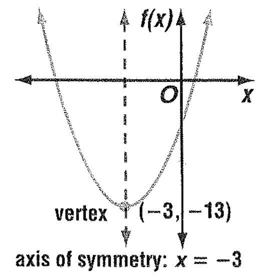
Today, most astronauts have advanced degrees in physics, chemistry, or the earth sciences.

For more information, contact:

NASA Center for Aerospace Information

800 Elkridge Landing Rd.
Linthicum Heights, MD 21090

The graph of any quadratic function is a **parabola.** The graph of the Vomit Comet is part of a parabola. All parabolas have an **axis of symmetry.** The axis of symmetry is the line about which the parabola is symmetric. That is, if you could fold the coordinate plane along the axis of symmetry, the portions of the parabola on each side of the line would match. The axis of symmetry is named by the equation of the line. All parabolas have a **vertex** as well. The vertex is the point of intersection of the parabola and the axis of symmetry. Notice that this parabola intersects the $x$-axis twice. The $x$-coordinates of these intersection points are called the **zeros** of the function.




**Example** **2**

**CONNECTION**

**Physics**

**An arrow is shot upward with an initial velocity of 64 feet per second. The height of the arrow $h(t)$ in terms of the time $t$ since the arrow was released is $h(t) = 64t - 16t^2$.**

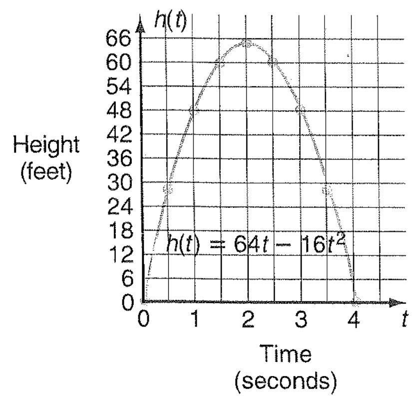
**a. Draw the graph of the function relating the height of the arrow to the time.**

**b. Name the axis of symmetry and the vertex.**

**c. How long after the arrow is released does it reach its maximum height? What is that height?**

a. First find and graph the ordered pairs that satisfy the function $h(t) = 64t - 16t^2$. Then graph the parabola suggested by the points.

| $t$ | $h(t)$ |
|---|---|
| 0 | 0 |
| 0.5 | 28 |
| 1.0 | 48 |
| 1.5 | 60 |
| 2.0 | 64 |
| 2.5 | 60 |
| 3.0 | 48 |
| 3.5 | 28 |
| 4.0 | 0 |




*This is the graph of the function that describes the height at any given time. The actual path of the arrow is an entirely different parabola.*

b. The equation of the axis of symmetry is $x = 2$. Since the vertex is the point of intersection of the parabola and the axis of symmetry, it is at (2, 64).

c. You can see from the graph that the arrow reaches its maximum height at 2 seconds. The maximum height is 64 feet. *How does the vertex relate to the maximum height and the time of the maximum height?*

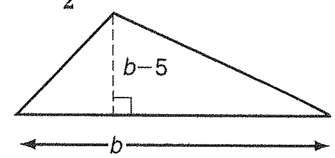
You can use quadratic functions to describe different mathematical situations as well as situations occurring in everyday life.

## Example 3

**INTEGRATION**
**Geometry**

**Write a function to represent the area of a triangle if the height is 5 cm less than the base. Then sketch the graph. Use $A = \frac{1}{2}bh$.**

Let $b$ represent the measure of the base of the triangle. Then $b - 5$ represents the height. Since the area is a function of the length of the base, let $f(b)$ represent the area.



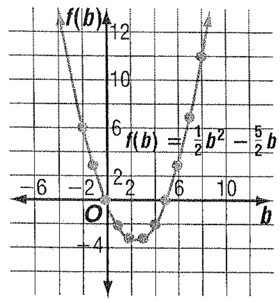

$$f(b) = \frac{1}{2}b(b - 5)$$

$$= \frac{1}{2}b^2 - \frac{5}{2}b$$

Find several ordered pairs that satisfy the quadratic function $f(b) = \frac{1}{2}b^2 - \frac{5}{2}b$. Then graph these ordered pairs and draw a parabola.

*Which points on the graph could actually represent possible lengths of the base and areas of the triangles?*

| b | f(b) |
|---|---|
| −1 | 3 |
| 0 | 0 |
| 1 | −2 |
| 2 | −3 |
| 3 | −3 |
| 4 | −2 |
| 5 | 0 |
| 6 | 3 |
| 7 | 7 |
| 8 | 12 |




When a quadratic function is set equal to zero, the result is a **quadratic equation.** A quadratic equation is an equation that can be written in the form $ax^2 + bx + c = 0$, where $a \neq 0$. Since the largest exponent of the variable is 2, we say that a quadratic equation has a degree of 2. A quadratic equation contains only one variable, and all of the exponents are positive integers.

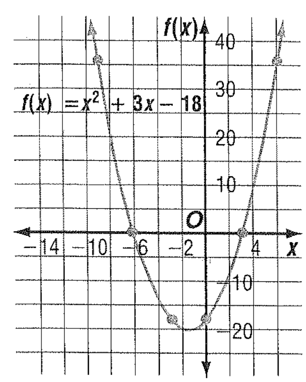
The **roots,** or **solutions,** of a quadratic equation are values of the variable that satisfy the equation. There are several methods you can use to find the roots of a quadratic equation. One method is to graph its related quadratic function. The related quadratic function of a quadratic equation in the form $ax^2 + bx + c = 0$ is $f(x) = ax^2 + bx + c$. The zeros of the function are the solutions of the equation, since $f(x) = 0$ at those points.

## Example 4

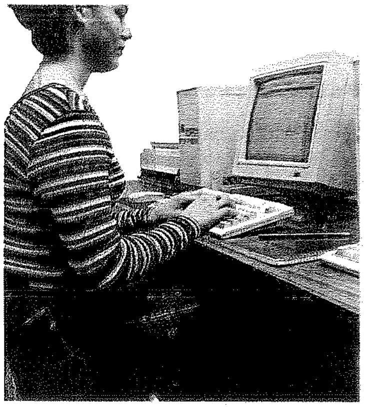
**Solve $x^2 + 3x - 18 = 0$ by graphing.**

Graph the quadratic function $f(x) = x^2 + 3x - 18$ by finding and graphing the ordered pairs that satisfy the function. Then graph the parabola by connecting the points. Solve the equation by noting the points at which the graph intersects the $x$-axis.

| x | f(x) |
|---|---|
| −9 | 36 |
| −6 | 0 |
| −3 | −18 |
| 0 | −18 |
| 3 | 0 |
| 6 | 36 |




The table shows that $f(x) = 0$ when $x = -6$ and $x = 3$. The graph crosses the $x$-axis at $-6$ and 3. Thus, the solutions of the equation are $-6$ and 3.



