Focus On

PREREQUISITE SKILLS

To be successful in this chapter, you'll need to understand these concepts and be able to apply them. Refer to the lesson in parentheses if you need more review before beginning the chapter.

Solve equations for a specific variable.

Solve each equation for *y*.

1. $3x - 2y = 12$

2. $y + x = 0$

3. $0.25y - 0.5x = 4.5$

4. $2.25x + 1.5y = 1$

Graph linear equations.

Graph each equation.

5. $4y - 5x = 10$

6. $x + y = -2$

7. $y + 2x = 6$

Determine if two lines are parallel, perpendicular, or neither.

Find the slope of the graph for each equation in the pair. Then state whether the lines are parallel, perpendicular, or neither.

8. $x + y = 0; x - y = 0$

9. $2x + 3y = 6; 2x - 3y = 6$

10. $x - 4y = 5; 4y - x = 2$

11. $2x + 3y = 6; 3x - 2y = 6$

Draw graphs of inequalities in two variables.

Graph each inequality.

12. $x - y > 2$

13. $2x + y \leq -1$

14. $y < |x| + 4$

Focus On

READING SKILLS

In this chapter, you will learn to solve ***systems of linear equations and inequalities***. In the dictionary, a system is defined as "an interacting group of items forming a unified whole." A system of equations or inequalities is two or more equations or inequalities that act together to define a mathematical situation or relationship between variables. A system of equations or inequalities is often graphed on one coordinate plane to show the relationship between the variables and to find the solution to the system.

3–1A Graphing Technology Systems of Equations

A Preview of Lesson 3–1

You can use a graphing calculator to graph systems of equations because several equations can be graphed on the screen at the same time. If a system of equations has a solution, it is located where the graphs intersect. The coordinates of the intersection point, (x, y), can be determined by using the TRACE function.

Example 1 **Solve the system of equations to the nearest hundredth.**

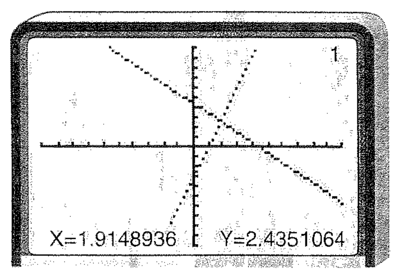
$y = -x + 4.35$

$y = 3x - 3.25$

Begin by graphing both equations in the standard viewing window.

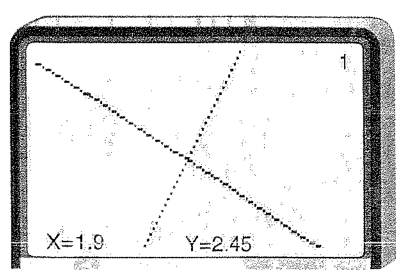
Enter: [Y=] [(–)] [X,T,θ,n] [+] 4.35 [ENTER] 3 [X,T,θ,n] [–] 3.25 [ZOOM] 6

Use the TRACE function to determine the coordinates of the point of intersection. Press [TRACE] and use the arrow keys to place the cursor as close as possible to the intersection point. Observe the coordinates at the bottom of the screen. Then ZOOM IN to determine the coordinates of the intersection point with greater accuracy.



Enter: [ZOOM] 2 [ENTER]

Use TRACE to position the cursor at the point of intersection and ZOOM IN again. Observe the coordinates. Now move the cursor one time to the left or right. Any digits that remain unchanged as you move are accurate.



Repeat the process of zooming and checking digits until you have the number of accurate digits that you desire.

The solution is approximately (1.90, 2.45). *Verify this result.*

The graphing calculator also has an INTERSECT feature that automatically finds the coordinates of the point of intersection.

Enter: [2nd] [CALC] 5 [ENTER] [ENTER] [ENTER]

You can use the TABLE function on the TI-83 to solve systems of equations.

Example 2 **Solve the system of equations to the nearest hundredth.**

$x + y = 4$

$2x + 3y = 9$

First solve each equation for y. $y = -x + 4$

$y = -\frac{2}{3}x + 3$

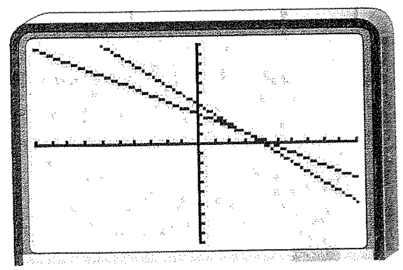
Graph both equations in the standard viewing window.

It is not necessary to graph the equations first.

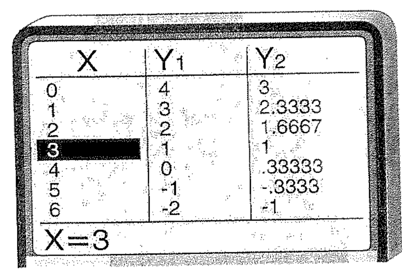
Enter: [Y=] [(–)] [X,T,θ,n] [+] 4

[ENTER] [(] [(–)] 2 [÷] 3

[)] [X,T,θ,n] [+] 3 [ZOOM] 6



Then press [2nd] [TABLE]. On the screen, you will see the coordinates of points on both lines. Use the arrow keys to scroll up or down and watch the trend of the coordinates. When you find a row at which $Y_1 = Y_2$, you have found the solution.



X	Y1	Y2
0	4	3
1	3	2.3333
2	2	1.6667
3	1	1
4	0	.33333
5	-1	-.3333
6	-2	-1

X=3

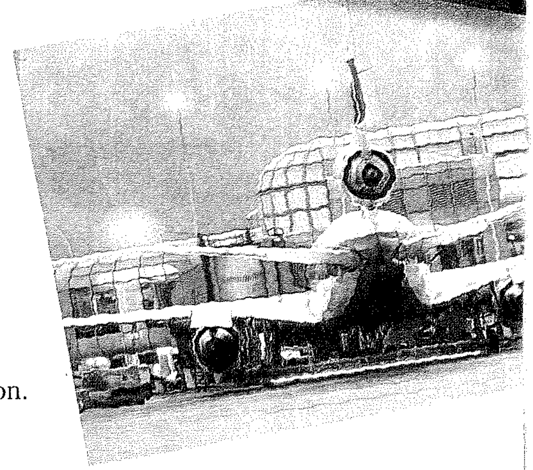
The solution is (3, 1). *Verify this result.*

EXERCISES

Use a graphing calculator to solve each system of equations to the nearest hundredth.

1. $y = 3x - 2$
   $y = -0.5x + 5$
2. $y = \frac{1}{4}x + 3$
   $y = -2x + 21$
3. $2x + 3y = 8$
   $3x - 8y = -13$
4. $y = -x + 7$
   $8 = 2x - y$
5. $y = 0.125x - 3.005$
   $y = -2.58$
6. $\frac{1}{2}x + \frac{1}{3}y = 1$
   $3x + 2y = 6$
7. $3.14x + 2.03y = 1.99$
   $9.32x - 3.77y = -4.21$
8. $12y = 4x - 16$
   $9y - 3x = 3$

3-1 Graphing Systems of Equations



What YOU'LL LEARN

- To solve systems of equations by graphing.

Why IT'S IMPORTANT

You can graph systems of equations to solve problems involving business and nutrition.

Real World APPLICATION

Business

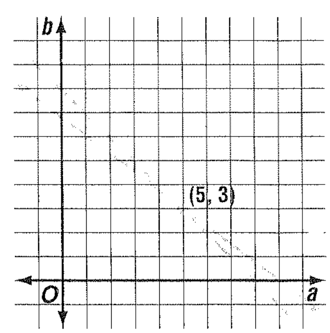
Terri Shackelford is the chief executive officer (CEO) of a commuter airline company. One day, the business section of a newspaper reported that a competing airline had purchased eight new aircraft. They purchased two different sizes of planes for a total of \$13.5 million. Ms. Shackelford calls the manufacturer of the planes and learns that the smaller planes cost \$1.5 million and the larger planes cost \$2 million. She would like to know how many of each size plane her competitor purchased.

