EXERCISES

Practice

Find the value of c that makes each trinomial a perfect square.

15. $x^2 + 2x + c$
16. $x^2 + 18x + c$
17. $t^2 + 40t + c$
18. $r^2 - 9r + c$
19. $a^2 - 100a + c$
20. $x^2 + 15x + c$

Find the exact solution for each equation by completing the square.

21. $x^2 + 3x - 18 = 0$
22. $x^2 + 2x - 120 = 0$
23. $x^2 - 8x + 11 = 0$
24. $x^2 + 7x - 17 = 0$
25. $x^2 + 9x + 20.25 = 0$
26. $9x^2 + 96x + 256 = 0$
27. $x^2 + 4x + 11 = 0$
28. $2x^2 - 7x + 12 = 0$
29. $3x^2 + 7x + 7 = 0$
30. $16x^2 + 9x + 20 = 0$
31. $x^2 - 3x - 20 = 0$
32. $2x^2 - x - 31 = 0$
33. $12x^2 - 13x - 35 = 0$
34. $x^2 + 19x - 12 = 0$
35. $ax^2 + bx + c = 0$
36. $px^2 + rx + m = 0$

Programming

37. The graphing calculator program at the right determines if an equation written in standard form contains a perfect square trinomial. It uses the same process that you use when completing the square to determine if the expression is a perfect square trinomial.

Use the guess-and-check strategy in the program at the right to find the value for k that makes each expression a perfect square trinomial.

a. $x^2 + kx + 64$
b. $x^2 - 14x + k$
c. $4x^2 - 12x + k$
d. $16x^2 + 40x + k$
e. $4x^2 + kx + 1$
f. $kx^2 - 20x + 4$

```
PROGRAM:PERFSQR
: Prompt A,B,C
: If A<0 or C<0
: Then
: Goto 2
: End
: If √(A)≠(int √(A))
: Then
: Goto 2
: End
: If √(C)≠(int √(C))
: Then
: Goto 2
: End
: If B^2=4*A*C
: Then
: Disp "CONGRATULATIONS!",
  "IT IS A","PERFECT SQUARE."
: Else
: Goto 2
: Stop
: Lbl 2
: Disp "TRY AGAIN","IT IS
  NOT A","PERFECT SQUARE."
```

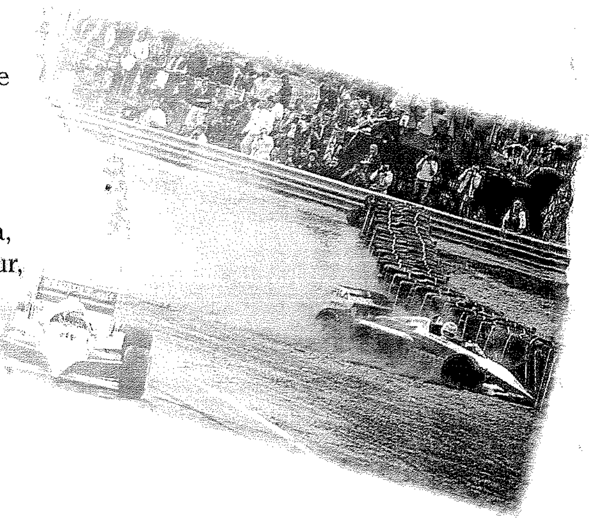
Critical Thinking

38. Find the values of m that make $f(x) = x^2 + (m + 5)x + (5m + 1)$ a perfect square trinomial function. How many roots does it have?

Applications and Problem Solving

39. **Physics** When a moving car hits an object, the damage it can cause can be measured by the *collision impact.* For a certain car, the collision impact I can be represented by the formula $I = 2(s)^2$, where s represents the speed in kilometers per minute.
   a. Sketch a graph of the function.
   b. What is the collision impact if the speed is 1 km/min? 2 km/min? 4 km/min?
   c. As the speed doubles, explain what happens to the value of the collision impact.

40. **Law Enforcement** The police can use the length of skid marks to help determine the speed of a vehicle before the brakes were applied. If the skid marks were on concrete, the formula $\frac{s^2}{24} = d$ can be used to approximate the speed of the vehicle. In the formula, s represents the speed in miles per hour, and d represents the length of the skid marks in feet. If the length of a car's skid marks on dry concrete were 50 feet, how fast was the car traveling when the brakes were applied?



Mixed Review

41. **Physics** A ball is thrown straight up with an initial velocity of 56 feet per second. The height of the ball t seconds after it is thrown is given by the formula $h(t) = 56t - 16t^2$. (Lesson 6–2)
   a. What is the height of the ball after 1 second?
   b. What is its maximum height?
   c. After how many seconds will it return to the ground?

42. **Forestry** The formula $V = 0.0027Ld^2 + 0.0027L\left(d + \frac{L}{A}\right)^2$ can be used to estimate how many cubic feet of wood will come from one log. In the equation, V represents the volume of the log in cubic feet, L represents the length of the log in feet, d represents the diameter of the top of the log in inches, and A represents the length of the log required for 1 inch of taper. Suppose that a 16-foot long log has a 1-inch taper over an 8-foot length. What diameter log does the forester want to find if he needs to get 150 cubic feet of wood out of one log? (Lesson 6–1)

43. Simplify $\sqrt[4]{5m^3n^5} \cdot \sqrt[4]{125m^2n^3}$. (Lesson 5–6)

44. **SAT Practice** In one week, a pet shop sold 8 ten-pound bags of birdseed and 12 twenty-pound bags of birdseed. What was the average (arithmetic mean) weight of the birdseed sales that week?

   A 10 B 12 C 15 D 16 E 20

45. **Consumerism** The prices of 14 video cameras are listed below.

\$877	\$819	\$1100	\$1450	\$812	\$973	\$1399
\$890	\$1409	\$949	\$900	\$775	\$1299	\$1399

   Make a box-and-whisker plot of the prices. (Lesson 4–8)

46. Graph the system of inequalities. Name the vertices of the polygon formed. Find the maximum and minimum values of the given function. (Lesson 3–5)

   $y \leq 7$ $x \leq 5$
   $y \geq -x + 6$ $f(x, y) = 2x - 3y$
   $y \leq x + 4$

47. Use Cramer's rule to solve $3m - 4n = 0$ and $n + 7 = -m$. (Lesson 3–3)

For Extra Practice, see page 889.

48. Graph a line that only goes through Quadrants II and III and passes through the point $(-4, 27)$. (Lesson 2–3)

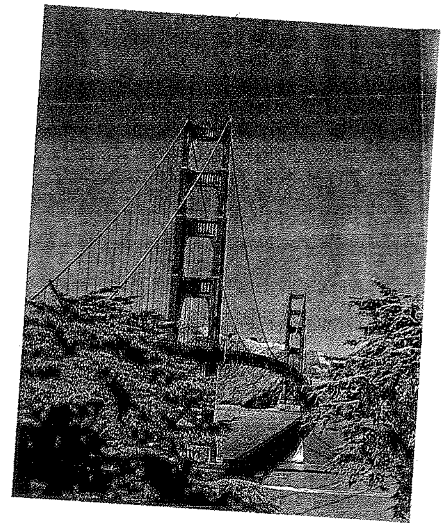
6-4 The Quadratic Formula and the Discriminant

What YOU'LL LEARN

- To solve quadratic equations by using the quadratic formula, and
- to use discriminants to determine the nature of the roots of quadratic equations.

Why IT'S IMPORTANT

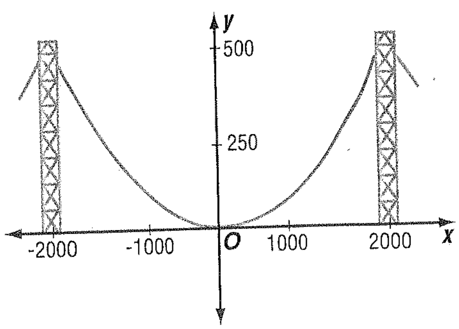
You can use the quadratic formula to solve quadratic equations that involve health and probability.



Real World APPLICATION

Landmarks

The Golden Gate Bridge in San Francisco, California, is a magnificent structure that was completed on May 27, 1937, after more than four years of construction and a cost of approximately $27 million. The Golden Gate is the tallest bridge in the world, with its towers extending 746 feet above the water and the floor of the bridge extending 200 feet above the water.



The two supporting cables that pass over the tops of the towers are each 7650 feet long and 36.5 inches in diameter. They are the largest bridge cables ever made. These supporting cables approximate the shape of a parabola with the lowest point reaching about 6 feet above the floor of the bridge. This parabola can be approximated by the quadratic function $y = 0.00012244898x^2 + 6$, where x represents the distance from the axis of symmetry and y represents the height of the cables.

How would you solve the quadratic equation $0.00012244898x^2 + 6 = 0$? In Lessons 6–1, 6–2, and 6–3, you learned several ways to solve quadratic equations. But it would be difficult to use any of those techniques to solve this equation. You might ask, "Isn't there a formula that will work for any quadratic equation?" The answer is yes! This formula can be derived by solving the general form of a quadratic equation for x.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad \textit{Divide each side by } a.$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \quad \textit{Subtract } \frac{c}{a} \textit{ from each side.}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} \quad \textit{Complete the square.}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \quad \textit{Simplify.}$$

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \quad \textit{Take the square root of each side.}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \quad \textit{Simplify.}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \textit{Subtract } \frac{b}{2a} \textit{ from each side.}$$

This equation is known as the **quadratic formula.**

The Quadratic Formula	**The solutions of a quadratic equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the following formula.** $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 1

Solve $r^2 - 7r = 18$ by using the quadratic formula.