Check your solutions by substituting each solution into the equation to see if it is satisfied.

$$x^2 + 3x - 18 = 0 \qquad\qquad x^2 + 3x - 18 = 0$$
$$(-6)^2 + 3(-6) - 18 \stackrel{?}{=} 0 \qquad\qquad (3)^2 + 3(3) - 18 \stackrel{?}{=} 0$$
$$0 = 0 \ \checkmark \qquad\qquad 0 = 0 \ \checkmark$$

A spreadsheet is a tool that can be used to manipulate numbers easily. Accountants, insurance underwriters, and loan officers all use spreadsheets to simplify calculations.

## EXPLORATION SPREADSHEET

To solve the quadratic equation $2x^2 + x - 3 = 0$ by using a spreadsheet, first enter the coefficients 2, 1, and $-3$ into the cells. Establish a starting value for $x$. Then choose an increment for the values of $x$ to increase. You can change any of these values by typing a different number into the cell. The spreadsheet automatically computes and updates all other values in the cells. The bottom row of cells contains the values of the equation for the given values of $x$. The spreadsheet below indicates that the roots are $-1.5$ and 1. *You can verify these solutions by graphing.*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Quadratic Equation | | | | | | | | | |
| 2 | X^2+ | 1 | X+ | −3 | = 0 | | | | |
| Starting Value = | | | −2 | | | | | | |
| Increment = | | | 0.5 | | | | | | |
| | | | | | | | | | |
| X | −2 | −1.5 | −1 | −0.5 | 0 | 0.5 | 1 | 1.5 | 2 |
| Y | 3 | 0 | −2 | −3 | −3 | −2 | 0 | 3 | 7 |

**Your Turn**

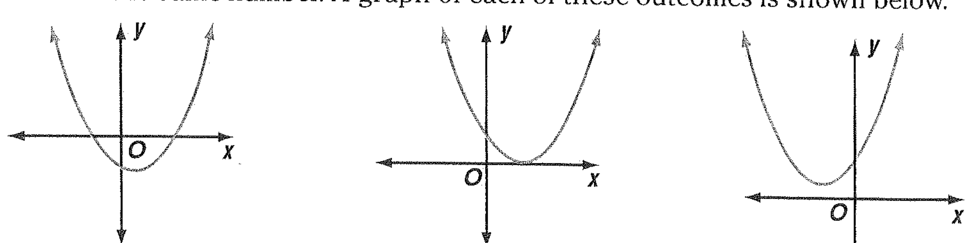
**a.** Work through the other examples in this lesson by using a spreadsheet.

**b.** Describe the difference between using a graphing calculator and using a spreadsheet to solve quadratic equations.

**c.** How did you determine the starting values and increments for $x$?

**d.** Find the roots of the equation $2x^2 + 2x - 12 = 0$ by using a spreadsheet.

There are three possible outcomes when solving a quadratic equation. The equation will have two real solutions, one real solution, or no real solutions. When a quadratic equation has one real solution, it really has two solutions that are the same number. A graph of each of these outcomes is shown below.




two real solutions — one real solution — no real solutions

Example 5 illustrates a quadratic equation for which the solutions are the same number.

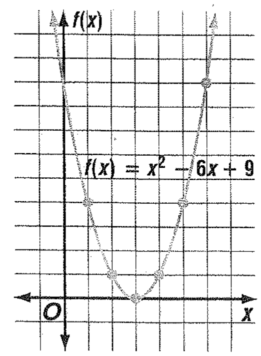
**Example 5** **Solve $x^2 - 6x = -9$ by graphing.**

Write the equation in standard form.

$x^2 - 6x = -9 \quad \rightarrow \quad x^2 - 6x + 9 = 0$

Find and graph the ordered pairs that satisfy the related function $f(x) = x^2 - 6x + 9$.

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $f(x)$ | 4 | 1 | 0 | 1 | 4 | 9 |



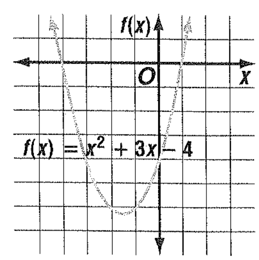

Notice that the graph has only one $x$-intercept, 3. Thus, the solution is 3.

## CHECK FOR UNDERSTANDING

**Communicating Mathematics**

**Study the lesson. Then complete the following.**

1. **Determine** if $3z^4 - 5z^3 + 6z - 6 = 0$ is a quadratic equation. Explain.
2. **Define** each term and explain how they are related.
   a. solution    b. root    c. zero of a function    d. $x$-intercept
3. Refer to Example 3.
   a. **List** three possible sets of dimensions for the triangle described.
   b. **Describe** the restrictions that must be placed on the graph of the function if you were to use it to solve the equation $\frac{1}{2}b^2 - \frac{5}{2}b = 0$.
4. **State** the difference between a quadratic equation and a quadratic function.
5. **Explain** why a quadratic equation cannot have more than two solutions.
6. **Determine** the solution of the equation $x^2 + 3x - 4 = 0$ from its related graph shown at the right. Explain how you found your answer.




**Guided Practice**

**Identify the quadratic term, the linear term, and the constant term in each function.**

7. $f(x) = x^2 + x - 4$
8. $f(x) = -4x^2 - 8x - 9$

**Graph each function. Name the vertex and the axis of symmetry.**

9. $g(x) = x^2 - 4x + 4$
10. $f(x) = x^2 + 9$
11. $h(x) = x^2 + 6x + 9$

**Solve each equation by graphing.**

12. $d^2 + 5d + 6 = 0$
13. $a^2 - 4a + 4 = 0$
14. $2x^2 - x - 3 = 0$
15. A quadratic function $f(x)$ has values $f(3) = -4$, $f(6) = -7$, and $f(9) = 8$. Between which two $x$ values is $f(x)$ sure to have a zero? Explain how you know.