She could find the answer by guess and check, but she did not become a CEO by guessing. She decides that using equations might be more straightforward. Let a represent the number of smaller planes purchased, and let b represent the number of larger planes purchased. We can then write the following equations.

$a + b = 8$ *A total of 8 planes was purchased.*

$1.5a + 2b = 13.5$ *The total cost of the planes was \$13.5 million.*

By graphing these two equations, we can find the number of each type of plane. Each point of each line has coordinates that satisfy the equation of that line. The one point that satisfies both equations is the point at which the two lines intersect, namely, (5, 3).



Therefore, the competing company purchased 5 of the smaller planes and 3 of the larger planes.

Together the equations $a + b = 8$ and $1.5a + 2b = 13.5$ are called a **system of equations.** The *solution* of this system is (5, 3). To check this solution, replace a with 5 and b with 3 in each equation.

$$a + b = 8 \qquad\qquad 1.5a + 2b = 13.5$$
$$5 + 3 \stackrel{?}{=} 8 \qquad\qquad 1.5(5) + 2(3) \stackrel{?}{=} 13.5$$
$$8 = 8 \qquad\qquad 7.5 + 6 \stackrel{?}{=} 13.5$$
$$13.5 = 13.5$$

The solution (5, 3) is correct.

EXPLORATION — GRAPHING CALCULATORS

Use a graphing calculator to graph each system of equations. You must first solve each equation for y.

1. $\frac{1}{2}x + \frac{1}{3}y = 2$
$x - y = -1$

2. $2x + 3y = 5$
$-6x - 9y = -15$

3. $y = \frac{x}{2}$
$2y = x + 4$

a. Describe the graphs of each system of equations.

b. How are the graphs of the systems of equations similar?

c. How are the graphs of the systems of equations different?

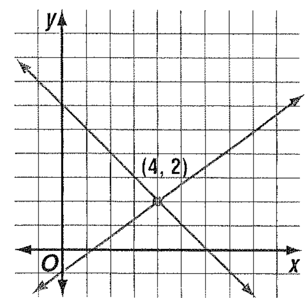
When two lines have different slopes, the graphs of the equations are intersecting lines.

Example 1

Solve the system of equations by graphing.
$\boldsymbol{x + y = 6}$
$\boldsymbol{3x - 4y = 4}$

The slope-intercept form of $x + y = 6$ is $y = -x + 6$.

The slope-intercept form of $3x - 4y = 4$ is $y = \frac{3}{4}x - 1$.



In this case, the lines have different slopes and intersect at (4, 2). The solution of the system is (4, 2).

Check:

$$x + y = 6 \qquad 3x - 4y = 4$$
$$4 + 2 \stackrel{?}{=} 6 \qquad 3(4) - 4(2) \stackrel{?}{=} 4$$
$$6 = 6 \ \checkmark \qquad 12 - 8 \stackrel{?}{=} 4$$
$$4 = 4 \ \checkmark$$

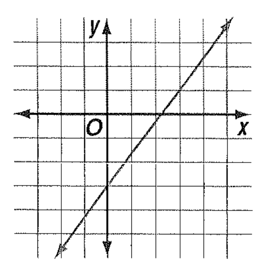
A system of equations that has at least one solution is called a **consistent system** of equations. If a system has exactly one solution, it is an **independent system.** So, the system in Example 1 is consistent and independent.

Example 2

Solve the system of equations by graphing.
$\boldsymbol{12x - 9y = 27}$
$\boldsymbol{8x - 6y = 18}$

The slope-intercept form of $12x - 9y = 27$ is $y = \frac{4}{3}x - 3$.

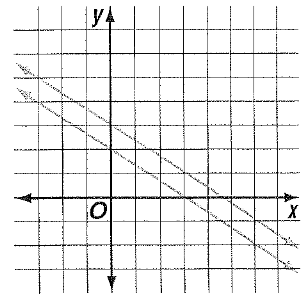
The slope-intercept form of $8x - 6y = 18$ is $y = \frac{4}{3}x - 3$.



Since the equations are equivalent, their graphs are the same line. Any ordered pair representing a point on that line will satisfy both equations. So, there are *infinitely many* solutions to this system.

A system is **dependent** if it has an infinite number of solutions. So, the system in Example 2 is consistent and dependent.

Example 3 **Solve the system of equations by graphing.**
$4x + 6y = 18$
$6x + 9y = 18$



The slope-intercept form of $4x + 6y = 18$ is $y = -\frac{2}{3}x + 3$.

The slope-intercept form of $6x + 9y = 18$ is $y = -\frac{2}{3}x + 2$.

The empty set is also called the null set. The empty or null set can be represented as ∅ or {}.

The lines have the same slope, but different y-intercepts. Their graphs are parallel lines. Since they never intersect, there are no solutions to this system. *The solution set is the empty set, ∅.*

A system with no solutions, like the one in Example 3, is called an **inconsistent system.**

The chart below summarizes the possibilities for the graphs and solutions of two linear equations in two variables, which are illustrated in Examples 1–3.

Example	Graphs of Equations	Slopes and Intercepts	Name of System of Equations	Number of Solutions
1	lines intersect	different slopes	consistent, independent	one
2	lines coincide	same slope, same intercepts	consistent, dependent	infinite
3	lines parallel	same slope, different intercepts	inconsistent	zero

A business can use equations to represent both its costs and its income. A graph of the resulting system of equations clearly illustrates when the business is making a profit and when it is not.

Example 4

Real World APPLICATION

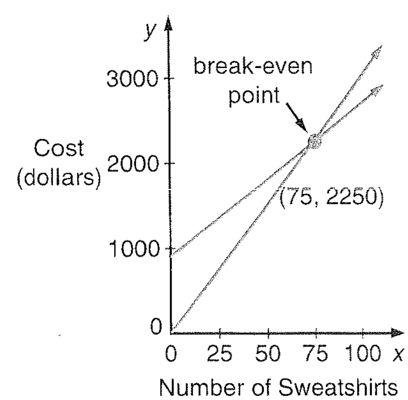
Business

Lina Sanchez is starting a business in Vail, Colorado. She plans to make souvenir sweatshirts to sell to skiers. She has initial start-up costs of \$900. Each sweatshirt costs \$18 to produce, and she plans to sell the sweatshirts for \$30. How many sweatshirts must Lina sell before she starts to make a profit?

The cost of making x sweatshirts is represented by $y = 900 + 18x$.

The income from x sweatshirts is represented by $y = 30x$.

Graph this system of equations. The graphs of these two equations intersect at (75, 2250). This point is called the *break-even point.* If Lina sells fewer than 75 sweatshirts, she loses money. If she sells more than 75 sweatshirts, she makes a profit.

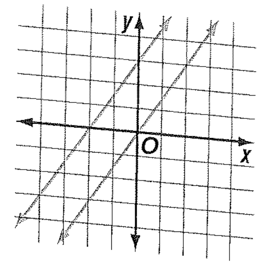


CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. Refer to the graph at the right.
   a. **Explain** whether the graph represents a system of equations that is *consistent and independent*, *consistent and dependent*, or *inconsistent*.
   b. **Describe** the slope and y-intercepts of the graphs.



2. **Sketch** and describe the graphs of $y = -2$ and $x = 4$. What is the solution to this system of linear equations?
3. **Explain** why a system of linear equations cannot have exactly two solutions.
4. **Write** a system of equations that is consistent.
5. Refer to the graph in Example 4.
   a. Determine if Lina Sanchez would have a profit or a loss if she sold 100 sweatshirts. Estimate the amount of the profit or loss.
   b. How many sweatshirts must Lina sell in order to make a profit of $600?

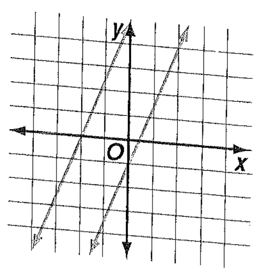
MATH JOURNAL

6. **Write** a paragraph explaining how to identify whether the graph of a system of linear equations would be two intersecting lines, two distinct parallel lines, or two coincident lines.