First, write the equation in standard form. $r^2 - 7r = 18 \rightarrow r^2 - 7r - 18 = 0$

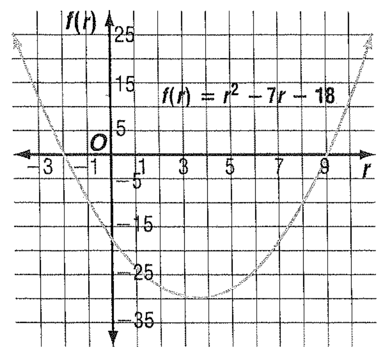
Then, substitute the values of a, b, and c directly into the quadratic formula. *$a = 1$, $b = -7$, and $c = -18$*

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(-18)}}{2(1)}$$

$$= \frac{7 \pm \sqrt{121}}{2} \text{ or } \frac{7 \pm 11}{2}$$

$$r = \frac{7 + 11}{2} \text{ or } 9 \quad \text{and} \quad r = \frac{7 - 11}{2} \text{ or } -2$$



The graph of the related function shows that there are two solutions.

The solutions are 9 and −2.
Check these solutions.

Sometimes roots are irrational. You can express these roots exactly by writing them in radical form.

Example 2

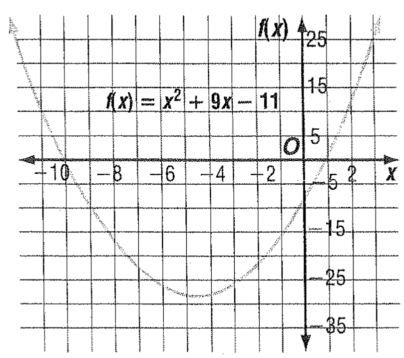
Solve $x^2 + 9x - 11 = 0$ by using the quadratic formula.

The equation is in standard form. Substitute the values of a, b, and c directly into the quadratic formula. *$a = 1$, $b = 9$, and $c = -11$*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-9 \pm \sqrt{9^2 - (4)(1)(-11)}}{2(1)} \text{ or } \frac{-9 \pm \sqrt{125}}{2}$$

The solutions are $\frac{-9 + 5\sqrt{5}}{2}$ and $\frac{-9 - 5\sqrt{5}}{2}$.



The graph of the related function shows that there are two solutions.

You can use a scientific calculator to help you find approximate solutions. First evaluate the radical part of the expression $\frac{-9 \pm \sqrt{9^2 - (4)(1)(-11)}}{2(1)}$ and store the result in memory.

Enter: 9 [x^2] [−] 4 [×] 11 [+/−] [=] [2nd] [$\sqrt{x}$] [STO▸] 11.18033989

Now find the solutions.

Enter: 9 [+/−] [+] [RCL] [=] [÷] 2 [=] *1.090169944*

Enter: 9 [+/−] [−] [RCL] [=] [÷] 2 [=] *−10.09016994*

The two solutions are approximately 1.1 and −10.1.

How do these solutions compare with the graph?

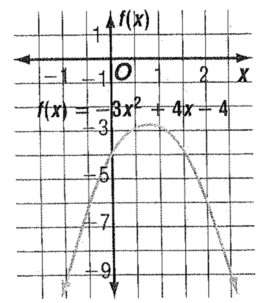
When using the quadratic formula, if the radical contains a negative value, the solutions will be imaginary. Imaginary solutions always appear in conjugate pairs.

Example 3 **Solve $-3x^2 + 4x - 4 = 0$ by using the quadratic formula.**

Substitute the values of a, b, and c directly into the quadratic formula.

$a = -3$, $b = 4$, and $c = -4$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-4 \pm \sqrt{(4)^2 - 4(-3)(-4)}}{2(-3)}$$
$$= \frac{-4 \pm \sqrt{-32}}{-6}$$
$$= -\frac{4 \pm 4i\sqrt{2}}{-6} \text{ or } \frac{2 \pm 2i\sqrt{2}}{3}$$



The solutions are the imaginary numbers $\frac{2 + 2i\sqrt{2}}{3}$ and $\frac{2 - 2i\sqrt{2}}{3}$.

A graph of the related function shows that the solutions are imaginary.

Example 4 **CONNECTION Physics**

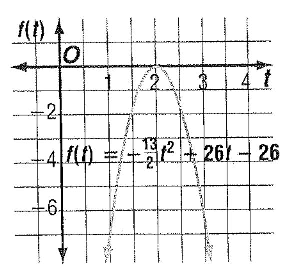
A car is traveling at 26 meters per second (m/s) and accelerating at -13 m/s^2. After traveling 26 m, the driver brings the car to a complete stop. The equation $26 = 26t - \frac{13}{2}t^2$, where t is the time it takes to stop, can be used to represent this situation. How long did it take the driver to stop the car?

First, write the equation in standard form. $-\frac{13}{2}t^2 + 26t - 26 = 0$

Then, substitute directly into the quadratic formula.

$a = -\frac{13}{2}$, $b = 26$, and $c = -26$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-26 \pm \sqrt{(26)^2 - 4\left(-\frac{13}{2}\right)(-26)}}{2\left(-\frac{13}{2}\right)}$$
$$= \frac{-26 \pm \sqrt{0}}{-13}$$
$$= 2$$



It took 2 seconds to stop the car.

A graph of the related function shows that there is one solution.

Study Examples 1, 2, 3, and 4 and observe the relationship between the expression under the radical, $b^2 - 4ac$, and the roots of the quadratic equation. This expression, $b^2 - 4ac$, is called the **discriminant.** The value of the discriminant determines the nature of the roots of a quadratic equation. The table below summarizes all the possibilities.

Example	Value of $b^2 - 4ac$	Discriminant a Perfect Square?	Nature of Root(s)	Nature of Related Graph
1	$b^2 - 4ac > 0$	yes	2 real, rational	intersects x-axis twice
2	$b^2 - 4ac > 0$	no	2 real, irrational	intersects x-axis twice
3	$b^2 - 4ac < 0$	—	2 imaginary	does not intersect x-axis
4	$b^2 - 4ac = 0$	—	1 real	intersects x-axis once

Example 5 **Find the value of the discriminant for each quadratic equation. Then describe the nature of the roots.**

a. $x^2 - 8x + 16 = 0$

$a = 1, b = -8, c = 16$

$$\begin{aligned} b^2 - 4ac &= (-8)^2 - 4(1)(16) \\ &= 64 - 64 \\ &= 0 \end{aligned}$$

The value of the discriminant is 0, so there is one real root.

b. $5x^2 + 42 = 0$

$a = 5, b = 0, c = 42$

$$\begin{aligned} b^2 - 4ac &= (0)^2 - 4(5)(42) \\ &= 0 - 840 \\ &= -840 \end{aligned}$$

The value of the discriminant is negative, so there are two imaginary roots.

c. $x^2 - 5x - 50 = 0$

$a = 1, b = -5, c = -50$

$$\begin{aligned} b^2 - 4ac &= (-5)^2 - 4(1)(-50) \\ &= 25 + 200 \\ &= 225 \end{aligned}$$

The value of the discriminant is 225, which is a perfect square. There are two real, rational roots.

d. $2x^2 - 9x + 8 = 0$

$a = 2, b = -9, c = 8$

$$\begin{aligned} b^2 - 4ac &= (-9)^2 - 4(2)(8) \\ &= 81 - 64 \\ &= 17 \end{aligned}$$

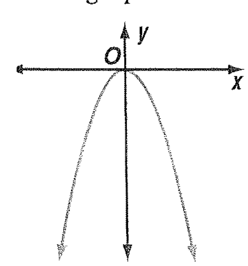
The value of the discriminant is 17, which is not a perfect square. There are two real, irrational roots.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **Explain** why the roots of a quadratic equation are imaginary if the value of the discriminant is less than 0.
2. **Draw** graphs to illustrate the relationship between the nature of the roots determined by the discriminant and the number of times the graph of the related function intersects the x-axis.
3. Refer to the application at the beginning of the lesson. Calculate the value of the discriminant for the equation of the supporting cables for the Golden Gate Bridge. What does it mean?
4. **Describe** the value of the discriminant for the equation whose graph is at the right. How many real roots are there?



MATH JOURNAL

5. **Assess Yourself** Of the methods you have used to solve quadratic equations—graphing, factoring, completing the square, and the quadratic formula—which do you prefer? Why?

Guided Practice

6. Which equation shows how to solve $3x^2 - x + 2 = 0$ by using the quadratic formula?

   a. $x = \frac{1 \pm \sqrt{3^2 - 4(3)(2)}}{2(3)}$

   b. $x = \frac{-2 \pm \sqrt{(-1)^2 - 4(3)(2)}}{2}$

   c. $x = \frac{1 \pm \sqrt{1^2 - 4(3)(2)}}{2(3)}$

   d. $x = \frac{-1 \pm \sqrt{(-1)^2 - 4(3)(2)}}{2}$

State the values of *a*, *b*, and *c* for each equation. Then find the value of the discriminant.

7. $3x^2 - 5x = 2$
8. $x^2 - 24x + 144 = 0$
9. $x^2 + 7 = -5x$

Find the value of the discriminant and describe the nature of the roots (real, imaginary, rational, irrational) of each quadratic equation. Then solve the equation. Express irrational roots as exact and approximate to the nearest hundredth.

10. $x^2 + 10x = -25$
11. $2x^2 = 72$
12. $x^2 + x - 5 = 0$
13. $x^2 + 5x + 10 = 0$
14. If the discriminant of the equation of a parabola is 2025, how many times does the graph of the equation intersect the x-axis? Justify your answer.
15. Solve $x^2 + 2x - 5 = 0$.
    a. How many roots does the function $y = x^2 + 2x - 5$ have? What are they?
    b. Approximately where does the graph of $y = x^2 + 2x - 5$ cross the x-axis?

EXERCISES

Practice

Find the value of the discriminant and describe the nature of the roots (real, imaginary, rational, irrational) of each quadratic equation. Then solve the equation. Express irrational roots as exact and approximate to the nearest hundredth.