## EXERCISES

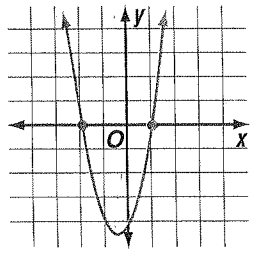
### Practice

**Identify the quadratic term, the linear term, and the constant term in each function.**

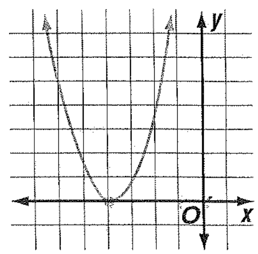
16. $g(x) = 5x^2 - 7x + 2$
17. $g(n) = 3n^2 - 1$
18. $f(n) = \frac{1}{3}n^2 + 4$
19. $f(z) = z^2 + 3z$
20. $f(x) = (x + 3)^2$
21. $f(t) = (3t + 1)^2 - 8$

**Use the related graph of each equation to determine its solutions.**

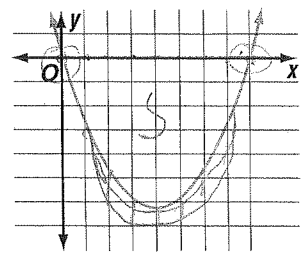
22. $2x^2 + 2x - 4 = 0$




23. $x^2 + 8x + 16 = 0$




24. $x^2 - 16x = 0$




**Graph each function. Name the vertex and the axis of symmetry.**

25. $f(x) = x^2$
26. $f(x) = x^2 + 12x + 36$
27. $g(x) = x^2 - 9x + 9$
28. $h(x) = x^2 + 4$
29. $f(x) = x^2 - 9$
30. $h(x) = x^2 - 10x + 27$
31. $f(x) = x^2 + 20x + 93$
32. $g(x) = x^2 - \frac{2}{5}x + \frac{26}{25}$
33. $f(x) = x^2 + 3x - 0.95$

**Solve each equation by graphing.**

34. $m^2 + 3m = 28$
35. $p^2 - 2p - 24 = 0$
36. $4n^2 - 7n - 15 = 0$
37. $c^2 + 4c + 4 = 0$
38. $n^2 - 3n = 0$
39. $2w^2 - 3w = 9$
40. $4v^2 - 8v - 5 = 0$
41. $2c^2 + 5c - 12 = 0$
42. $(3x + 4)(2x + 7) = 0$

### Critical Thinking

43. **Number Theory** Although no one has found a formula that generates prime numbers, eighteenth-century Swiss mathematician Leonhard Euler discovered that $y = x^2 + x + 17$ produces prime numbers up to a certain point.
    a. Based on the graph of $f(x) = x^2 + x + 17$, what are the roots? Explain your answer.
    b. Describe how the nature of the graph relates to the equation for generating prime numbers.

### Applications and Problem Solving

Real World

44. **World Records** In 1940, Emanual Zacchini of Italy was fired a record distance of 175 feet from a cannon while performing in the United States. Suppose his initial upward velocity was 80 feet per second. The height $y$ can be represented by the function $y = 80x - 16x^2$, where $x$ represents the number of seconds that have passed.
    a. Draw the graph of the function, relating heights Mr. Zacchini reaches to the time after he was shot from the cannon.
    b. How long after he was shot out of the cannon did he reach his maximum height? What was that height?

45. **Free-Falling** In 1942, I.M. Chisov of the USSR bailed out of an airplane without a parachute at 21,980 feet and survived. The function $h(t) = -16t^2 + 21{,}980$ describes the relationship between height $h(t)$ in feet and time $t$ in seconds.
    a. Graph the function.
    b. About how many seconds did he fall before reaching the ground?

## Mixed Review

46. **Electrical Engineering** The relationship between the flow of electricity $I$ in a circuit, the resistance to the flow $Z$, called impedance, and the electromotive force $E$, called voltage, is given by the formula $E = I \cdot Z$. Electrical engineers use $j$ to represent the imaginary unit. An electrical engineer is designing a circuit that is to have a current of $(6 - j8)$ amps. If the impedance of the circuit is $(14 + j8)$ ohms, find the voltage. (Lesson 5–9)

47. **Geometry** The area of a square is $169x^2$. The length of one of its sides minus 14 is equal to 77. Solve for $x$. (Lesson 5–8)

48. Write an augmented matrix for $4x + 3y = 10$ and $5x - y = 3$. Then solve the system of equations. (Lesson 4–6)

49. Evaluate the determinant of $\begin{bmatrix} \frac{1}{2} & -1 & 3 \\ -3 & 5 & -2 \\ 3 & 2 & 4 \end{bmatrix}$ using expansion by minors. (Lesson 4–5)

50. **SAT Practice** If $x > 0$, then $\frac{\sqrt{16x^2 + 64x + 64}}{x + 2} =$

    **A** 2 **B** 4 **C** 8 **D** 16

    **E** It cannot be determined from the information given.

51. Write an equation in slope-intercept form and in standard form for the line that has a slope of $-3$ and passes through $(-7, 5)$. (Lesson 2–4)

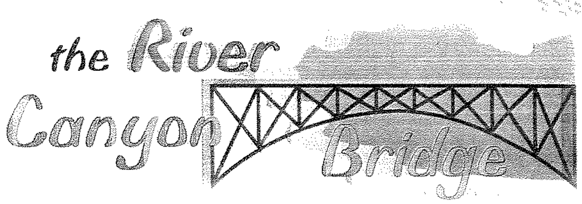
52. Graph a line that passes through (3, 0) and is perpendicular to $y - 3x = 1$. (Lesson 2–3)

53. Sonia's aquarium holds 31.5 L of water. The container she uses to fill the aquarium holds 1.5 L. How many full containers of water does it take to fill the aquarium? (Lesson 1–4)

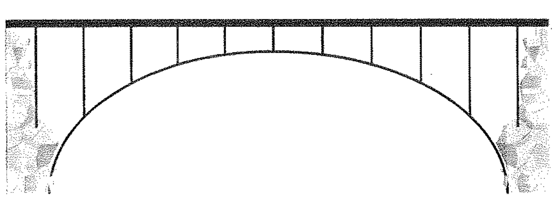
For Extra Practice, see page 889.

## WORKING ON THE Investigation

Refer to the Investigation on pp. 328–329.




After reviewing your proposal for four types of bridges, your company's consultant suggests a bridge that is supported by an arch as shown below.



The arch would be in the shape of a parabola that opens downward with its vertex at the center of the bridge. The bridge would span the canyon and be anchored to the canyon walls. It would be supported by a set of short struts that are anchored to the parabolic arch.