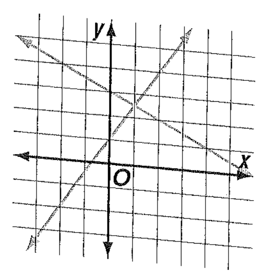
Guided Practice

State the number of solutions to each system of equations. State whether the system is *consistent and independent, consistent and dependent, or inconsistent*. If the system is consistent and independent, estimate the solution.

7.



8.



Graph each system of equations and state its solution. Also, state whether the system is *consistent and independent, consistent and dependent, or inconsistent*.

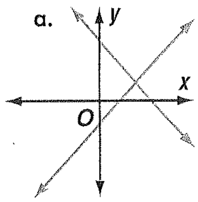
9. $5x - y = 3$
   $y = 5x - 3$

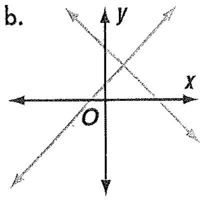
10. $2x - 3y = 7$
    $2x + 3y = 7$

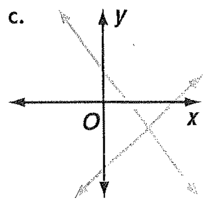
11. $x + y = 6$
    $3x + 3y = 3$

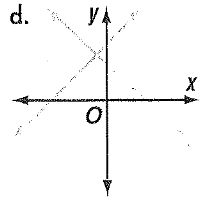
12. $\frac{1}{2}x - y = 0$
    $\frac{1}{4}x + \frac{1}{2}y = -2$

13. Which graph illustrates the solution to the system of equations $x + y = 4$ and $y = x - 2$?

a.


b.


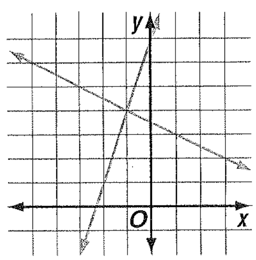
c.


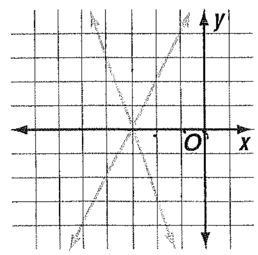
d.


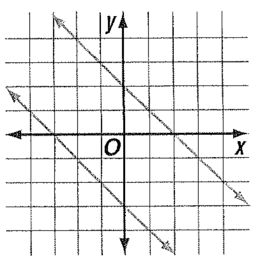
EXERCISES

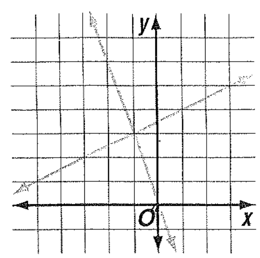
Practice

State the number of solutions to each system of equations. State whether the system is *consistent and independent, consistent and dependent,* or *inconsistent.* If the system is consistent and independent, estimate the solution.

14.


15.


16.


17.


Graph each system of equations and state its solution. Also, state whether the system is *consistent and independent, consistent and dependent,* or *inconsistent.*

18. $2x + 3y = 12$
$2x - y = 4$

19. $3x - y = 8$
$x - y = 8$

20. $4x - 6y = 5$
$2x - 3y = 5$

21. $x + 2y = 6$
$2x + y = 9$

22. $x + 1 = y$
$2x - 2y = 8$

23. $2x + 4y = 8$
$x + 2y = 4$

24. $3x - 8y = 4$
$6x - 42 = 16y$

25. $3x + 6 = 7y$
$x + 2y = 11$

26. $\frac{3}{4}x + \frac{1}{6}y = \frac{2}{3}$
$9x + 2y = 8$

27. $\frac{2}{3}x + y = -3$
$y - \frac{1}{3}x = 6$

28. $\frac{4}{3}x + \frac{1}{5}y = 3$
$\frac{2}{3}x - \frac{3}{5}y = 5$

29. $9x + 8y = 8$
$\frac{3}{4}x + \frac{2}{3}y = 8$

30. $3x + 4y = 7$
$1.5x + 2y = 3.5$

31. $1.2x + 2.5y = 4$
$0.8x - 1.5y = -10$

32. $5x - 7y = 70$
$-10x + 14y = 120$

Find values of m and n that satisfy the condition given for each system.

33. $5x - 3y = 8$
    $mx + ny = 4$
    consistent and dependent

34. $4x - 5y = 10$
    $mx - y = n$
    consistent and independent

35. $3x + 4y = 8$
    $mx + 2y = n$
    inconsistent

36. $2x + 7y = 5$
    $x + my = n$
    consistent and dependent

37. **Geometry** The sides of an angle are parts of two lines whose equations are $y = -\frac{3}{2}x - 6$ and $y = \frac{2}{3}x + 7$. Find the coordinates of the vertex of the angle.

38. **Geometry** The length of the base of an isosceles triangle is 2 centimeters shorter than the length of either of the other sides. If the perimeter of the triangle is 16 centimeters, find the length of each side of the triangle.

Graphing Calculator

Use a graphing calculator to solve each system of equations to the nearest hundredth.

39. $3.6x + 4.8y = -7.2$
    $5.8x - 7.1y = 32.9$

40. $-14x + 18y = 75$
    $9.1x - 11.7y = 36$

41. $3.6x - 2y = 4$
    $-2.7x + y = 3$

42. $7x + 13.5y = 31$
    $9.8x + 18.9y = 43.4$

Critical Thinking

43. State the conditions for which the system below is: (a) consistent and dependent, (b) consistent and independent, and (c) inconsistent.
    $ax + by = c$
    $dx + ey = f$

Applications and Problem Solving

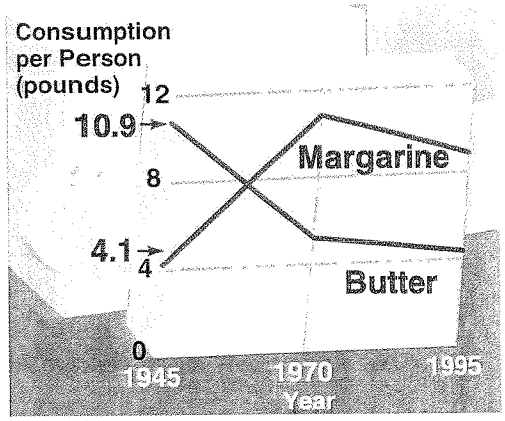
Real World

44. **Consumer Awareness** During a recent week, two grocery stores had sales on cereal. At Store A, all cereals were $33\frac{1}{3}\%$ off, while at Store B, the cereals were \$1 off. Assume that the regular prices of the cereals were the same at each store.
    a. Graph a system of equations representing the sales at the two stores.
    b. What does the point of intersection represent?
    c. For what regular prices would Store A's sale be a better deal?
    d. For what regular prices would Store B's sale be a better deal?

45. **Football** Mani Peters is the kicker for the Winston College football team. In one game, Mani kicked the ball 9 times for a total of 13 points. If 2 of the 9 kicks were no good, how many field goals (worth 3 points each) and how many points after touchdowns (worth 1 point each) did Mani make?

46. **Consumer Awareness** The Photo Shop charges \$1.60 to develop a roll of film plus 11¢ for each print, while Photos R Us charges \$1.20 to develop a roll plus 11¢ a print. Under what conditions is it best to use the Photo Shop and when is it best to use Photos R Us?

47. **Nutrition** The graph shows how the consumption of butter and margarine has changed over the years.



a. During the early 1950s, which was more popular, butter or margarine?
b. Estimate the year that the consumption of butter and margarine was equal.
c. A recent study has shown that the consumption of margarine raises cholesterol levels. Do you think that this information has changed the consumption of margarine? Explain.
d. Do you think that the consumption of butter and margarine will be the same again sometime in the future? Explain.