16. $x^2 + 12x + 32 = 0$
17. $2x^2 - 12x + 18 = 0$
18. $x^2 - 4x + 1 = 0$
19. $3x^2 + 5x - 2 = 0$
20. $x^2 - 2x + 5 = 0$
21. $3x^2 + 11x + 4 = 0$
22. $x^2 - 12x + 42 = 0$
23. $x^2 = 6x$
24. $2x^2 + 7x - 11 = 0$
25. $5x^2 - 8x + 9 = 0$
26. $0.4x^2 + x = 0.3$
27. $2x^2 - 13x = 7$
28. $x^2 - 16x + 4 = 0$
29. $4x^2 - 9x = -7$

Critical Thinking

30. **Probability** Suppose you picked an integer at random from 1 through 12 as a value for c in the equation $y = x^2 + 6x + c$. What is the probability that the resulting equation will have imaginary roots?

Applications and Problem Solving

Real World

31. **Health** A person's blood pressure depends on his or her age. For women, normal systolic blood pressure is given by the formula $P = 0.01A^2 + 0.05A + 107$, where P is the normal blood pressure in millimeters of mercury (mm Hg) and A is the age. For men, the normal systolic blood pressure is given by the formula $P = 0.006A^2 - 0.02A + 120$.

(continued on the next page)

a. Graph both functions on the same set of axes. Describe the differences between the graphs and explain how the differences reflect the blood pressures of men and women.

b. Find the normal blood pressure of a woman who is 35 years old.

c. Find the approximate age of a man whose blood pressure is 134 mm Hg.

32. **Business** Bryan Starr is a 17-year-old high school senior from Upper Arlington, Ohio, who started his own lawn service when he was 12. His business has grown substantially and he has been able to put away money for college, buy a truck, and invest in new lawn equipment to keep up with the growing demand for his services. Suppose his weekly revenue R can be represented by the formula $R = -p^2 + 50p - 125$, where p is the average price he charges for each lawn.

    a. Sketch a graph of the related function. Explain why it behaves like it does, considering Bryan's business.

    b. Explain how Bryan could earn $400 each week.

    c. What price should he charge to earn the maximum revenue? What would be his revenue?

    d. Use the discriminant to find if there is a price he could charge that would make his weekly revenue $600. Explain.

Mixed Review

33. **Geometry** The area of a square plus its perimeter minus 12 is equal to 0. Find x if the area is $4x^2$. (Lesson 6–3)

34. Solve $3t^2 + 4t = 15$ by factoring. (Lesson 6–2)

35. Simplify $\sqrt{x^2 + 6x + 9}$. (Lesson 5–5)

36. **ACT Practice** If one of the roots of polynomials is $-\frac{5}{2}$ and the polynomial is represented by $6x^2 + kx + 20 = 0$, then what is the value of k?

    A -23 B $-\frac{4}{3}$ C $\frac{4}{3}$ D 7 E 23

37. Solve the system $x - y = 10$ and $2y - 3x = -1$ by using a matrix equation. (Lesson 4–6)

For Extra Practice, see page 890.

38. Solve $|2x - 5| \leq 9$. (Lesson 1–7)

SELF TEST

Solve each equation by graphing. (Lesson 6–1)

1. $z^2 + 4z + 3 = 0$
2. $m^2 + 6m = 27$

Solve each equation by factoring. (Lesson 6–2)

3. $x^2 + 5x - 36 = 0$
4. $2x^2 + 7x = -3$

Solve each equation by completing the square. (Lesson 6–3)

5. $x^2 + 6x = 55$
6. $x^2 - 7x + 21 = 0$

7. Find the value of the discriminant for the equation $x^2 - 8x + 2 = 0$ and describe the nature of the roots. (Lesson 6–4)

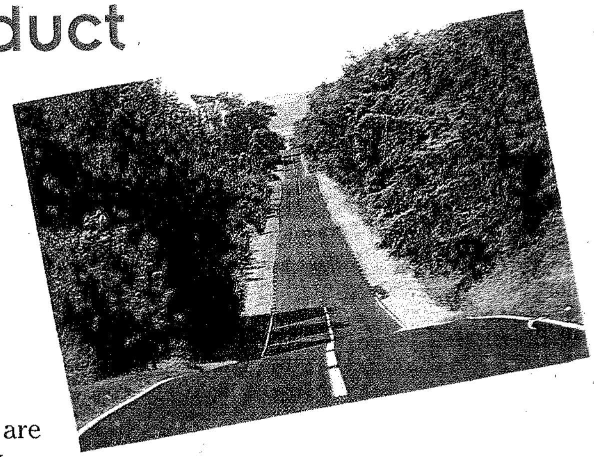
Solve each equation by using the quadratic formula. (Lesson 6–4)

8. $3x^2 + 5x - 1 = 0$
9. $2x^2 - 25x + 72 = 0$

10. **Horticulture** The length of a tropical garden at a local conservatory is 5 feet more than its width. A walkway 2 feet wide surrounds the outside of the garden. If the total area of the walkway and garden is 594 square feet, find the dimensions of the garden. (Lesson 6–2)

6-5

Sum and Product of Roots



What YOU'LL LEARN

- To find the sum and product of the roots of quadratic equations, and
- to find a quadratic equation to fit a given condition.

Why IT'S IMPORTANT

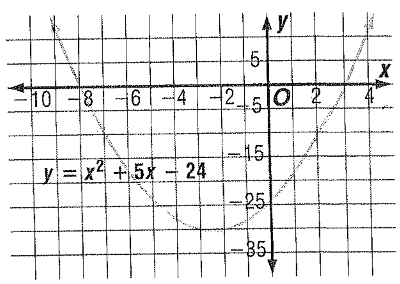
You can use the sum and product of roots to write quadratic equations involving space flight and engineering.

Real World APPLICATION

Engineering

One of the major tasks of civil engineers is to design roads that are safe and comfortable. In highway design, the quadratic function $y = ax^2 + bx + c$ is called a *transition curve* because it has properties that provide a smooth transition between peaks and valleys.

A road with an initial gradient, or slope, of 3% can be represented by the formula $y = ax^2 + 0.03x + c$, where y is the elevation and x is the distance along the curve. Suppose the elevation of the road is 1105 feet at points 200 feet and 1000 feet along the curve. You can find the equation of the transition curve. *This problem will be solved in Example 4.*



As in the situation above, you may know the roots of a quadratic equation without knowing the equation itself. For example, suppose the roots of a quadratic equation are 3 and −8 and you want to find the equation. In a previous lesson, you used factoring to solve an equation. You applied the zero product property and set both equations equal to 0 to find the solutions. You can work backward to find the equation when you know the solutions.

$x = 3$ or $x = -8$	*Start with the solutions.*
$x - 3 = 0$ $\quad x + 8 = 0$	*Rewrite equations equal to 0*
$(x - 3)(x + 8) = 0$	*Multiplicative property of zero*
$x^2 + 5x - 24 = 0$	*Multiply.*

The last equation has roots 3 and −8 and is written in standard form. The sum and product of the roots can help to develop the equation in another way.

Add the roots. $3 + (-8) = -5$ *−5 is the opposite of the coefficient of x.*

Multiply the roots. $3(-8) = -24$ *−24 is the constant term.* $x^2 + 5x - 24 = 0$

This pattern can be generalized for any quadratic equation by using the roots defined by the quadratic formula. Let s_1 and s_2 represent the roots.

$$s_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad s_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad \textit{Quadratic formula}$$

Their sum can be represented as follows.

$$s_1 + s_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \textit{Add the roots.}$$

$$= \frac{-2b + 0}{2a} \text{ or } -\frac{b}{a} \quad \textit{Simplify.}$$

The sum of the roots is $-\frac{b}{a}$.

The product of the roots may be represented as follows.

$$s_1(s_2) = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{b^2 - (b^2 - 4ac)}{4a^2} \quad \textit{Multiply.}$$

$$= \frac{b^2 - b^2 + 4ac}{4a^2} \quad \textit{Use the distributive property.}$$

$$= \frac{4ac}{4a^2} \text{ or } \frac{c}{a}$$

The product of the roots is $\frac{c}{a}$.

The rule below can help you find a quadratic equation if you know the roots.

Sum and Product of Roots

If the roots of $ax^2 + bx + c = 0$ with $a \neq 0$ are s_1 and s_2, then $s_1 + s_2 = -\frac{b}{a}$ and $s_1 \cdot s_2 = \frac{c}{a}$.

Example 1 **Write a quadratic equation that has roots $\frac{3}{4}$ and $-\frac{12}{5}$.**

Find the sum and product of the roots. Begin by expressing the sum and product of the roots with the same denominator.

$$s_1 + s_2 = \frac{3}{4} + \left(-\frac{12}{5}\right)$$

$$= \frac{15}{20} - \frac{48}{20} \text{ or } -\frac{33}{20}$$

$$s_1 \cdot s_2 = \left(\frac{3}{4}\right)\left(-\frac{12}{5}\right)$$

$$= -\frac{36}{20}$$

So, $-\frac{33}{20} = -\frac{b}{a}$ and $-\frac{36}{20} = \frac{c}{a}$.

Therefore, $a = 20$, $b = 33$, and $c = -36$. The equation is $20x^2 + 33x - 36 = 0$.

Check by solving the equation by factoring.

$$20x^2 + 33x - 36 = 0$$
$$(4x - 3)(5x + 12) = 0$$
$$4x - 3 = 0 \quad \text{or} \quad 5x + 12 = 0$$
$$4x = 3 \qquad 5x = -12$$
$$x = \frac{3}{4} \ \checkmark \qquad x = -\frac{12}{5} \ \checkmark$$

The method used in Example 1 can also be used with equations whose roots are imaginary.

Example 2 **Write a quadratic equation that has roots $7 - 3i$ and $7 + 3i$.**

$$s_1 + s_2 = (7 - 3i) + (7 + 3i)$$
$$= 14 \quad -\frac{b}{a} = \frac{14}{1}$$
$$s_1(s_2) = (7 - 3i)(7 + 3i)$$
$$= 49 + 9 \text{ or } 58 \quad \frac{c}{a} = \frac{58}{1}$$

Since $-\frac{b}{a} = \frac{14}{1}$ and $\frac{c}{a} = \frac{58}{1}$, $a = 1$, $b = -14$, $c = 58$. Replace a, b, and c in $ax^2 + bx + c = 0$ with these values. The resulting equation is $x^2 - 14x + 58 = 0$.