1. Draw a scaled blueprint of this bridge on grid paper. Let the road lie along the $x$-axis and let the vertex of the parabola lie on the $y$-axis.
2. Make a table of values for points that lie along the arch of the bridge.
3. Look for a pattern in your table and estimate at what points the arch of the bridge will touch the sides of the canyon.
4. How does this design compare with the bridges you considered?

Add the results of your work to your Investigation Folder.

6-2

# Solving Quadratic Equations by Factoring

**What YOU'LL LEARN**

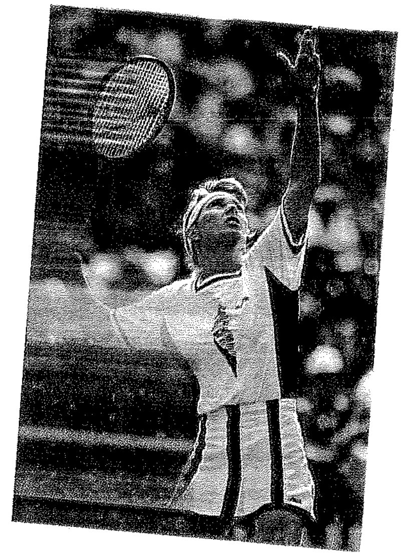
- To solve problems by using the guess-and-check strategy, and
- to solve quadratic equations by factoring.

**Why IT'S IMPORTANT**

You can use quadratic equations to solve problems involving tennis, meteorology, and air travel.

**Real World APPLICATION**

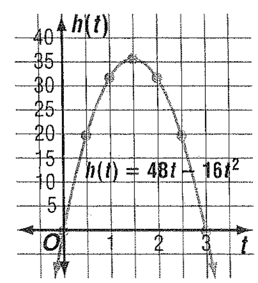
### Tennis



In 1993, German tennis player Steffi Graf won three of the four biggest tournaments in tennis: Wimbledon, the French Open, and the U.S. Open. When an object, like a tennis ball, is hit straight up into the air, the height of the object is given by the function $h(t) = v_0t - 16t^2$, where $h(t)$ represents the height of the object, $v_0$ represents the initial velocity, and $t$ represents the time that the object has traveled. If a tennis ball is hit upward with an initial velocity of 48 feet per second, how long does it take for the ball to fall to the ground? To find the solution, substitute the known values into the formula $h(t) = v_0t - 16t^2$ to get the resulting function $h(t) = 48t - 16t^2$.

There are several methods that you can use to find the solution. In the last lesson, you learned to find the solution by graphing.

| $t$ | 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 |
|---|---|---|---|---|---|---|---|
| $h(t)$ | 0 | 20 | 32 | 36 | 32 | 20 | 0 |




For this function, the zeros are 0 and 3. This means that the ball starts out at 0 feet and travels for 3 seconds before reaching the ground again.

Another way to solve a quadratic equation is by **factoring.** To solve by factoring, you must use the **zero product property.**

| *Zero Product Property* | **For any real numbers $a$ and $b$, if $ab = 0$, then either $a = 0$, $b = 0$, or both.** |
|---|---|

We can solve the same equation by factoring that we solved above by graphing.

$0 = 48t - 16t^2$
$0 = 16t(3 - t)$

Since the product of $16t$ and $3 - t$ is 0, either $16t$ or $3 - t$ equals 0. So set each factor equal to 0 and solve.

$16t = 0$ or $3 - t = 0$ *Zero product property*
$t = 0$ $t = 3$

The solutions (or roots) are 0 and 3. These are the same as the zeros obtained when you graphed the related function, $h(t) = 48 - 16t^2$. These answers seem reasonable since a tennis ball might not stay in the air for more than 3 seconds.

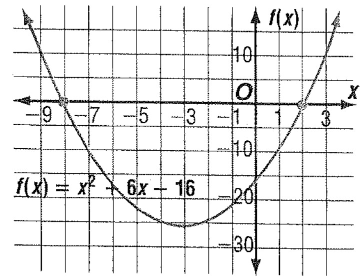
The **guess-and-check** strategy can be useful when factoring quadratic equations. You may have to try several combinations of factors before finding the right one.

**Example 1** **PROBLEM SOLVING** **Guess and Check**

**Solve $x^2 + 6x - 16 = 0$ by factoring.**

Start guessing by using factors of 16: 2, 4, 8, or 16.

| Guess | Check | Correct? |
|---|---|---|
| $(x - 4)(x + 4)$ | $x^2 - 16 = 0$ | *No, the linear term is 0.* |
| $(x - 8)(x + 2)$ | $x^2 - 6x - 16 = 0$ | *No, the linear term is negative.* |
| $(x + 8)(x - 2)$ | $x^2 + 6x - 16 = 0$ | *Yes* |




You can now solve the equation by using the zero product property to find the values of $x$.

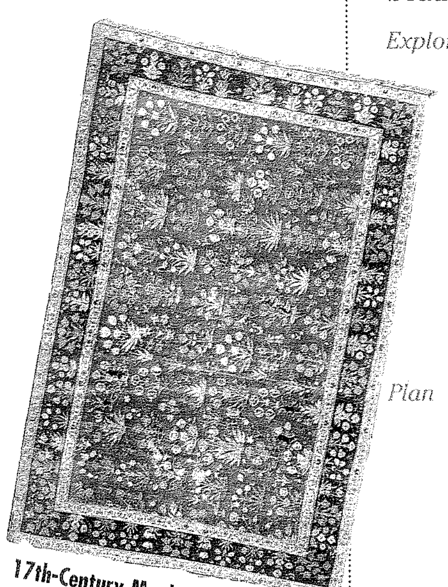
$$(x + 8)(x - 2) = 0$$
$$x + 8 = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = -8 \qquad\qquad x = 2$$

The solutions are −8 and 2. You can check these solutions by graphing the related quadratic function and checking the $x$-intercepts.

When using factoring to solve a real-life problem, you need to examine each solution carefully to see if it is reasonable for that situation.