Internet Connection

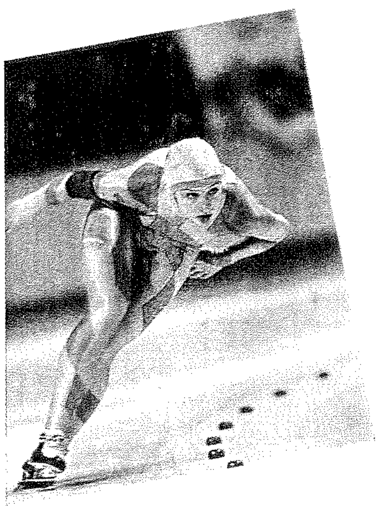
Data Update For more information on food consumption, visit: **www.algebra2.glencoe.com**

48. **Guess and Check** Mr. and Mrs. Leshin have fewer than ten children. The sum of the squares of the number of boys and the number of girls in the family equals 25. How many children do Mr. and Mrs. Leshin have?

Mixed Review

49. Graph $y = |x| + 1$. (Lesson 2–6)

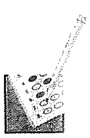
50. **Speed Skating** In the 1988 Winter Olympics, Bonnie Blair set a world record for women's speed skating by skating approximately 12.79 meters per second in the 500-meter race. Suppose she could maintain that speed. Write an equation that represents how far she could travel in t seconds. What type of function does the equation represent? (Lesson 2–6)



51. **Economics** A developer surveyed families in a suburb of Raleigh, North Carolina, to determine their monthly household income and the percent of their income spent on housing. The table below shows the data from eight families. (Lesson 2–5)

Monthly Income ($)	870	1430	1920	2460	2850	3240	3790	4510
Percent Spent on Housing	44	39	40	35	43	38	37	33

a. Write a prediction equation for this relationship.
b. Predict the percent of income spent on housing by a family with a monthly income of $3000.



52. **ACT Practice** The formula for calculating sales tax is $S = Ar$, where S is the sales tax, A is the cost of the item, and r is the sales-tax rate. If the cost of a large-screen TV is $600 and the sales tax is $42, what is the sales-tax rate?

A 0.07% **B** 0.70% **C** 1.43% **D** 7.00% **E** 14.29%

53. Graph a line that passes through $(4, -2)$ and is perpendicular to a line whose slope is -3. (Lesson 2–3)

54. Find the slope of a line passing through (5, 4) and (2, 2). (Lesson 2–3)

55. Write $x = \frac{1}{3}y + 3$ in standard form. (Lesson 2–2)

56. Evaluate $[25 - (5 - 2)^2 + 5] \div 7$. (Lesson 1–1)

For Extra Practice, see page 880.

3-2 Solving Systems of Equations Algebraically

What YOU'LL LEARN

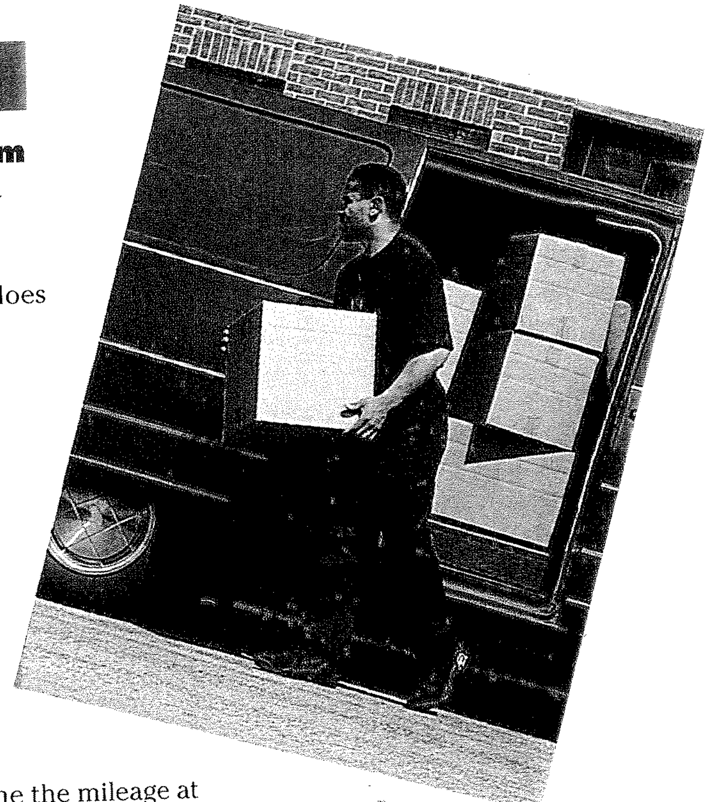
- To use the substitution and elimination methods to solve systems of equations.

Why IT'S IMPORTANT

You can use systems of equations to solve problems involving literature and population growth.

Real World APPLICATION

Consumerism



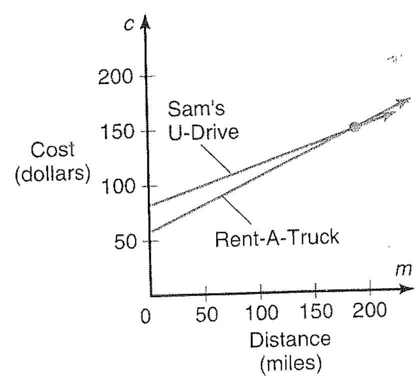
Austin is moving to a new condominium. He is sure that he can complete the move in one day, but he does not know how many trips he will need to make between his old and new residences. He plans to rent a 16-foot moving van for a day. When he checks the cost, he finds the following.

Rent-A-Truck: $59.95 a day plus 49¢ per mile
Sam's U-Drive: $81 per day plus 38¢ per mile

Austin needs to determine the mileage at which it is better to rent from Rent-A-Truck and the mileage at which it is better to rent from Sam's U-Drive.

To solve the problem, let m represent the miles driven and let c represent the total cost. Then we can write and graph the following equations.

$c = 59.95 + 0.49m$ *Cost for Rent-A-Truck*
$c = 81 + 0.38m$ *Cost for Sam's U-Drive*



It is very difficult to determine an exact solution from the graph. However, we can use the graph to estimate the solution. An estimate of the solution of this system of equations is (190, 150).

The cost will be the same if the mileage is somewhere around 190 miles. For lesser distances, Rent-A-Truck is cheaper, and for greater distances, Sam's U-Drive is cheaper.

For systems of equations like this one, it may be easier to solve the system by using algebraic methods rather than by graphing. Two algebraic methods are the **substitution method** and the **elimination method.** *This system of equations will be solved algebraically in Example 1.*

The Modeling Mathematics activity below suggests how to solve a system of equations using the substitution method.

MODELING MATHEMATICS

Systems of Equations

Materials: equation mat cups and counters

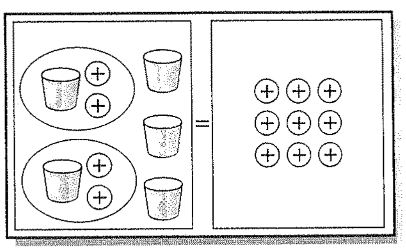
Use modeling to solve the system of equations.

$3x + 2y = 9$
$y = x + 2$

Your Turn

a. Let one cup represent x. If $y = x + 2$, how can you represent y?

b. Represent $3x + 2y = 9$ on the equation mat. On one side of the mat, place three cups to represent $3x$ and two representations of y from step a. On the other side of the mat, place nine positive counters.



c. Use what you know about equation mats and zero pairs to solve the equation. What value of x solves the system of equations?

d. Use the value of x from step c and the equation $y = x + 2$ to find the value of y.

e. What is the solution of the system of equations?

In the substitution method, one equation is solved for one variable in terms of the other. Then, this expression for the variable is substituted in the other equation.

Example 1

Real World APPLICATION

Consumerism

Refer to the application at the beginning of the lesson.

a. Determine the mileage for which the cost for Rent-A-Truck and Sam's U-Drive would be the same. Find the cost.

b. Under what conditions should Austin rent from Rent-A-Truck and under what conditions should Austin rent from Sam's U-Drive?

a. The two equations are $c = 59.95 + 0.49m$ and $c = 81 + 0.38m$. To solve this system of equations, use the substitution method.