You can also use the sum and product of roots to check the solutions of an equation.

Example 3 **Solve $2x^2 - 7x + 3 = 0$. Check by using the sum and product of the roots.**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(3)}}{2(2)} \quad a = 2,\ b = -7, \text{ and } c = 3$$
$$= \frac{7 \pm \sqrt{25}}{4} \text{ or } \frac{7 \pm 5}{4}$$

The solutions are $\frac{7 + 5}{4}$ and $\frac{7 - 5}{4}$ or 3 and $\frac{1}{2}$.

Check: *The sum of the roots, $s_1 + s_2$, should be $-\frac{b}{a}$ or $\frac{7}{2}$.*

$3 + \frac{1}{2} = \frac{7}{2}$ ✓

The product or the roots, s_1s_2, should be $\frac{c}{a}$ or $\frac{3}{2}$.

$3\left(\frac{1}{2}\right) = \frac{3}{2}$ ✓

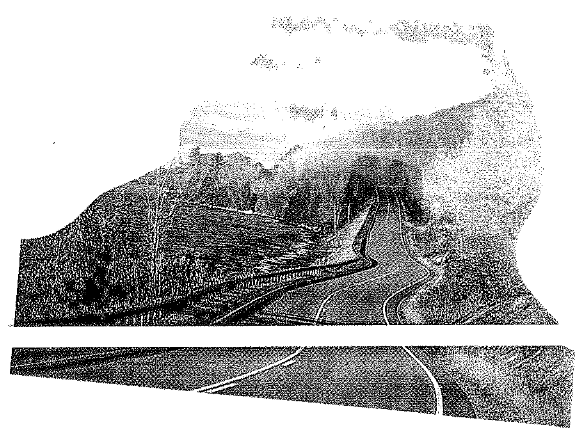
You can use the sum and product of roots to solve real-world problems.

Example 4

Real World APPLICATION

Engineering

Refer to the application at the beginning of the lesson. Find the equation of the transition curve if the formula for a road with a gradient of 3% is $y = ax^2 + 0.03x + c$.



Explore Read the problem and determine two points on the road. The points on the road are (200, 1105) and (1000, 1105). So $y = 1105$ when $x = 200$ and $x = 1000$.

(continued on the next page)

Plan Since $y = 1105$, rewrite the equation equal to 0.

$$1105 = ax^2 + 0.03x + c \qquad \textit{Substitute 1105 for } y.$$
$$0 = ax^2 + 0.03x + (c - 1105) \qquad \textit{Rewrite equation equal to 0.}$$

Let $b = 0.03$, $s_1 = 200$, and $s_2 = 1000$.

Solve Use the sum and product of the roots to find the equation.

$$s_1 + s_2 = -\frac{b}{a} \qquad s_1 \cdot s_2 = \frac{c}{a}$$
$$200 + 1000 = -\frac{(0.03)}{a} \qquad 200 \cdot 1000 = \frac{c - 1105}{a}$$

Why substitute $c - 1105$ for c?

$$1200 = -\frac{0.03}{a} \qquad 200{,}000 = \frac{c - 1105}{-0.000025}$$
$$a = -0.000025 \qquad -5 = c - 1105$$
$$1100 = c$$

Thus, the equation of the transition curve is $y = -0.000025x^2 + 0.03x + 1100$.

Examine Check to see if the equation makes sense. Substitute $x = 200$ and $x = 1000$ into the equation $y = -0.000025x^2 + 0.03x + 1100$.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **Write** the sum and the product of the roots of a quadratic equation expressed in terms of a, b, and c.
2. **Explain** how to use the zero product property to form a quadratic equation when its roots are known.
3. **Describe** how to create an equation when only the sum and product of the roots are known.

Math Journal

4. When might it be difficult to find an equation using the sum and product of roots? Give some examples.

Guided Practice

State the sum and product of the roots of each quadratic equation.

5. $x^2 - 12x + 22 = 0$
6. $x^2 - 22 = 0$
7. $3x^2 + 75 = 0$
8. $2x^2 - \frac{1}{4}x = \frac{-8}{15}$

Write a quadratic equation that has the given roots.

9. $4 \pm \sqrt{5}$
10. $\sqrt{5} \pm 8i$
11. $-7, \frac{2}{3}$
12. $\frac{-3}{4}, \frac{5}{8}$

Solve each equation. Check by using the sum and product of the roots.

13. $x^2 - 49 = 0$
14. $2x^2 + 15x = 27$
15. $16x^2 - 81 = 0$
16. Suppose a quadratic equation has two real roots, r_1 and r_2. If the sum of the roots is $-\frac{19}{2}$ and the product of the roots is -30, write a quadratic equation that has roots r_1 and r_2.

EXERCISES

Practice

Write a quadratic equation that has the given roots.

17. $6, -9$
18. $5, -1$
19. $2, \frac{5}{8}$
20. $6, 6$
21. $-\frac{2}{5}, \frac{2}{5}$
22. $-\frac{2}{5}, \frac{2}{7}$
23. $-4, -\frac{2}{3}$
24. $\frac{4}{3}, -\frac{1}{6}$
25. $4 \pm \sqrt{3}$
26. $\frac{3}{7} \pm 2i$
27. $\frac{-2 \pm 5i}{4}$
28. $\frac{-2 \pm 3\sqrt{5}}{7}$

Solve each equation. Check by using the sum and product of the roots.

29. $-2x^2 - 11x - 12 = 0$
30. $-3x^2 + 22x - 24 = 0$
31. $x^2 - 8x = 0$
32. $x^2 - 16 = 0$
33. $x^2 + \frac{1}{6}x - \frac{1}{3} = 0$
34. $\frac{1}{2}x^2 - \frac{13}{20}x - \frac{3}{20} = 0$
35. $x^2 - 8x - 18 = 0$
36. $2x^2 + 10x = -10$
37. $4x^2 - 31x - 45 = 0$
38. $3x^2 + 17 = 0$

39. Write a quadratic equation whose roots satisfy the following conditions.
    a. The sum of the roots is 4. The product of the roots is $\frac{13}{12}$.
    b. The sum of the roots is $\frac{1}{6}$. The product of the roots is $\frac{5}{21}$.

Critical Thinking

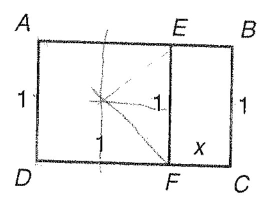
40. Find a value k such that -3 is a root of $2x^2 + kx - 21 = 0$.
41. Find a value k such that $\frac{1}{2}$ is a root of $2x^2 + 11x = -k$.
42. List all possible integral roots of $x^2 + bx - 24 = 0$. Assume that b represents an integer.

Applications and Problem Solving

Real World

43. **Space Flight** The United States is currently researching a project called the National Aerospace Plane, which would be able to regularly fly passengers directly into space. If the space plane is successful, it will be able to take off from an airport and rocket to orbit by accelerating to Mach 25 or 17,700 miles per hour. Scientists exploring the concepts of velocity and acceleration in space launch a model rocket that returns to Earth 18 seconds after takeoff. Use the formula $h = v_i t - \frac{1}{2}gt^2$, where $g = 9.8$ m/s^2, to determine the initial velocity of the rocket. *Remember that the rocket starts on Earth when $t = 0$.*

44. **Geometry** Imagine that square *AEFD* is cut off the end of rectangle *ABCD* at the right. The remaining rectangle *EBCF* has the same ratio of length to width as the original rectangle *ABCD*. A rectangle with those similarities is called a "golden rectangle." Because of their pleasing shape, golden rectangles can be found in ancient and modern architecture.



    a. Find the ratio of length to width for each rectangle.
    b. Set the ratios equal, write the equation, and solve for x. *Remember that measures cannot be negative.*
    c. Substitute the value of x into the ratios you found in part a. What number do you get? This number is known as the *golden ratio.*

Mixed Review

45. Solve $x^2 - 2x - 35 = 0$ by using the quadratic formula. (Lesson 6–4)

46. Solve $m^2 + 3m - 180 = 0$ by completing the square. (Lesson 6–3)

47. Graph the function $f(x) = x^2 + 8x - 5$. Name the vertex and axis of symmetry. (Lesson 6–1)

48. Simplify $\sqrt{108} - \sqrt{48} + (\sqrt{3})^3$. (Lesson 5–6)

49. Factor $4x^2 - 9$. (Lesson 5–4)

50. Solve the system $3a - b + 2c = 7$, $-a + 4b - c = 3$, and $a + 4b - c = 1$ by using augmented matrices. (Lesson 4–7)

51. **ACT Practice** If x and k are both greater than zero, and if $4x^2k^2 + xk - 33 = 0$, then what is the value of k in terms of x?

A $-\frac{3}{x}$ B $\frac{-11}{4x}$ C $\frac{3}{4x}$ D $\frac{3}{x}$ E $\frac{11}{4x}$

For **Extra Practice**, see page 890.

52. Graph $y = -|5x - 12| + 1$. (Lesson 2–6)

Mathematics and SOCIETY

Mathematics and Juggling

The excerpt below appeared in an article in *New Scientist* on March 18, 1995.

MATHEMATICS AND MUSIC, IT IS SAID, often go together. Mathematicians are also reputed to be unusually good at chess. But there is a less well-known activity that also enjoys a time-honoured association with mathematics: juggling. Although the connection may seem tenuous at first sight, there has recently been a rush to apply mathematics to juggling, as mathematicians have come up with a clever way to invent new juggling patterns. . . . The pure mathematics of juggling concerns itself only with the patterns of throws, ignoring detail such as the precise timings, the exact trajectories and even the objects being thrown. Some theoretically minded jugglers recently introduced an idea called "site swaps" to describe the patterns in a compact form. And last year, four mathematicians—turned it into a mathematical theory. . . . (Site swap notation) is an easy way to remember a wide range of patterns that look very impressive when performed. The advantage for mathematicians is that a neat, compact notation makes it easier to count or classify patterns. ■

1. What are the characteristics of the motions of objects being juggled?
2. If you were to construct a mathematical theory about juggling, what are some of the variables that might be involved?
3. Use your creativity to design a new juggling pattern that could actually be performed. What are some factors that limit any pattern design?