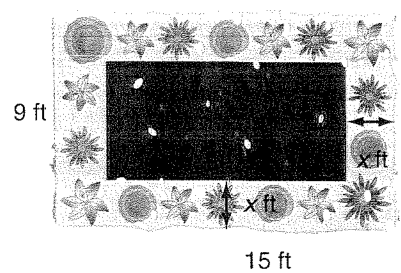
**Example 2** **Real World APPLICATION** **Antiques**

**International auction houses sell hundreds of oriental rugs each year to collectors from all over the world. A 17th-century Mughal carpet was recently purchased from a museum for $253,000, despite having moth damage, corrosion, and holes. The rug has a border of uniform width depicting lilies, asters, and roses and a red center called a raspberry field. The auction brochure for the rug describes it as measuring 9 feet by 15 feet, with a raspberry field of 91 square feet. If you were interested in purchasing the rug, you might want to know how wide that beautiful border is. How wide is it?**



17th-Century Mughal carpet, sold for $253,000

*Explore* Read the problem and make a drawing that models the Mughal carpet.




Let $x$ feet be the width of the border.
The length of the red center is $15 - 2x$ feet.
The width of the red center is $9 - 2x$ feet.

*Plan* Write an equation. The area of the red center can be expressed as the product of the length and width.

$A = \ell w$

$= (15 - 2x)(9 - 2x)$ *Substitute for $\ell$ and $w$.*

$= 135 - 48x + 4x^2$ *Multiply the binomials.*

$= 4x^2 - 48x + 135$

*Solve* Since the area of the red center is 91 square feet, replace $A$ with 91 and solve this equation.

**TECHNOLOGY Tips**

You can use a graphing calculator and the TRACE function or 2nd CALC root to find the zeros of the graphed function.

$$A = 4x^2 - 48x + 135$$

$$91 = 4x^2 - 48x + 135 \quad \textit{Replace A with 91.}$$

$$0 = 4x^2 - 48x + 44 \quad \textit{Subtract 91 from each side.}$$

$$0 = x^2 - 12x + 11 \quad \textit{Divide each side by 4.}$$

$$0 = (x - 11)(x - 1) \quad \textit{Factor.}$$

$$x - 11 = 0 \quad \text{or} \quad x - 1 = 0$$

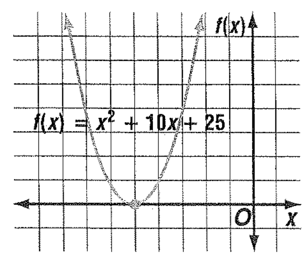
$$x = 11 \qquad x = 1$$

The solutions are 11 and 1. Use $x = 1$ since 11 is an unreasonable answer. Thus, the width of the floral border is 1 foot.

*Examine* If the border is 1 foot wide all the way around, the width of the center is $9 - 2(1)$ or 7 and the length is $15 - 2(1)$ or 13. Since $7 \times 13 = 91$, the answer makes sense.

You have seen that a quadratic equation may have one solution. Factoring shows this is true because the two factors of the quadratic function are the same.

**Example 3** **Solve $x^2 + 10x = -25$ by factoring.**




*The graph of the related function intersects the x-axis in one point.*

$$x^2 + 10x = -25$$

$$x^2 + 10x + 25 = 0$$

$$(x + 5)(x + 5) = 0$$

$$x + 5 = 0 \quad \text{or} \quad x + 5 = 0$$

$$x = -5 \qquad x = -5$$

The only solution is $-5$.

No matter which method you use, you can always check your solutions by substituting the values into the equation and simplifying.

**Check:** $x^2 + 10x = -25$

$$(-5)^2 + 10(-5) \stackrel{?}{=} -25$$

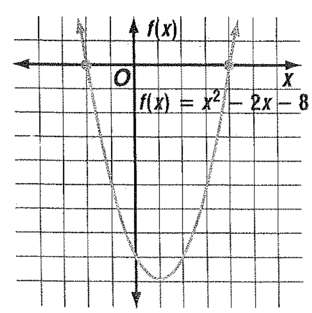
$$-25 = -25 \ \checkmark$$

## CHECK FOR UNDERSTANDING

**Communicating Mathematics**

**Study the lesson. Then complete the following.**

1. **Explain** why the solution of 11 feet is unreasonable in Example 2.
2. **State** the zeros of the function graphed at the right. Verify by solving the related equation by factoring.



3. **You Decide** Chelsea was trying to find the zeros of the function $f(x) = x^2 - 13x + 36$. Carmen explained that she was making a mistake, but Chelsea insisted she was right. "All you have to do is make the $f(x)$ equal to zero and solve for $x$," Chelsea said. Look at her work below.

$x^2 - 13x + 36 = 0$

$(x - 9)(x + 4) = 0$

$x - 9 = 0$ or $x + 4 = 0$

$x = 9$ $\qquad$ $x = -4$

Carmen said there was still something wrong. Who is correct? Explain.

**Guided Practice**

**Solve each equation.**

4. $(y - 8)(y + 6) = 0$
5. $(3y + 7)(y + 5) = 0$

**Solve each equation by factoring.**

6. $q^2 - 5q - 24 = 0$
7. $r^2 - 16r + 64 = 0$
8. $a^3 = 81a$
9. $3y^2 + y - 14 = 0$

10. **Physics** According to the the *Guinness Book of World Records,* the longest pendulum in the world is 73 feet $9\frac{3}{4}$ inches. It was installed in Tokyo, Japan, in 1983. The time in seconds $t$ for a pendulum to swing back and forth is given by the formula $t^2 = 1.23L$, where $L$ is the length of the pendulum in feet.
    a. What is the length of time needed for this pendulum to swing back and forth once?
    b. How long would it take for a pendulum that is 6 feet long to swing back and forth once?
    c. How long should the pendulum be if you want it to swing back and forth exactly 15 times a minute?

## EXERCISES

**Practice**

**Solve each equation.**

11. $(a + 4)(a + 1) = 0$
12. $z(z - 1) = 0$
13. $(2x + 6)(x + 3) = 0$
14. $(3y - 5)(2y + 7) = 0$

**Solve each equation by factoring.**

15. $x^2 - x = 12$
16. $d^2 - 5d = 0$
17. $z^2 - 12z + 36 = 0$
18. $y^2 + y - 30 = 0$
19. $r^2 - 3r = 4$
20. $3c^2 = 5c$
21. $18u^2 - 3u = 1$
22. $4y^2 = 25$
23. $9y^2 + 16 = -24y$
24. $4x^2 - 13x = 12$

**Solve each equation by graphing or by factoring.**

25. $b^2 + 3b = 40$
26. $4a^2 - 17a + 4 = 0$
27. $4s^2 - 11s = 3$
28. $6r^2 + 7r = 3$
29. $12m^2 + 25m + 12 = 0$
30. $18n^2 - 3n = 15$
31. $n^3 = 9n$
32. $x^3 = 64x$
33. $35z^3 + 16z^2 = 12z$
34. $18r^3 + 16r = 34r^2$

**Critical Thinking**

35. A parabola has intercepts at $x = -8$, $x = 4$, and $y = -8$. What is the equation of the parabola?

## Applications and Problem Solving

Real World

36. **Air Travel** An airlines route map states the formula $(VM)^2 = 1.22A$, where *VM* represents the number of miles to the horizon you can see from an airplane if you are flying on a clear day. The altitude, in feet, is represented by *A*.

a. Suppose you are flying at an altitude of 36,000 feet on a clear day. About how many miles are there to the horizon?

b. Suppose the distance to the horizon is 236 miles. What is your altitude?

c. The Sears Tower in Chicago is the tallest building in the world. The observation deck is 1454 feet above ground. About how far could you see on a clear day if you were standing on the observation deck?