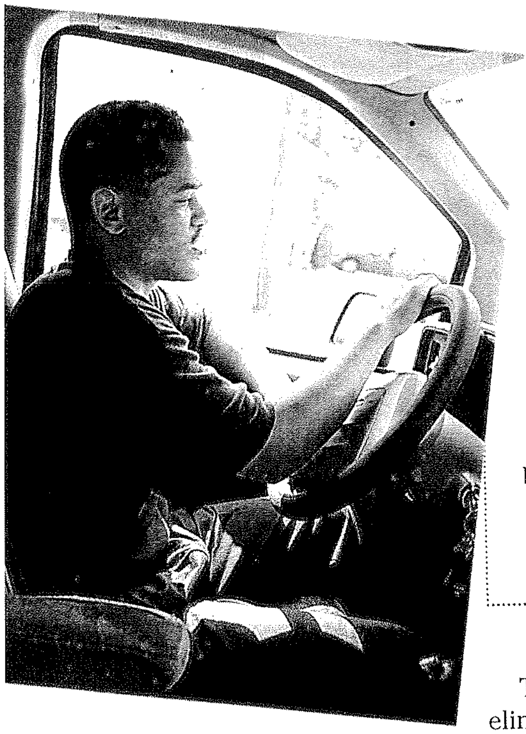
$$c = 59.95 + 0.49m$$

$$81 + 0.38m = 59.95 + 0.49m \quad \textit{Substitute } (81 + 0.38m) \textit{ for } c.$$

$$8100 + 38m = 5995 + 49m \quad \textit{Multiply by 100. Why?}$$

$$2105 = 11m$$

$$191.36 \approx m \quad \textit{Solve for } m.$$



Substitute 191.36 for m in $c = 59.95 + 0.49m$.

$c = 59.95 + 0.49m$
$c \approx 59.95 + 0.49(191.36)$ *Substitute 191.36 for m.*
$c \approx 153.72$ *Solve for c.*

The solution is about (191.36, 153.72). Therefore, at about 191.36 miles, the cost of $153.72 is equal for both Rent-A-Truck and Sam's U-Drive. *Compare this result to the estimate we obtained from the graph.*

b. Austin will need to decide how many trips he thinks he will need to make. If he thinks he will be driving fewer than 191.36 miles, he should rent from Rent-A-Truck. If he thinks he will be driving more than 191.36 miles, he should rent from Sam's U-Drive.

The second algebraic method is the elimination method. To use the elimination method effectively, first compare the coefficients of the variables.

Example 2 **Use the elimination method to solve the system of equations.**
$3a - 2b = -3$
$3a + b = 3$

In each equation, the coefficient of a is 3. If one equation is subtracted from the other, the variable a will be eliminated.

$3a - 2b = -3$
$(-)\ 3a + b = 3$ *Subtract.*
$-3b = -6$ *The variable a is eliminated.*
$b = 2$ *Solve for b.*

Now, find a by substituting 2 for b in either original equation.

First Equation
$3a - 2b = -3$
$3a - 2(2) = -3$ *Substitute 2 for b.*
$3a - 4 = -3$
$3a = 1$
$a = \frac{1}{3}$

or

Second Equation
$3a + b = 3$
$3a + 2 = 3$
$3a = 1$
$a = \frac{1}{3}$

The solution is $\left(\frac{1}{3}, 2\right)$.

Check by replacing (a, b) with $\left(\frac{1}{3}, 2\right)$ in each equation.

First Equation: $3\left(\frac{1}{3}\right) - 2(2) \stackrel{?}{=} -3$
$-3 = -3$ ✓

Second Equation: $3\left(\frac{1}{3}\right) + 2 \stackrel{?}{=} 3$
$3 = 3$ ✓

In the following example, adding or subtracting the two equations will not eliminate either variable.

Example 3 **Use the elimination method to solve the system of equations.**

$$\mathbf{3x + 5y = -4}$$
$$\mathbf{2x - 3y = 29}$$

To use the elimination method we must write equivalent equations containing the same coefficient for either x or y.

Method 1

Multiply the first equation by 3 and the second by 5. Then the variable y can be eliminated by addition.

$3x + 5y = -4$ → Multiply by 3. → $9x + 15y = -12$

$2x - 3y = 29$ → Multiply by 5. → $10x - 15y = 145$

Now, add to eliminate y.

$$\begin{aligned} 9x + 15y &= -12 \\ (+)\ 10x - 15y &= 145 \quad \textit{Add.} \\ \hline 19x \qquad &= 133 \quad \textit{The variable x is eliminated.} \\ x &= 7 \end{aligned}$$

Find y by substituting 7 for x in $3x + 5y = -4$.

$$\begin{aligned} 3(7) + 5y &= -4 \quad \textit{Substitute 7 for x.} \\ 21 + 5y &= -4 \\ 5y &= -25 \\ y &= -5 \quad \textit{Solve for y.} \end{aligned}$$

The solution is $(7, -5)$. *Check this solution.*

Method 2

We could also solve the system by eliminating x first. Multiply the first equation by 2 and the second by -3. Then add.

$3x + 5y = -4$ → Multiply by 2. → $6x + 10y = -8$

$2x - 3y = 29$ → Multiply by -3. → $(+)\ -6x + 9y = -87$

$$\begin{aligned} 19y &= -95 \\ y &= -5 \end{aligned}$$

Finally, solve for x.

$$\begin{aligned} 3x + 5(-5) &= -4 \quad \textit{Substitute −5 for y.} \\ 3x - 25 &= -4 \\ 3x &= 21 \\ x &= 7 \quad \textit{Solve for x.} \end{aligned}$$

The solution is $(7, -5)$.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **Describe** how we could have solved Example 1 by using elimination.
2. **Explain** when you might use substitution rather than elimination.
3. Consider the system of equations $y = 4x + 3$ and $y = 2x - 5$.
   a. Use the symmetric and transitive properties of equality to write a statement about $4x + 3$ and $2x - 5$.
   b. Explain how you could solve the system of equations now.
   c. If you graph the system of equations, where would the lines intersect?
4. **You Decide** When Helen solves the system of equations $4y - 8x = 28$ and $y = 2x - 5$, the result is $-20 = 28$. Helen decides that the graphs of the equations are the same line. Juanita says that the graphs of the equations are parallel lines. Who is correct, and why?

MODELING MATHEMATICS

5. Use cups and counters to model and solve the system of equations.
   $x + 2y = -5$
   $y = x + 2$

Guided Practice

Solve each system of equations by using substitution.

6. $x - 2y = 1$
   $x + y = 4$
7. $2p + 3q = 2$
   $p - 3q = -17$

Solve each system of equations by using elimination.

8. $m + n = 6$
   $m - n = 5$
9. $5x + 3y = 0$
   $4x + 5y = 13$

Solve each system of equations. Use either algebraic method.

10. $4a - b = 26$
    $8a - b = 54$
11. $6x + 9y = -45$
    $2x + 3y = -15$
12. $3s - 2t = 10$
    $4s + t = 6$
13. $\frac{1}{4}x + y = \frac{7}{2}$
    $2x - y = 4$
14. **Geometry** If the measure of one angle is five less than two-thirds of the measure of its supplementary angle, find the measure of the angles.

EXERCISES

Practice

Solve each system of equations by using substitution.

15. $4x - 3y = 18$
    $3x + y = 7$
16. $x + 3y = 13$
    $-3x + 2y = 27$
17. $3r + 9s = 36$
    $r = 8s - 10$
18. $n = 3m + 7$
    $4m + 9n = 1$
19. $4x + 6y = -9$
    $2x - 10y = -11$
20. $4x + 3y = 4$
    $6x - 6y = -1$

Solve each system of equations by using elimination.