6–6A Graphing Technology Families of Parabolas

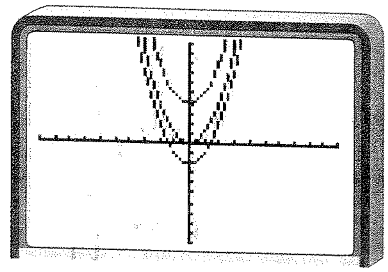
A Preview of Lesson 6–6

The equations for the parabolas in a family are closely related. In the general form of a quadratic equation, $y = a(x - h)^2 + k$, a, h, and k may change. Changing the value of a, h, or k results in a different parabola in the family.

Example 1 **Graph the following equations on the same screen in the standard viewing window. Describe any similarities and differences among the graphs.**

The parent graph in this example is the graph of $y = x^2$.

$y = x^2$, $y = x^2 + 4$, $y = x^2 - 2$



Enter: [Y=] [X,T,θ,n] [x^2] [ENTER]
[X,T,θ,n] [x^2] [+] 4 [ENTER]
[X,T,θ,n] [x^2] [−] 2 [ZOOM] 6

You can also graph the three equations by entering the following single equation.

Enter: [Y=] [X,T,θ,n] [x^2] [+] [2nd] [{] 0 [,]
4 [,] [(−)] 2 [2nd] [}] [ZOOM] 6

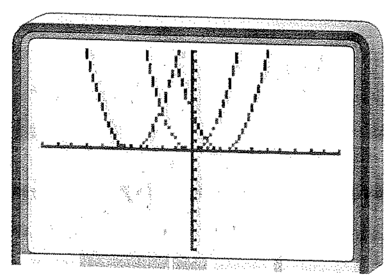
The graphs have the same shape and all open upward. The vertex of each graph is on the y-axis. However, the graphs have different vertical positions.

Example 1 shows how changing the value of k in the equation $y = a(x - h)^2 + k$ translates the parabola along the y-axis. If $k > 0$, the parabola is translated k units upward, and if $k < 0$, it is translated k units downward. How do you think changing the value of h will change the graphs in a family of parabolas?

Example 2 **Graph the following equations in the standard viewing window and describe any similarities and differences among the graphs.**

The parent graph in this example is the graph of $y = x^2$.

$y = x^2$, $y = (x + 4)^2$, $y = (x - 2)^2$



Enter: [Y=] [X,T,θ,n] [x^2] [ENTER] [(]
[X,T,θ,n] [+] 4 [)] [x^2] [ENTER] [(]
[X,T,θ,n] [−] 2 [)] [x^2] [ZOOM] 6

You can also graph the three equations by entering the following single equation.

Enter: [Y=] [(] [X,T,θ,n] [+] [2nd]
[{] 0 [,] 4 [,] [(−)] 2 [2nd] [}] [)] [x^2] [ZOOM] 6

The graphs have the same shape and all open upward. The vertex of each graph is on the x-axis. However, the graphs have different horizontal positions.

Example 2 demonstrates that changing the value of h in $y = a(x - h)^2 + k$ translates the graph horizontally. If $h > 0$, the graph translates to the right h units. If $h < 0$, the graph translates to the left h units.

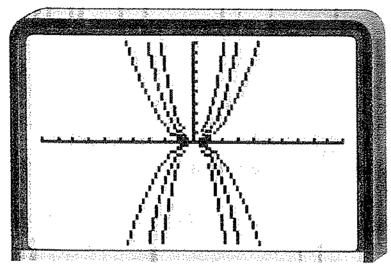
Changing the value of a in $y = a(x - h)^2 + k$ affects the direction of the opening and the shape of the graph. If $a > 0$, the graph opens upward, and if $a < 0$, the graph opens downward. If $|a| < 1$, the graph is wider than the graph of $y = x^2$, and if $|a| > 1$, then the graph is narrower than the graph of $y = x^2$. Graphs of equations with a values that have the same absolute value, such as $y = 2x^2$ and $y = -2x^2$, have the same shape.

Example 3

The parent graph in this example is the graph of $y = x^2$.

Graph the following equations in the standard viewing window and describe any similarities and differences among the graphs.
$y = x^2, y = -x^2, y = 2x^2, y = -2x^2, y = 0.5x^2, y = -0.5x^2$

Enter: [Y=] [X,T,θ,n] [x^2] [ENTER]
[(−)] [X,T,θ,n] [x^2] [ENTER]
2 [X,T,θ,n] [x^2] [ENTER]
[(−)] 2 [X,T,θ,n] [x^2] [ENTER]
.5 [X,T,θ,n] [x^2] [ENTER]
[(−)] .5 [X,T,θ,n] [x^2] [ZOOM] 6



The graphs of $y = x^2$, $y = 2x^2$, and $y = 0.5x^2$ open upward, while the graphs of $y = -x^2$, $y = -2x^2$, and $y = -0.5x^2$ open downward. The graphs of $y = 2x^2$ and $y = -2x^2$ are narrower than the graph of $y = x^2$, while the graphs of $y = 0.5x^2$ and $y = -0.5x^2$ are wider than the graph of $y = x^2$.

EXERCISES

Study the lesson. Then complete the following.

1. Describe the effect that changing the value of k in an equation of the form $y = a(x - h)^2 + k$ has on the graph of the equation. Give an example.
2. Describe the effect that changing the value of h in an equation of the form $y = a(x - h)^2 + k$ has on the graph of the equation. Give an example.
3. How do the graphs of $y = a(x - h)^2 + k$ and $y = -a(x - h)^2 + k$ compare? Give an example.

Examine each pair of equations below and predict the graphs for each. Then use a graphing calculator to confirm your results. Write one sentence that compares the two graphs.

4. $y = x^2, y = (x + 6)^2$
5. $y = x^2, y = (x - 8)^2$
6. $y = x^2, y = x^2 + 1.5$
7. $y = x^2, y = x^2 - 11$
8. $y = -x^2, y = -5x^2$
9. $y = x^2, y = -2x^2$
10. $y = x^2, y = -\frac{1}{2}x^2 + 4$
11. $y = -\frac{1}{3}x^2, y = -\frac{1}{3}x^2 + 2$
12. $y = x^2, y = -6(x + 1)^2 - 11$
13. $y = (x + 2)^2 + 1, y = (x + 2)^2 - 4$
14. $y = 2(x + 3)^2 + 1,$ $y = 4(x + 3)^2 + 1$
15. $y = 2(x - 4)^2 + 3,$ $y = \frac{1}{2}(x - 4)^2 - 5$

6-6 Analyzing Graphs of Quadratic Functions

What YOU'LL LEARN

- To graph quadratic functions of the form $y = a(x - h)^2 + k$, and
- to determine the equation of a parabola by using points on its graph.

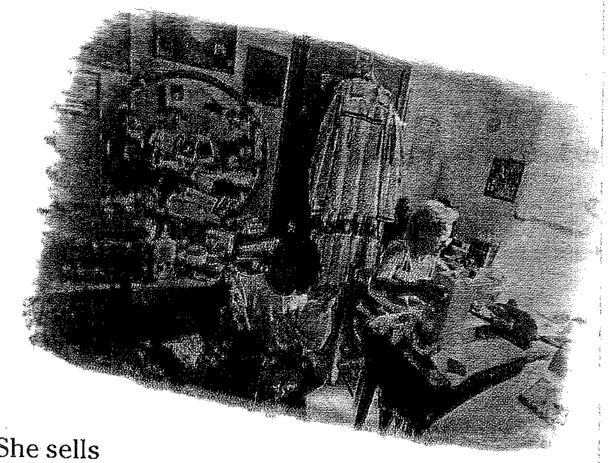
Why IT'S IMPORTANT

You can graph quadratic functions to solve problems involving biology and number theory.

Real World APPLICATION

World Cultures

The Cherokee Nation is a federally recognized sovereign nation that has its own court system, legislature, and tax commission.

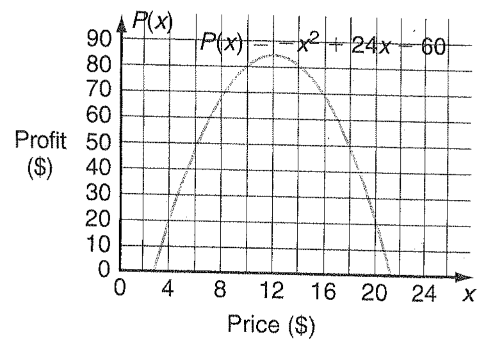


Lorene Drywater, a Cherokee Nation worker in Tahlequah, Oklahoma, makes buffalo grass dolls, a craft she learned from her mother. She sells the dolls to tourists for additional income. As anyone who sells things can tell you, deciding on an appropriate price is very important. If your price is too low, you will not make much of a profit. If your price is too high, you will also probably not make much of a profit because fewer people will buy what you are selling. The best price is the price that leads to the maximum profit.

FYI

Cherokee Nation Industries, Inc. employs nearly 300 workers, most of whom are Cherokee. These skilled technicians work on equipment for commercial aircraft, M1 Abrams tanks, multilaunch rocket systems, and NATO spy planes.

Suppose Ms. Drywater's profit $P(x)$ can be found by $P(x) = -x^2 + 24x - 60$, where x represents the price of each doll. What price should she charge to receive the maximum profit?



You can see from this graph that the vertex is (12, 84) and the axis of symmetry is $x = 12$. Thus, she should charge \$12 for each doll to receive the maximum profit of \$84. Is there a way to find this information without graphing the function?