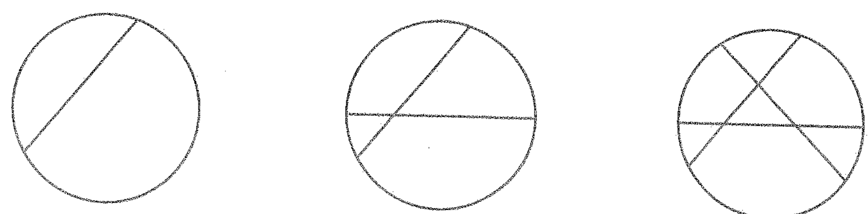
37. **Meteorology** Weather forecasters can determine the approximate time that a thunderstorm will last if they know the diameter *d* of the storm in miles. The time *t* in hours can be found by using the formula $216t^2 = d^3$.

a. Draw the graph of $y = 216t^2 - 5^3$ and use it to estimate how long a thunderstorm will last if its diameter is 5 miles.

b. Find how long a thunderstorm will last if its diameter is 5 miles and compare this time with your estimate in part a.

38. **Guess and Check** Find two integers whose sum is 15 and whose product is 54.

39. **Geometry** The diagram below shows the relationship between the number of chords drawn through a circle and the maximum number of parts into which chords can divide a circle.



1 chord 2 parts | 2 chords 4 parts | 3 chords 7 parts

This relationship can be described by the formula $p = \frac{1}{2}x^2 + \frac{1}{2}x + 1$, where *p* represents the number of parts and *x* represents the number of chords. How many chords would you have to draw to divide a circle into 37 parts?

## Mixed Review

40. **Geometry** The length of a rectangle is 2 centimeters greater than its width. (Lesson 6–1)

a. Write a function to represent the area of the rectangle. Then graph the function.

b. Use the graph to estimate the area of a rectangle whose width is 4 cm. Compare your estimate to the actual area.

41. Solve $6y^2 = -96$. (Lesson 5–9)

42. **SAT Practice** If the average (arithmetic mean) of 3 distinct positive integers is 12, what is the greatest possible value of any one of the integers?

A 32 B 33 C 34 D 35 E 36

43. Find the inverse of $\begin{bmatrix} 4 & -2 \\ -3 & 6 \end{bmatrix}$. (Lesson 4–4)

44. Graph the system $3x - 2y = 10$ and $y - x = -1$ and state its solution. (Lesson 3–1)

45. Solve $3(2m + 9n) - (4 + 6m) = -5m$ for $m$. (Lesson 2–2)

46. Simplify $\frac{2}{3}\left(\frac{1}{2}a + 3b\right) + \frac{1}{2}\left(\frac{2}{3}a + b\right)$. (Lesson 1–2)

For Extra Practice, see page 889.

# 6-3 Completing the Square

**What YOU'LL LEARN**

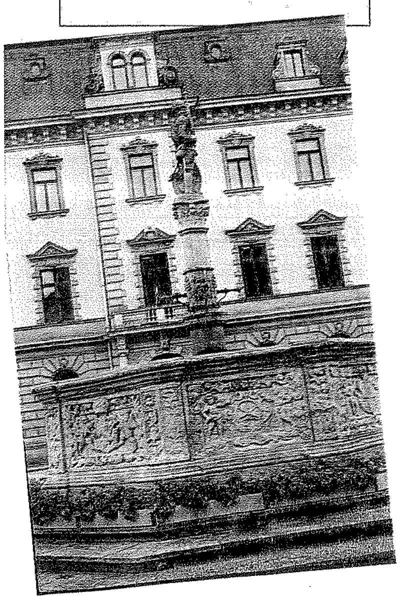
- To solve quadratic equations by completing the square.

**Why IT'S IMPORTANT**

You can complete the square to solve quadratic equations that involve architecture and law enforcement.

**GLOBAL CONNECTIONS**

The largest non-palatial residence is St. Emmeram Castle in Regensburg, Germany. It is valued at more than $177 million and contains 517 rooms with a floor space of 231,000 square feet. Only 95 rooms are personally used by the family of the late Prince Johannes von Thurn und Taxis.

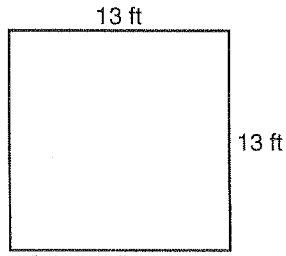


**Real World APPLICATION**

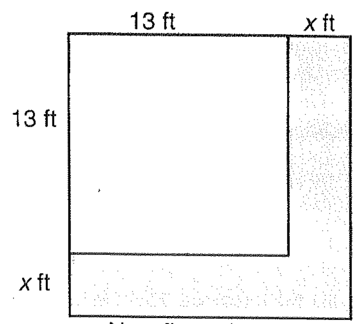
## Architecture

An architect for Windham Homes is changing the floor plan of a house to meet the needs of a new customer. The present floor plan calls for a square dining area that measures 13 feet by 13 feet. The customer would also like for the dining area to be square, but with an area of 250 square feet. How much will this add to the dimensions of the room?

A strip must be added to the length and width of the dining room as shown below. The width of the strip that will be added is represented by $x$ feet. The length and width of the new dining area would then be $(13 + x)$ feet.