21. $x - 3y = -12$
    $2x + 11y = -7$
22. $3x - 4y = 1$
    $5x + 2y = 45$
23. $4p + 5q = 7$
    $3p - 2q = 34$
24. $5c - 6d = -27$
    $7c + 3d = -15$
25. $\frac{1}{3}x + \frac{1}{2}y = 7$
    $\frac{2}{3}x - y = -2$
26. $\frac{2}{5}x - \frac{1}{2}y = 6$
    $\frac{4}{5}x + \frac{3}{2}y = -8$

Solve each system of equations. Use either substitution or elimination.

27. $3x - 7y = 5$
 $x + 3y = -1$

28. $2p - 5q = -53$
 $6p + 7q = 39$

29. $2a - b = 8$
 $6a - 3b = -9$

30. $3u + 5v = -12$
 $2u - 3v = -8$

31. $y = 5x + 37$
 $2x - 3y = -20$

32. $5s - t = 2$
 $t = 4s + 3$

33. $y = 3x - 27$
 $y = \frac{1}{2}x - 7$

34. $2.5m - 1.3n = 0.9$
 $10m - 5.2n = 3.6$

35. $\frac{1}{4}x + \frac{3}{5}y = -3$
 $\frac{3}{4}x - \frac{2}{5}y = 13$

36. $\frac{3}{5}s - \frac{1}{6}t = 1$
 $\frac{1}{5}s + \frac{5}{6}t = 11$

37. $1.5a - 0.2b = -8.3$
 $0.4a + 0.4b = -0.4$

38. $4m + 9n = 3$
 $8m - 3n = -8$

39. **Geometry** Find the coordinates of the vertices of the triangle whose sides are contained in the lines whose equations are $5x - 3y = -7$, $x + 2y = 9$, and $3x - 7y = 1$.

40. **Geometry** Find the coordinates of the vertices of the parallelogram whose sides are contained in the lines whose equations are $2x + y = -12$, $2x - y = -8$, $2x - y - 4 = 0$, and $4x + 2y = 24$.

Programming

41. The graphing calculator program at the right will help you solve systems of equations of the form $Ax + By = C$ and $Dx + Ey = F$.

```
PROGRAM:SLVSYSTM
: Disp "ENTER COEFFICIENTS"
: Prompt A,B,C,D,E,F
: If AE-BD=0
: Then
: Goto 1
: End
: (CE-BF)/(AE-BD) → X
: (AF-CD)/(AE-BD) → Y
: Disp "THE SOLUTION IS"
: Disp "X=",X
: Disp "Y=",Y
: Stop
: Lbl 1
: If CE-BF=0 or AF-CD=0
: Then
: Disp "INFINITE","SOLUTIONS"
: Else
: Disp "NO SOLUTION"
```

Run the program to find the solution for each system of equations.

a. $x - 3y = 6$
 $2x + 6y = 24$

b. $x + 4y = 2$
 $-x + y = -7$

c. $2x - y = 36$
 $3x - 0.5y = 26$

d. $2x + y = 45$
 $3x - y = 5$

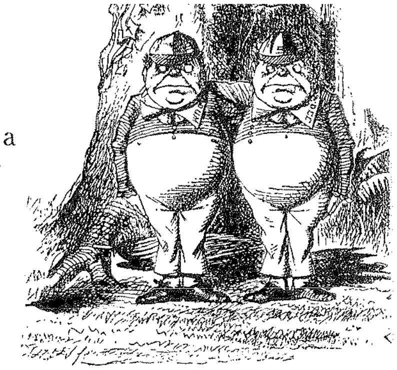
Critical Thinking

42. Solve the system of equations. (*Hint:* Let $n = \frac{1}{x}$ and $m = \frac{1}{y}$.)
 $\frac{1}{x} + \frac{3}{y} = \frac{3}{4}$
 $\frac{3}{x} - \frac{2}{y} = \frac{5}{12}$

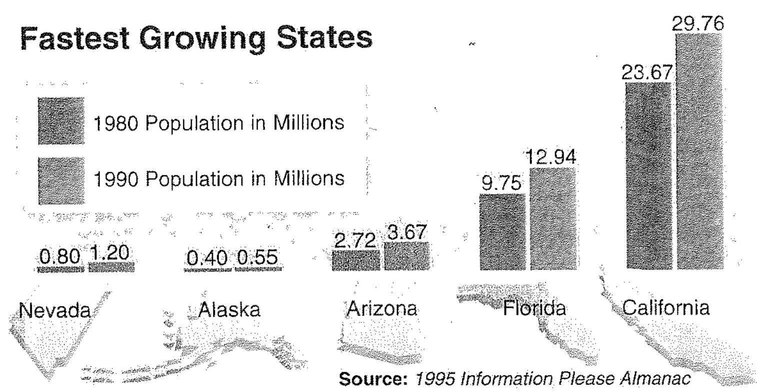
Applications and Problem Solving

43. **Sales** The drama club at Lincoln High School sells hot chocolate and coffee at the school's football games to make money for a special trip. At one game, they sold \$200 worth of hot drinks. They need to report how many of each type of drink they sold for their club records. Macha knows that they used 295 cups that night. If hot chocolate sells for 75¢ and coffee sells for 50¢, how many of each type of hot drinks did they sell?

44. Problem Solving Colin and Nate are dressing up as the characters Tweedledum and Tweedledee from Lewis Carroll's *Through the Looking Glass* for a costume party. They are making up riddles similar to those in the book to pose to the other party guests. In one riddle, Tweedledum says, "The sum of your weight and twice mine is 361 pounds." Tweedledee answers, "Contrariwise, the sum of your weight and twice mine is 362 pounds." What are the weights of Tweedledum and Tweedledee?



45. Population Growth The chart below shows the states with the greatest percentage of population growth during the 1980s. In 1990, the population of New York was 17.99 million, which represented a growth of 0.43 million over its 1980 population. Assume that each state continues to gain the same number of residents every ten years.

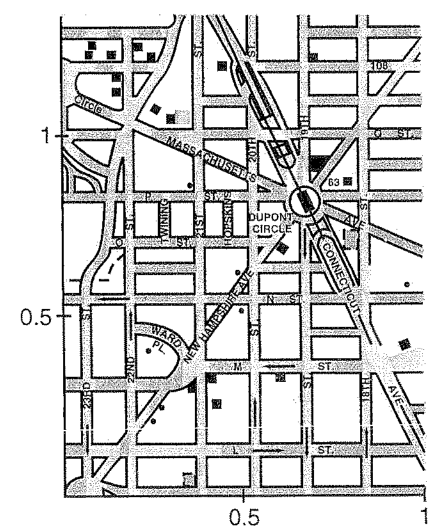


a. Write an equation that represents the population of New York d decades after 1990.

b. Write an equation that represents the population of Arizona d decades after 1990.

c. When will the populations of Arizona and New York be equal?

46. City Planning A portion of the subway in Washington, D.C., heads out of the main part of town in a northwesterly direction. It goes under New Hampshire Avenue as shown at the right. If distances are measured in kilometers, the path of the subway can be represented by the equation $y = -2.5x + 2.5$, and the path of New Hampshire Avenue can be represented by the equation $y = x$. What are the coordinates of the point at which the subway goes under New Hampshire Avenue?



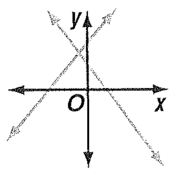
Longest Subway Networks

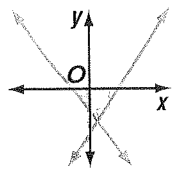
1. Washington, D.C., 612 km
2. London, 430 km
3. New York, 370 km
4. Paris, 301 km
5. Moscow, 225 km

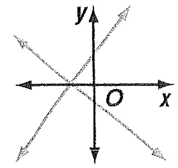
47. Consumer Awareness A new car rental company wants to offer two rental options similar to the truck rental plans offered in the application at the beginning of the lesson. Design two rate structure options that will offer different rates per mile, but will be equal for customers driving 150 miles.