You know that the graph of $y = ax^2 + bx + c$ is a parabola. Quadratic functions can also be expressed in the general form $y = a(x - h)^2 + k$. We can write $P(x) = -x^2 + 24x - 60$ in the general form by completing the square.

$$\begin{aligned} P(x) &= -x^2 + 24x - 60 \\ &= -1(x^2 - 24x) - 60 \\ &= -1(x^2 - 24x + 144) - 60 + 144 \\ &= -1(x - 12)^2 + 84 \end{aligned}$$

Add and subtract $\left(\frac{-24}{2}\right)^2$ or 144 to obtain an equivalent equation.

In this equation, $h = 12$ and $k = 84$. Compare these values with the coordinates of the vertex and axis of symmetry you found above by graphing. What pattern do you notice?

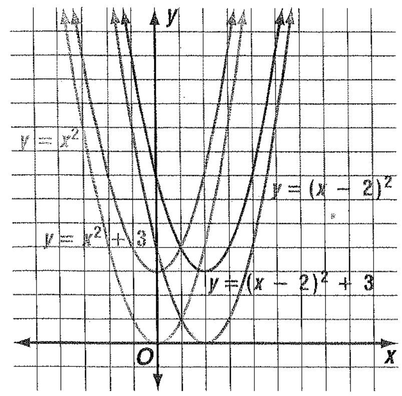
Fabulous FIRSTS

Wilma Mankiller (1945–)

Wilma Mankiller was the first and only woman elected chief of the Cherokee Nation, the second largest Native American tribe in the U.S. She served as chief for ten years. She resigned her position in 1995.

The graphs of $y = x^2$, $y = (x - 2)^2$, $y = x^2 + 3$, and $y = (x - 2)^2$ are shown at the right on the same set of axes. Study these graphs.

Equation	Vertex	Axis of Symmetry
$y = x^2$	(0, 0)	$x = 0$
$y = (x - 2)^2$	(2, 0)	$x = 2$
$y = x^2 + 3$	(0, 3)	$x = 0$
$y = (x - 2)^2 + 3$	(2, 3)	$x = 2$



Notice that the graphs all have the same shape. The difference is their position.

We can express the equations for these parabolas in the general form $y = (x - h)^2 + k$. When a quadratic function is written in this form, the vertex is (h, k), and the equation of the axis of symmetry is $x = h$.

Equation	General Form	h	k
$y = x^2$	$y = (x - 0)^2 + 0$	0	0
$y = (x - 2)^2$	$y = (x - 2)^2 + 0$	2	0
$y = x^2 + 3$	$y = (x - 0)^2 + 3$	0	3
$y = (x - 2)^2 + 3$	$y = (x - 2)^2 + 3$	2	3

In Chapter 4, you learned that a translation slides a figure on the coordinate plane without changing its shape or size. As the values of h and k change, the graph of $y = a(x - h)^2 + k$ is the graph of $y = x^2$ translated $|h|$ units left or right and $|k|$ units up or down. If h is positive, the parabola is translated to the right. If h is negative, it is translated to the left. Likewise, for k, the translation is up if k is positive and down if k is negative.

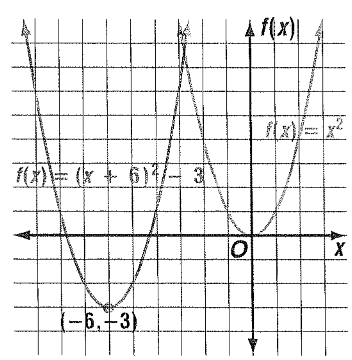
Example 1 **Name the vertex and the axis of symmetry for the graph of $f(x) = (x + 6)^2 - 3$. Then graph the function. How is this graph different from the graph of $f(x) = x^2$?**

This function can be rewritten as $f(x) = [x - (-6)]^2 + (-3)$. Then $h = -6$ and $k = -3$. The vertex is at $(-6, -3)$, and the axis of symmetry is $x = -6$.

Finding several points on the graph makes graphing easier.

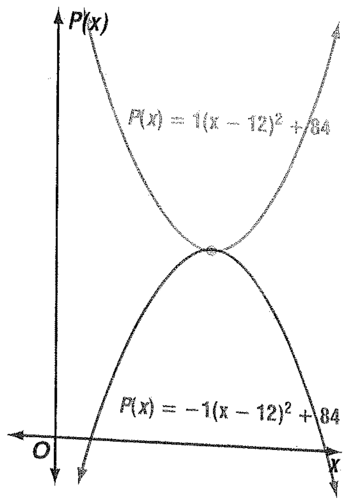
It is helpful to choose points that are close to h and on either side of h.

x	$(x + 6)^2 - 3$	$f(x)$
−8	$(-8 + 6)^2 - 3$	1
−7	$(-7 + 6)^2 - 3$	−2
−6	$(-6 + 6)^2 - 3$	−3
−5	$(-5 + 6)^2 - 3$	−2
−4	$(-4 + 6)^2 - 3$	1
−3	$(-3 + 6)^2 - 3$	6



The shape of the graph is the same as the shape of the graph of $f(x) = x^2$, but it is translated 6 units left and 3 units down. *Notice that points with the same y-coordinates are the same distance from the axis of symmetry, $x = -6$.*

How does the value of a in the general form $y = a(x - h)^2 + k$ affect a parabola? Consider the equation in the application at the beginning of the lesson.



Graph $P(x) = -1(x - 12)^2 + 84$ and $P(x) = 1(x - 12)^2 + 84$ on the same coordinate plane and compare the graphs.

The graphs have the same vertex and are shaped the same. The graph of $P(x) = -1(x - 12)^2 + 84$ opens downward, and the graph of $P(x) = 1(x - 12)^2 + 84$ opens upward.

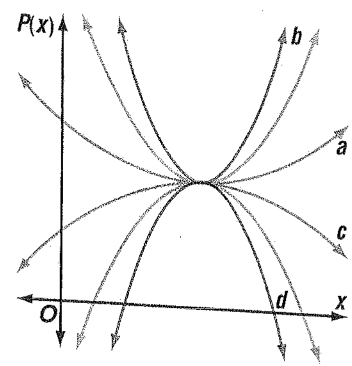
Graph the following functions to further investigate the relationships between similar functions. What do you find?

a. $P(x) = \frac{1}{4}(x - 12)^2 + 84$

b. $P(x) = 2(x - 12)^2 + 84$

c. $P(x) = -\frac{1}{4}(x - 12)^2 + 84$

d. $P(x) = -2(x - 12)^2 + 84$



All of the graphs for quadratic functions a, b, c, and d have the vertex (12, 84) and the axis of symmetry $x = 12$. When a is negative, the graph opens downward. When a is positive, the graph opens upward. As the value of $|a|$ increases, the graph becomes narrower.

The chart below summarizes the characteristics of the graph of $y = a(x - h)^2 + k$.

$y = a(x - h)^2 + k$	a is positive.	a is negative.
Vertex	(h, k)	(h, k)
Axis of Symmetry	$x = h$	$x = h$
Direction of Opening	upward	downward
As the value of $\lvert a \rvert$ increases, the graph of $y = a(x - h)^2 + k$ narrows.		

Example 2 **Graph $f(x) = -5x^2 + 80x - 319$. Name the vertex, axis of symmetry, and direction of opening for the graph.**

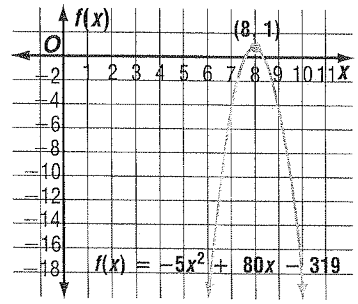
Write the function in the form $f(x) = a(x - h)^2 + k$ by completing the square.

$$\begin{aligned} f(x) &= -5x^2 + 80x - 319 \\ &= -5(x^2 - 16x) - 319 \\ &= -5(x^2 - 16x + 64) - 319 - (-5)(64) \quad \textit{Why subtract } (-5)(64)? \\ &= -5(x - 8)^2 + 1 \end{aligned}$$

The general form of this function is $f(x) = -5(x - 8)^2 + 1$. So, $a = -5$, $h = 8$, and $k = 1$. The vertex is at (8, 1), and the axis of symmetry is $x = 8$. Since $a = -5$, the graph opens downward and is narrower than the graph of $f(x) = (x - 8)^2$.

(continued on the next page)

x	$-5(x-8)^2+1$	$f(x)$
6	$-5(6-8)^2+1$	−19
7	$-5(7-8)^2+1$	−4
8	$-5(8-8)^2+1$	1
9	$-5(9-8)^2+1$	−4
10	$-5(10-8)^2+1$	−19

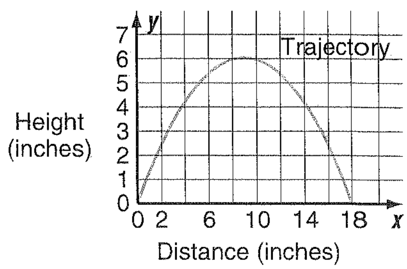


One way to find the equation of a parabola is by using the values of the vertex and one other point on the graph.

Example 3

CONNECTION Biology

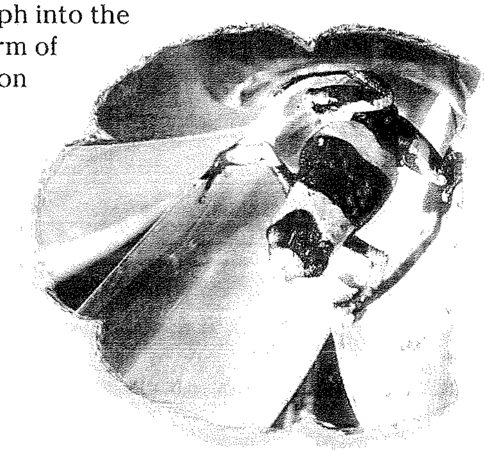
A deadly frog found in an area of rain forest in western Colombia can be lethal even to the touch because it exudes a toxic substance. Suppose the graph below represents the path, or trajectory, that one of the frogs takes while hopping through the rain forest. Write the equation of the parabola.