Original floor plan



New floor plan

We can use the formula $A = s^2$ to find the area of the new dining room. Each side measures $13 + x$, and the new area is 250 square feet.

$$A = s^2$$

$$250 = (13 + x)^2 \quad \text{Replace } A \text{ with 250 and } s \text{ with } 13 + x.$$

$$\pm\sqrt{250} = \sqrt{(13 + x)^2} \quad \text{Take the square root of each side.}$$

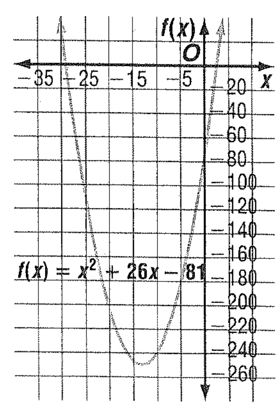
$$\pm\sqrt{250} = 13 + x$$

$$\pm\sqrt{250} - 13 = x$$

Use a scientific calculator to find approximate values for $x$.

**Enter:** 250 [√x] [−] 13 [=] *2.811388301*

**Enter:** 250 [√x] [+/−] [−] 13 [=] *−28.8113883*




Since we need the increased width of the dining area, we can ignore the negative solution. The width of the increase would be approximately 2.81 feet.

You can graph the related function to show that the answer is reasonable. Rewrite $250 = (13 + x)^2$ so that one side is zero. Then graph the related function.

$$250 = (13 - x)^2$$

$$(13 + x)^2 - 250 = 0 \rightarrow f(x) = (13 + x)^2 - 250$$

$$= x^2 + 26x - 81$$

The estimated zeros are about −30 and 3. Thus, the answer 2.8 is reasonable.

The quadratic equation $250 = (13 + x)^2$ contained one side that was a perfect square, $(13 + x)^2$. This allowed us to solve it by taking the square root of each side. When an equation does not contain a perfect square, you may create a perfect square by applying a process called **completing the square.**

In a perfect square, there is a relationship between the coefficient of the middle term and the constant term.

**Specific Case**

$$(x + 9)^2 = x^2 + 18x + 81$$

$$9 = \frac{1}{2}(18) \rightarrow 9^2 = 81$$

**General Case**

$$(x + c)^2 = x^2 + 2cx + c^2$$

$$c = \frac{1}{2}(2c) \rightarrow c^2$$

To complete the square in the expression below, you would use the same process. Using the pattern of coefficients, take half the coefficient of the linear term and square it.

$$x^2 - 8x + \underline{\ ?\ }$$

$$\left(-\frac{8}{2}\right)^2 \rightarrow (-4)^2 \text{ or } 16$$

*$x^2 - 8x + 16$ is a perfect square trinomial, which can be written as $(x - 4)^2$.*

In Example 1, use the pattern of coefficients to make the trinomial a perfect square.

**Example 1** **Find the value of $c$ that makes $x^2 + 14x + c$ a perfect square.**

$c = \left(\frac{14}{2}\right)^2$ *Square half the coefficient of the linear term.*

$= 7^2$ or 49

The value of $c$ is 49. Therefore, the trinomial $x^2 + 14x + 49$ is a perfect square. It can be written $(x + 7)^2$.

**Example 2** **Solve $x^2 - 6x = 40$ by completing the square.**

First, you must find the term that completes the square on the left side of the equation. Then add that term to each side.

$x^2 - 6x + \square = 40 + \square$

$x^2 - 6x + 9 = 40 + 9$ $\quad\left(-\frac{6}{2}\right)^2 = 9$

$(x - 3)^2 = 49$ *Factor the perfect square trinomial.*

$x - 3 = \pm 7$ *Take the square root of each side.*

$x - 3 = 7$ or $x - 3 = -7$

$x = 10$ $\quad x = -4$

The solutions are 10 and $-4$. *Check by factoring.*

A good way to help you visualize completing the square is to model the process using algebra tiles.

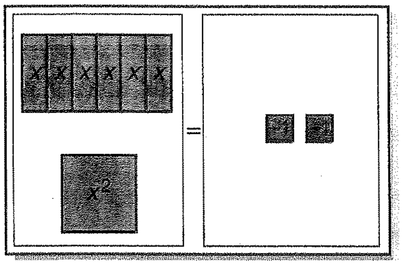
## MODELING MATHEMATICS

### Completing the Square

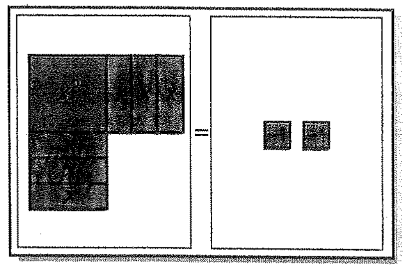
**Materials:** algebra tiles  equation mat

Use algebra tiles to complete the square for the equation $x^2 + 6x + 2 = 0$.

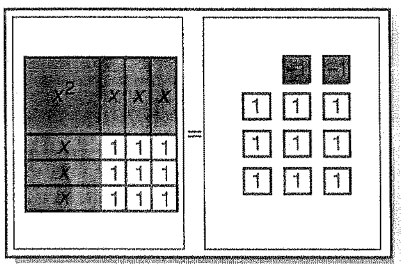
**Step 1** Subtract 2 from each side of the equation to model the equation $x^2 + 6x = -2$.



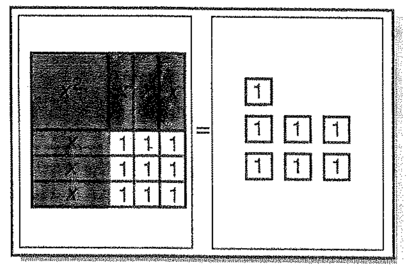

**Step 2** Begin to arrange the $x^2$-tile and $x$-tiles into a square.



**Step 3** To complete the square, add 9 1-tiles to the left side of the mat. Since it is an equation, add 9 1-tiles to the right side.




**Step 4** Remove the zero pairs on the right side of the mat. After completing the square, the equation is $x^2 + 6x + 9 = 7$ or $(x + 3)^2 = 7$.




**Your Turn**

a. Use algebra tiles to complete the square for the equation $x^2 + 4x + 1 = 0$.

b. The equation $x^2 + 5x - 2 = 0$ has an odd number for the coefficient of $x$. Complete the square by using algebra tiles or by making a drawing.

c. Write a paragraph explaining how you could complete the square with models without first rewriting the equation. Include a drawing.

When the coefficient of the second-degree term is not 1, you must first divide the equation by that coefficient before completing the square.