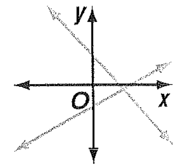
Mixed Review

48. Which graph illustrates the solution to the system of equations $y = 2x - 5$ and $x + y = -3$? (Lesson 3–1)

a. 

b. 

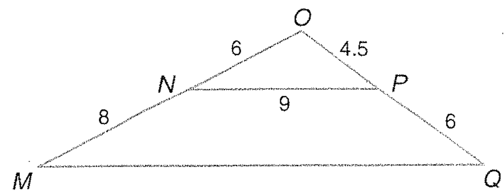
c. 

d. 

49. **Business** The Fix-It Auto Repair Shop has a sign that states that the labor costs are \$40 per hour or any fraction thereof. What type of function does this relationship represent? (Lesson 2–6)

50. If $g(x) = |4x + 17|$, find the value of $g(-2)$. (Lesson 2–6)

51. **ACT Practice** In the figure, ΔMOQ is similar to ΔNOP. What is MQ?



A 12 **B** 12.5 **C** 19 **D** 21 **E** 27

52. Find the slope and y-intercept of the graph of $-5x = 6 - 2y$. (Lesson 2–4)

53. Find the value of k in the equation $kx - 3y = 12$ if (2, 2) is a solution of the equation. (Lesson 2–4)

54. Find the x- and y-intercepts of the graph of $x = 9$. (Lesson 2–3)

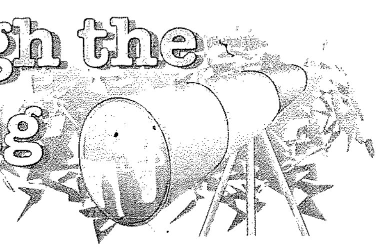
55. Find the mean, median, and mode of the prime numbers between 0 and 35. (Lesson 1–3)

For Extra Practice, see page 881.

WORKING ON THE In·ves·ti·ga·tion

Refer to the Investigation on pages 60–61.

Through the Looking Glass



By making observations with different instruments and comparing the results, naturalists can make more reliable conclusions than by using just one instrument.

1. You have four different sizes of calibration scopes. Record the dimensions of each of the four tubes.
2. Suppose you were standing 50 feet from an animal. Looking through Tube A, the animal's image fills your view exactly. Explain what you need to do to be able to look through Tube B so that the animal's image fills your view exactly. Explain what you need to do to be able to look through Tube D so that the animal's image fills your view exactly.
3. Suppose you had a Tube E that was 3 times as long and twice as wide as Tube A. Exactly where would you need to stand so that the animal's image fills your view exactly?

Add the results of your work to your Investigation Folder.

3-3 Cramer's Rule

***What* YOU'LL LEARN**

- To find the values of second-order determinants, and
- to solve systems of equations by using Cramer's rule.

***Why* IT'S IMPORTANT**

You can use Cramer's rule to solve problems involving sports and politics.

INTEGRATION

Geometry

Two sides of a parallelogram are contained in the lines whose equations are $2.3x + 1.2y = 2.1$ and $4.1x - 0.5y = 14.3$. To find the coordinates of a vertex of the parallelogram, we must solve the system of equations. However, solving this system by using substitution or elimination would require many calculations. *This problem will be solved in Example 3.*

Another method for solving systems of equations is **Cramer's rule.** The rule gives us a quick way to find the solution to a system of two equations with two variables. It is especially useful when the coefficients are large or involve fractions or decimals. Cramer's rule makes use of **determinants.** A determinant is a square array of numbers or variables enclosed between two parallel vertical bars. The numbers or variables written within a determinant are called **elements.** The determinant below has two rows and two columns and is called a **second-order determinant.**

rows → $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ ← *columns*

A second-order determinant is evaluated as follows.

Value of a Second-Order Determinant	$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

FYI

The theory of determinants is attributed to German mathematician Gottfried Wilhelm Leibnitz. His work expanded upon the earlier work of a Japanese mathematician, Seki Kowa.

Notice that the value of the determinant is found by calculating the difference of the products of the two diagonals.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{matrix} bc \\ ad \end{matrix} \Rightarrow ad - bc$$

Example 1 **Find the value of each determinant.**

a. $\begin{vmatrix} 3 & 5 \\ 2 & 6 \end{vmatrix}$

$$\begin{vmatrix} 3 & 5 \\ 2 & 6 \end{vmatrix} = 3(6) - 5(2)$$
$$= 8$$

b. $\begin{vmatrix} -2 & 7 \\ 5 & 8 \end{vmatrix}$

$$\begin{vmatrix} -2 & 7 \\ 5 & 8 \end{vmatrix} = -2(8) - 7(5)$$
$$= -51$$

To discover how Cramer's rule uses determinants to solve a system of linear equations, consider the following system.

$ax + by = e$
$cx + dy = f$ *a, b, c, d, e, and f represent constants, not variables.*

Solve for x by using elimination.

$$\begin{aligned} adx + bdy &= de && \text{Multiply the first equation by } d. \\ (-)\ bcx + bdy &= bf && \text{Multiply the second equation by } b. \\ adx - bcx &= de - bf && \text{Subtract.} \\ (ad - bc)x &= de - bf && \text{Factor.} \\ x &= \frac{de - bf}{ad - bc} && \text{Notice that } ad - bc \text{ must not be zero.} \end{aligned}$$

Solving for y in the same way produces the following expression.

$$y = \frac{af - ce}{ad - bc}$$

So, the solution to the system of equations $\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$ is $\left(\frac{de - bf}{ad - bc}, \frac{af - ce}{ad - bc}\right)$.

Notice that the two fractions have the same denominator. It can be written as a determinant. The numerators can also be written as determinants.

$$ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad de - bf = \begin{vmatrix} e & b \\ f & d \end{vmatrix} \qquad af - ce = \begin{vmatrix} a & e \\ c & f \end{vmatrix}$$

So, now we can find the solution to a system of two linear equations in two variables by using determinants. This method is called Cramer's rule.

Cramer's Rule	**The solution to the system** $\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$ **is** (x, y), **where** $x = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$, $y = \dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$, **and** $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

Example 2 **Use Cramer's rule to solve the system of equations.**

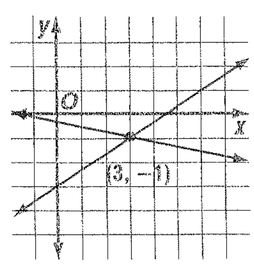
$2x - 3y = 9$
$x + 5y = -2$

$$x = \frac{\begin{vmatrix} 9 & -3 \\ -2 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix}} = \frac{9(5) - (-3)(-2)}{2(5) - (-3)(1)} = \frac{39}{13} = 3$$

$$y = \frac{\begin{vmatrix} 2 & 9 \\ 1 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix}} = \frac{2(-2) - 9(1)}{2(5) - (-3)(1)} = \frac{-13}{13} = -1$$

The solution is $(3, -1)$.

Check by graphing.



Example 3 Refer to the application at the beginning of the lesson. Find the coordinates of a vertex of the parallelogram.

INTEGRATION
Geometry

To find the coordinates of a vertex, you need to solve the system of equations.

$$2.3x + 1.2y = 2.1$$
$$4.1x - 0.5y = 14.3$$

Since the numbers in the equations would make solving the system by substitution or elimination difficult, use Cramer's rule.

$$x = \frac{\begin{vmatrix} 2.1 & 1.2 \\ 14.3 & -0.5 \end{vmatrix}}{\begin{vmatrix} 2.3 & 1.2 \\ 4.1 & -0.5 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} 2.3 & 2.1 \\ 4.1 & 14.3 \end{vmatrix}}{\begin{vmatrix} 2.3 & 1.2 \\ 4.1 & -0.5 \end{vmatrix}}$$

$$= \frac{2.1(-0.5) - 14.3(1.2)}{2.3(-0.5) - 4.1(1.2)} \qquad = \frac{2.3(14.3) - 4.1(2.1)}{2.3(-0.5) - (4.1)(1.2)}$$

$$= \frac{-18.21}{-6.07} \qquad = \frac{24.28}{-6.07}$$

$$= 3 \qquad = -4$$

The coordinates of a vertex of the parallelogram are (3, −4).