The vertex of the parabola is at (9, 6). So $h = 9$ and $k = 6$.

Substitute the values of h and k and the coordinates of one other point on the graph into the general form of the equation and solve for a.

Use (0, 0).



$y = a(x-h)^2 + k$

$0 = a(0-9)^2 + 6$ *Substitute 9 for h, 6 for k, 0 for x, and 0 for y.*

$-6 = a(-9)^2$

$a = -\frac{6}{81}$ or $-\frac{2}{27}$

The equation of the parabola is

$y = -\frac{2}{27}(x-9)^2 + 6$ or $y = -\frac{2}{27}x^2 + \frac{4}{3}x$.

FYI

The launch angle of a frog's jump is approximately 45°. This helps the frog cover maximum distance on flat ground.

It is also possible to write the equation of a parabola if you know three of its points.

Example 4

Write the equation of the parabola that passes through the points at (0, −3), (1, 4), and (2, 15).

Each point should satisfy the equation of the parabola. Using the form of the equation $y = ax^2 + bx + c$, find a, b, and c by substituting the coordinates of the points into the quadratic form of the equation.

Ordered Pairs	Substitution	Simplify
(0, −3)	$-3 = a(0)^2 + b(0) + c$	$-3 = c$
(1, 4)	$4 = a(1)^2 + b(1) + c$	$4 = a + b + c$
(2, 15)	$15 = a(2)^2 + b(2) + c$	$15 = 4a + 2b + c$

LOOK BACK

Refer to Lesson 3-7 for information on solving systems of equations in three variables.

Now solve the system of equations. From the first equation, $c = -3$, so substitute -3 for c in the other two equations.

$$4 = a + b - 3 \quad \rightarrow \quad 7 = a + b$$
$$15 = 4a + 2b - 3 \quad \rightarrow \quad 18 = 4a + 2b$$

Solve by using elimination.

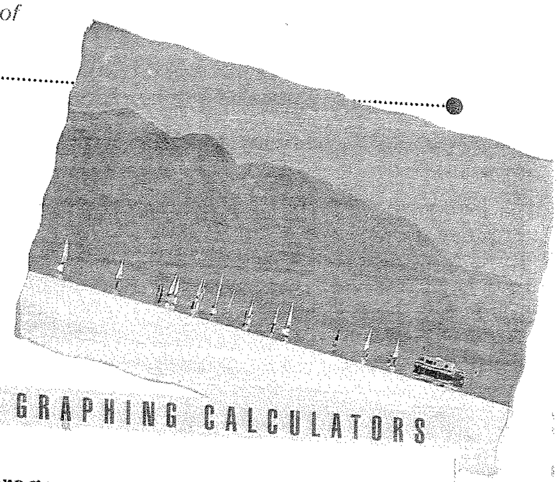
$$\begin{aligned} 7 &= a + b \\ 18 &= 4a + 2b \end{aligned} \quad \xrightarrow{\text{Multiply by } -2} \quad \begin{aligned} -14 &= -2a - 2b \\ 18 &= 4a + 2b \\ \hline 4 &= 2a \\ 2 &= a \end{aligned}$$

Now substitute 2 for a and solve for b.

$$7 = 2 + b$$
$$b = 5$$

The solution to the system is $(2, 5, -3)$. So the equation of the parabola is $y = 2x^2 + 5x - 3$. *Check to see if each of the three points satisfies the equation.*

You can use a graphing calculator to model real-world data.



EXPLORATION GRAPHING CALCULATORS

The data in the table represent the average number of minutes of exposure to the sun required to redden untanned Caucasian skin.
Source: *Mesa Tribune* (Mesa, Arizona)

Time of Day	Number of Minutes
9 A.M.	34
10 A.M.	20
11 A.M.	15
noon	13
1 P.M.	14
2 P.M.	18
3 P.M.	32
4 P.M.	60

Your Turn

a. Use the Edit option on the STAT menu feature to list the ordered pairs. Let x represent the number of hours since 8 A.M. Use the window $[-1, 9]$ with a scale factor of 1 and $[0, 65]$ with a scale factor of 5. Then press [2nd] [STAT PLOT] 1 [ENTER] [▼] [ENTER] [GRAPH].

b. To find a quadratic equation whose graph best fits the data, go to the CALC option on the STAT menu. Then choose 6, for QuadReg, and enter [2nd] [L1] [,] [2nd] [L2]. Then press [ENTER].

c. Write the equation of the graph that best fits the data. Then press [Y=] [VARS] 5 [▶] [▶] 7 [GRAPH].

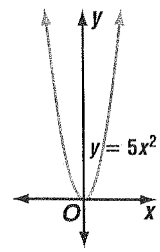
d. How well do you feel the graph of the equation fits the data? Justify your answer.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **Describe** the difference between the graphs of $y = 4(x + 6)^2 - 2$ and $y = -\frac{1}{2}(x + 6)^2 - 2$.
2. **List** the information you need to find the equation of a parabola.
3. **Analyze** the equation $y = -\frac{1}{6}(x + 4)^2 + 7$ and sketch its graph.
4. **Write** the equation of the function whose graph is 1 unit to the right and 4 units down from the graph shown at the right.



5. **You Decide** Leticia and Marisel both wrote the equation $f(x) = 2x^2 - 12x - 3$ in general form by completing the square. However, they each got different answers. Look at their solutions and find which one is correct. Describe the error in the other person's solution.

Leticia

$$\begin{aligned} f(x) &= 2x^2 - 12x - 3 \\ &= 2(x^2 - 6x) - 3 \\ &= 2(x^2 - 6x + 9) - 3 - 9 \\ &= 2(x - 3)^2 - 12 \end{aligned}$$

Marisel

$$\begin{aligned} f(x) &= 2x^2 - 12x - 3 \\ &= 2(x^2 - 6x) - 3 \\ &= 2(x^2 - 6x + 9) - 3 - 18 \\ &= 2(x - 3)^2 - 21 \end{aligned}$$

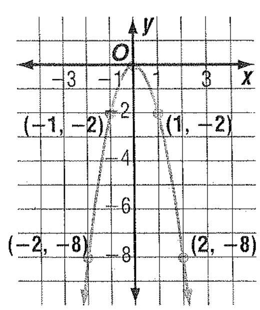
Guided Practice

Write each equation in the form $y = a(x - h)^2 + k$ if not already in that form. Then name the vertex, axis of symmetry, and direction of opening for the graph of each quadratic function.

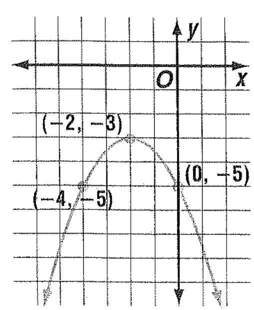
6. $f(x) = -2(x + 3)^2$
7. $f(x) = 5x^2 - 6$
8. $f(x) = x^2 - 4x + 5$
9. $f(x) = -3x^2 + 12x$

Write an equation for each parabola.

10.



11.



Write an equation for the parabola that passes through the given points.

12. (0, 2), (2, 2), (3, 4)
13. (1, 6), (−2, 27), (2, 11)

Graph each function.

14. $f(x) = 5(x + 3)^2 - 1$
15. $f(x) = -4x^2 + 16x - 11$
16. $f(x) = \frac{1}{3}(x - 1)^2 + 2$
17. Write the equation of a parabola with position 5 units below the parabola with equation $f(x) = 3x^2$.

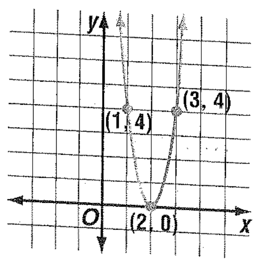
EXERCISES

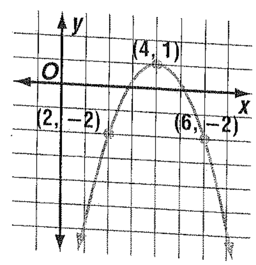
Practice

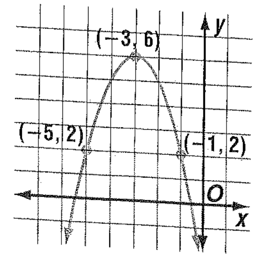
Write each equation in the form $y = a(x - h)^2 + k$ if not already in that form. Then name the vertex, axis of symmetry, and direction of opening for the graph of each quadratic function.

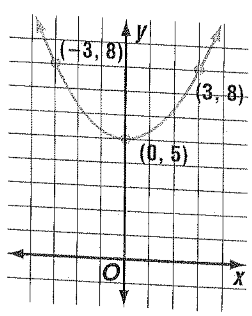
18. $f(x) = 4(x + 3)^2 + 1$
19. $f(x) = -(x + 11)^2 - 6$
20. $f(x) = -2(x - 2)^2 - 2$
21. $f(x) = 3(x - \frac{1}{2})^2 + \frac{1}{4}$
22. $f(x) = x^2 + 6x - 3$
23. $f(x) = -x^2 - 4x + 8$
24. $f(x) = 4x^2 + 24x$
25. $f(x) = -6x^2 + 24x$
26. $f(x) = 3x^2 - 18x + 11$
27. $f(x) = -2x^2 - 20x - 50$
28. $f(x) = -\frac{1}{2}x^2 + 5x - \frac{27}{2}$
29. $f(x) = \frac{1}{3}x^2 - 4x + 15$

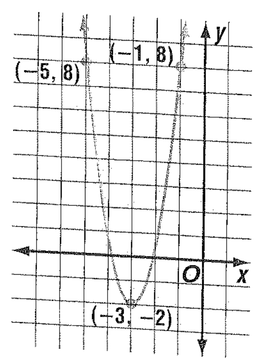
Write an equation for each parabola.