**Example 3** **Solve $4x^2 - 5x - 21 = 0$ by completing the square.**

$$4x^2 - 5x - 21 = 0$$

$$x^2 - \frac{5}{4}x - \frac{21}{4} = 0 \qquad \textit{Divide each side by 4.}$$

$$x^2 - \frac{5}{4}x = \frac{21}{4} \qquad \textit{Isolate the constant on one side.}$$

$$x^2 - \frac{5}{4}x + \frac{25}{64} = \frac{21}{4} + \frac{25}{64} \qquad \textit{Add } \left(-\frac{5}{4} \div 2\right)^2 \textit{ or } \frac{25}{64} \textit{ to each side.}$$

$$\left(x - \frac{5}{8}\right)^2 = \frac{361}{64} \qquad \textit{Factor.}$$

$$x - \frac{5}{8} = \pm\frac{19}{8}$$ *Take the square root of each side.*

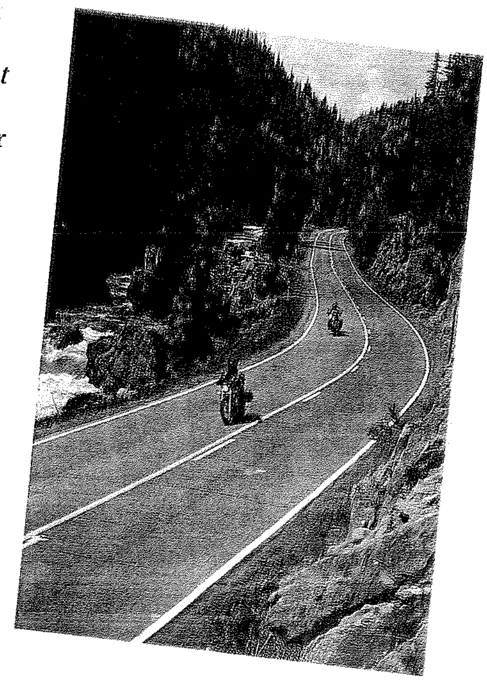
$$x = \frac{5}{8} \pm \frac{19}{8}$$

$$x = \frac{5}{8} + \frac{19}{8} \qquad x = \frac{5}{8} - \frac{19}{8}$$

$$= \frac{24}{8} \text{ or } 3 \qquad = -\frac{14}{8} \text{ or } -\frac{7}{4}$$

The solutions are 3 and $-\frac{7}{4}$. *Verify using the graph of the related function.*

Not all roots of quadratic equations will be rational numbers. Roots that are irrational numbers may be written as *exact* answers in radical form or as *approximate* answers in decimal form when a calculator is used.



**Example 4**

**CONNECTION**

**Physics**

**The distance $d$ that an object travels can be calculated when the initial speed $v_i$, elapsed time $t$, and the rate of constant acceleration $a$ are known. A formula that relates these factors is $d(t) = v_i t + \frac{1}{2}at^2$.**

**If a motorcycle has an initial speed of 30 m/s and a constant acceleration of 6 m/s$^2$, how much time will it take to travel 200 meters?**

$$d(t) = v_i t + \frac{1}{2}at^2$$

$$200 = 30t + \frac{1}{2} \cdot 6t^2$$ *Substitute the known values into the formula.*

$$200 = 30t + 3t^2$$

$$\frac{200}{3} = 10t + t^2$$ *Divide by 3.*

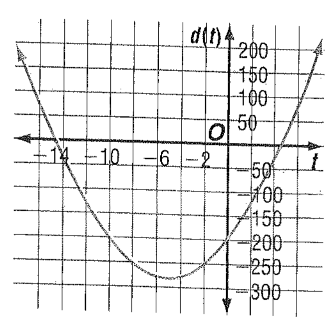
$$\frac{200}{3} + 25 = t^2 + 10t + 25$$ *Complete the square.*

$$\frac{275}{3} = (t + 5)^2$$ *Factor.*

$$\pm\sqrt{\frac{275}{3}} = t + 5$$ *Take the square root of each side.*

$$\pm\sqrt{\frac{275}{3}} - 5 = t$$ *Subtract 5 from each side.*

The solutions are $\sqrt{\frac{275}{3}} - 5$ or about 4.57 and $-\sqrt{\frac{275}{3}} - 5$, or about $-14.57$. Verify by looking at the graph of the related function.




We can eliminate $-14.57$ since negative time has no meaning in this example. Thus, the motorcycle will travel 200 meters in about 4.57 seconds.

Not all solutions to quadratic equations are real. In some cases, the solutions are complex numbers of the form $a + bi$, where $b \neq 0$.

**Example 5** **Solve $x^2 + 8x + 20 = 0$ by completing the square.**

$x^2 + 8x + 20 = 0$

$x^2 + 8x = -20$ *Subtract 20 from each side.*

$x^2 + 8x + 16 = -20 + 16$ $\left(\frac{8}{2}\right)^2 = 16$

$(x + 4)^2 = -4$ *Factor the left side.*

$x + 4 = \pm 2i$ *Take the square root of each side. Remember*

$x = -4 \pm 2i$ *that $\sqrt{-1} = i$.*

The roots are the complex numbers $-4 + 2i$ and $-4 - 2i$.

**LOOK BACK**

You can refer to Lesson 5-9 for information on complex numbers.

## CHECK FOR UNDERSTANDING

### Communicating Mathematics

**Study the lesson. Then complete the following.**

1. **Explain** how you would solve the equation $x^2 + 21x - 5 = 0$ by completing the square.
2. **Discuss** how you can tell if a quadratic equation has imaginary roots just by looking at a sketch of its graph.
3. **State** whether $b^2 + 5b + 23$ is a perfect square. Explain.
4. **Write** a paragraph explaining why we sometimes eliminate negative answers to real-world applications.

**MODELING MATHEMATICS**

5. Use algebra tiles or make a drawing to solve the equation $x^2 + 4x - 5 = 0$ by completing the square.

### Guided Practice

**Find the value of $c$ that makes each trinomial a perfect square.**

6. $x^2 + 12x + c$
7. $x^2 - 7x + c$

**Find the exact solutions for each equation by completing the square.**

8. $x^2 + 8x = 20$
9. $12t^2 - 17t = 5$
10. $r^2 + 14 = 8r$
11. $x^2 - 7x + 4 = 0$
12. $\frac{1}{2}x^2 - 4x + 8 = 0$
13. $x^2 + 2x + 6 = 0$

14. **Safety** Juanita is driving a truck at an initial velocity of 60 ft/s. She sees a stop sign 240 feet ahead of her. If she begins to decelerate at the rate of $7\frac{1}{2}$ ft/s$^2$, how long will it take her before she stops at the stop sign? Use the formula $s = v_i t + \frac{1}{2}at^2$.
*If acceleration is a positive number, what is deceleration?*