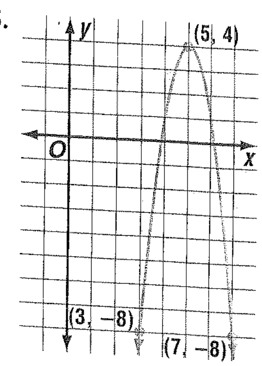
30.


31.


32.


33.


34.


35.


Write an equation for the parabola that passes through the given points.

36. (0, 0), (2, 6), (−1, 3)
37. (2, −3), (0, −1), $\left(-1, \frac{3}{2}\right)$
38. (1, 0), (3, 38), (−2, 48)
39. (−1, −10), (0, 6), (2, 88)

Graph each function.

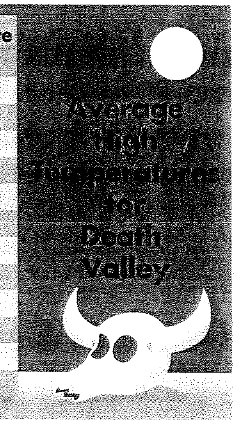
40. $f(x) = 3(x + 3)^2$
41. $f(x) = 2(x + 3)^2 - 5$
42. $f(x) = \frac{1}{2}(x + 3)^2 - 5$
43. $f(x) = \frac{1}{3}(x - 1)^2 + 3$
44. $f(x) = x^2 + 6x + 2$
45. $f(x) = -2x^2 + 16x - 31$
46. $f(x) = -5x^2 - 40x - 80$
47. $f(x) = 2x^2 + 8x + 10$
48. $f(x) = -9x^2 - 18x - 6$
49. $f(x) = -0.25x^2 - 2.5x - 0.25$

50. Write an equation for a parabola whose vertex is at (6, 1) and for which $a = 9$.
51. Write the equation of a parabola with position 2 units to the right and 9 units above the parabola with equation $f(x) = -2x^2$.

Graphing Calculators

52. A recent article listed the average high temperatures for Death Valley, California, the hottest place in the United States.
    a. Use a graphing calculator to model the data at the right. Find a quadratic equation whose graph best fits the data.
    b. Do you think the graph of the equation fits the data? Justify your answer.
    c. How would you expect a graph representing the average low temperatures to compare with the graph you found above?

Month	Temperature (°F)
January	65
February	73
March	81
April	88
May	100
June	110
July	116
August	113
September	106
October	91
November	75
December	66



Source: *USA Today*

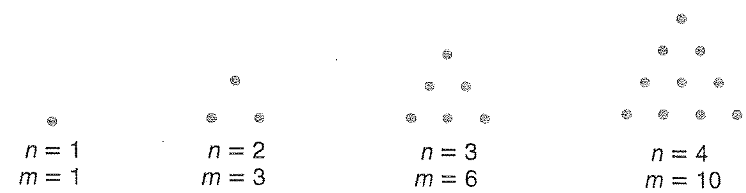
Critical Thinking

53. Given $f(x) = ax^2 + bx + c$ with $a \neq 0$, complete the square and rewrite the equation in the form $f(x) = a(x - h)^2 + k$. State an expression for h and k in terms of a, b, and c.

54. Sketch the family of graphs $y = (x + 2)^2 + 3$, $y = (x + 2)^2$, and $y = (x + 2)^2 - 3$. Fully describe the nature of the graphs and the roots of each equation.

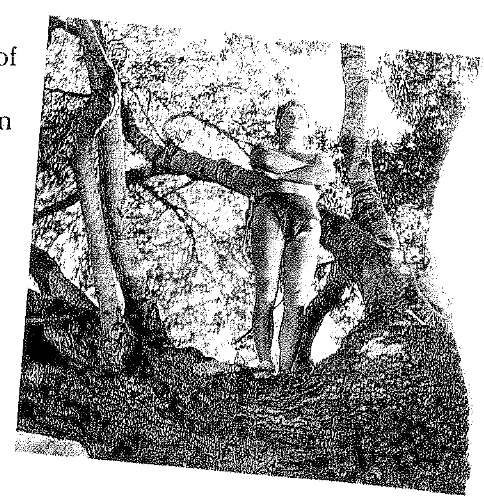
Applications and Problem Solving

55. **Number Theory** *Triangular numbers* are numbers that can be represented by a triangular array of m dots, with n dots on each side.



    a. The relationship between the number of dots on each side and the total number of dots can be modeled by a quadratic function. Identify a reasonable domain and range for this function.
    b. Write the quadratic function.
    c. Graph the function. What is the least triangular number?

56. **Entertainment** Did you know that Tarzan has been featured in 43 movies? The story of the man who swings through the trees has fascinated us for years. Suppose Tarzan is in a tree 14 meters above the ground and he decides to use a vine to swing to another tree. One second after he begins his swing, he is 10.5 meters above the ground. Three seconds after he begins his swing, he is 6.5 meters above the ground.



    a. Write three ordered pairs to represent the situation.
    b. Write an equation for the parabola that passes through these points.
    c. Sketch the graph of the parabola.
    d. It seems that Tarzan gets pretty close to the ground as he swings through the trees. How close does he actually get?

Mixed Review

57. **Horticulture** Helene Jonson has a rectangular garden 25 feet by 50 feet. She wants to increase the garden on all sides by an equal amount. If the area of the garden will be increased by 400 square feet, by how much will each dimension be increased? (Lesson 6–5)

58. Solve $4x^2 - 8x + 13 = 0$ by using the quadratic formula. (Lesson 6–4)

59. Graph $h(x) = x^2 - 2x + 5$. Name the vertex and the axis of symmetry. (Lesson 6–1)

60. **SAT Practice** If x is an integer, then which of the following must also be integers?

I. $\frac{9x + 9}{9x}$ II. $\frac{9x + 9}{x + 1}$ III. $\frac{9x^2 + x}{9x}$

A I only **B** II only **C** III only **D** I and II **E** II and III

61. Simplify $(-x + 4)(-2 - 3x)$. (Lesson 5–2)

62. Find $\begin{bmatrix} -6 & 3 \\ 4 & 7 \end{bmatrix} \cdot \begin{bmatrix} 2 & -5 \\ -3 & 6 \end{bmatrix}$. (Lesson 4–3)

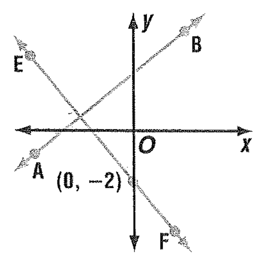
63. Find $-2\begin{bmatrix} -3 & 0 & 12 \\ -7 & \frac{1}{3} & 4 \end{bmatrix}$. (Lesson 4–1)

64. Solve the system of equations. (Lesson 3–7)

$x + 2y - z = -7$

$3x + y + z = -12$

$4z = 24$



65. Refer to the graph at the right. The slope of $\overleftrightarrow{AB}$ is $\frac{4}{3}$. Line EF is perpendicular to AB and has a y-intercept of -2. Write the equation of EF. (Lesson 2–4)

66. Solve $23x - 7 > 62$. (Lesson 1–6)

For Extra Practice, see page 890.

WORKING ON THE In•ves•ti•ga•tion

Refer to the Investigation on pages 328–329.

the River Canyon Bridge

The formula for a parabolic curve whose vertex lies along the y-axis is $ax^2 + y = b$, where b represents the y-coordinate of the vertex.

1 Knowing the value of b, how might you determine the value of a for a particular bridge design? Describe your method for finding a.

2 Depending on the arch of the bridge, the value of a varies. For the bridges you might consider, the value of a must be within certain ranges. Describe possible values for a and justify your answer.

3 Determine the values of a and b for each of your designs and for the design proposed by the consultant in the Investigation in Lesson 6–1. With this information, write an equation for the parabolic curve of the arch in each design.

Add the results of your work to your Investigation Folder.

6–7A Graphing Technology Quadratic Inequalities

A Preview of Lesson 6–7

To graph quadratic inequalities in two variables, we will use a procedure similar to that discussed in Lesson 2-7A on graphing linear inequalities. We will utilize the SHADE(feature found in the DRAW menu.

Example 1 **Graph $y \leq 0.5x^2 + x - 3$ in the standard viewing window.**

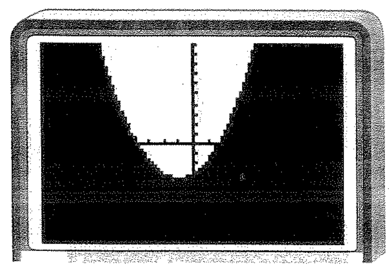
You will recall that you must enter two functions when graphing an inequality in two variables. The calculator shades between the designated functions. The first function entered is the lower boundary of the region to be shaded and the second function is the upper boundary of the region.

Since the inequality asks for all points such that *y is less than or equal to* $0.5x^2 + x - 3$, we will shade below the parabola. The lower boundary will be $y = -10$, and the upper boundary will be $y = 0.5x^2 + x - 3$.

Before you begin, clear any functions stored in the Y = list. Then clear the DRAW menu by pressing [2nd] [DRAW] [ENTER] [ENTER] [CLEAR].

Make sure Ymin is greater than or equal to −10.

Enter: [ZOOM] 6 [2nd] [DRAW] 7
[(−)] 10 [,] .5 [X,T,θ,n] [x^2]
[+] [X,T,θ,n] [−] 3 [)]
[ENTER]



To graph an inequality that has a > or ≥ sign, the function itself will be the lower boundary.

Example 2 **Graph $y \geq x^2 - 4x + 1$ in the standard viewing window.**

Since the inequality asks for all points such that *y is greater than or equal to* $x^2 - 4x + 1$, we will shade above the parabola. The lower boundary will be $x^2 - 4x + 1$, and the upper boundary will be $y = 10$. *Be sure to clear the DRAW menu first.*

Make sure Ymax is less than or equal to 10.

Enter: [ZOOM] 6 [2nd] [DRAW] 7
[X,T,θ,n] [x^2] [−] 4 [X,T,θ,n]
[+] 1 [,] 10 [)] [ENTER]