Check by using substitution.

First Equation: $2.3(3) + 1.2(-4) = 2.1$ ✓
Second Equation: $4.1(3) - 0.5(-4) = 14.3$ ✓

Systems of equations can also be solved by using a BASIC program.

EXPLORATION — BASIC

The BASIC program below finds the solution of the system of equations of the form $ax + by = c$ and $dx + ey = f$.

```
10  PRINT "ENTER THE COEFFICIENTS."
20  INPUT A, B, C, D, E, F
30  IF A*E-B*D = 0 THEN 80
40  LET X = (C*E-B*F)/(A*E-B*D)
50  LET Y= (A*F-C*D)/(A*E-B*D)
60  PRINT "(";X;",";Y;") IS A SOLUTION"
70  GOTO 10
80  IF C*E-B*F = 0 OR A*F-C*D=0 THEN 110
90  PRINT "NO SOLUTION."
100  GOTO 10
110  PRINT "INFINITE NUMBER OF SOLUTIONS."
120  GOTO 10
130  END
```

Your Turn

a. Use the program to solve the system of equations in Example 2.
b. Use the program to solve the system of equations in Example 3.
c. Study Steps 40 and 50. How does this program relate to Cramer's rule?

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **Evaluate** a determinant if both elements of a row or column are 0.
2. **Describe** the elements of a determinant when the value of the determinant is 0 and none of the elements is 0.
3. In Cramer's rule, if the value of the determinant in the denominator is 0, what must be true of the graph of the system of equations represented by the determinant?
4. Carmen used Cramer's rule and solved for x as follows.

$$x = \frac{\begin{vmatrix} 18 & -5 \\ -4 & 8 \end{vmatrix}}{\begin{vmatrix} 2 & -5 \\ 3 & 8 \end{vmatrix}}$$

Write the system of equations that she was solving.

Guided Practice

Find the value of each determinant.

5. $\begin{vmatrix} 5 & 2 \\ 4 & 1 \end{vmatrix}$
6. $\begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$
7. $\begin{vmatrix} \frac{2}{5} & 6 \\ \frac{1}{3} & 2 \end{vmatrix}$

Use Cramer's rule to solve each system of equations.

8. $2x - y = 1$
   $3x + 2y = 19$
9. $5x + 2y = 8$
   $2x - 3y = 7$
10. $\frac{1}{6}x - \frac{1}{9}y = 0$
    $x + y = 15$
11. $2m - 5n = 2$
    $3m + 4n = -5$
12. **Geometry** The two sides of an angle are contained in lines whose equations are $4x + y = -4$ and $2x - 3y = -9$. Find the coordinates of the vertex of the angle.

EXERCISES

Find the value of each determinant.

13. $\begin{vmatrix} 8 & 5 \\ 6 & -2 \end{vmatrix}$
14. $\begin{vmatrix} -2 & 4 \\ 8 & -7 \end{vmatrix}$
15. $\begin{vmatrix} -8 & 3 \\ -9 & 7 \end{vmatrix}$
16. $\begin{vmatrix} -6 & -2 \\ 8 & 5 \end{vmatrix}$
17. $\begin{vmatrix} 2 & -7 \\ -5 & 3 \end{vmatrix}$
18. $\begin{vmatrix} 21 & 43 \\ 17 & -29 \end{vmatrix}$
19. $\begin{vmatrix} -54 & 39 \\ 18 & -13 \end{vmatrix}$
20. $\begin{vmatrix} -3.2 & -5.8 \\ 4.1 & 3.9 \end{vmatrix}$
21. $\begin{vmatrix} 7 & -5.2 \\ 1.3 & 2.29 \end{vmatrix}$

Use Cramer's rule to solve each system of equations.

22. $5x + 7y = 13$
    $2x - 5y = 13$
23. $3a + 5b = 33$
    $5a + 7b = 51$
24. $2m + 7n = 4$
    $m - 2n = -20$
25. $3x - 2y = 4$
    $\frac{1}{2}x - \frac{2}{3}y = 1$
26. $1.5x + 0.7y = 0.5$
    $2.2x - 0.6y = -7.4$
27. $4u + 3v = 6$
    $8u - v = -9$

28. $\frac{1}{2}r + \frac{3}{4}s = -4$
$\frac{3}{4}r - \frac{7}{8}s = 10$

29. $\frac{1}{3}x + \frac{2}{5}y = 5$
$\frac{2}{3}x - \frac{1}{2}y = -3$

30. $\frac{1}{2}x - y = -1$
$\frac{3}{4}x + \frac{1}{2}y = -\frac{1}{4}$

31. $\frac{3}{4}a + \frac{1}{2}b = \frac{11}{12}$
$\frac{1}{2}a - \frac{1}{4}b = \frac{1}{8}$

32. $0.2a = 0.3b$
$0.4a - 0.2b = 0.2$

33. $3.5x + 4y = -5$
$2(x - y) = 10$

34. **Geometry** The sides of a triangle are contained in the lines represented by $x - 4y = -6$, $5x + y = 33$, and $2x - y = 2$. Find the coordinates of the vertices of the triangle.

Critical Thinking

35. When using Cramer's rule, how can you tell whether there is no solution or an infinite number of solutions?

Applications and Problem Solving

Real World

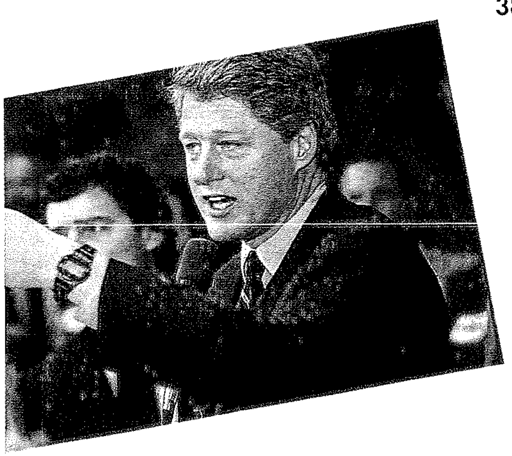
36. **Sports** The winning times for various speed-skating events in the 1998 Winter Olympics are given in the chart below. Suppose the men and women continue to decrease their times for the 1000-meter event by the same amount each Winter Olympics. In what Olympics will the women's time be faster than the men's in this event?

Winning Times for Speed Skaters in the 1998 Winter Olympics

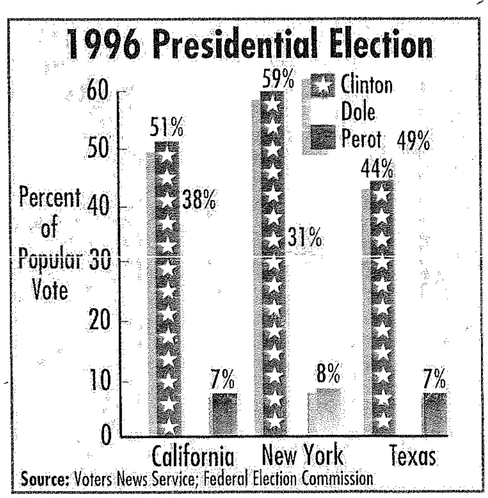
Event	Men's Time	Change from 1994 Winner	Women's Time	Change from 1994 Winner
500 m	35.59 s	−0.74 s	38.21 s	−1.04 s
1000 m	70.64 s	−1.79 s	76.51 s	−2.23 s
1500 m	107.87 s	−3.42 s	117.58 s	−4.61 s

Source: CBS Sportsline

37. **Exercise** Rosa rides the bus to work. Usually she rides 35 minutes and then walks 6 minutes to get to her place of employment. Since the bus travels 30 miles per hour and her walking speed is 5 miles per hour, she can easily determine that the total distance is 18 miles. One day when the weather is nice, she decides to get off the bus earlier and walk the rest of the way for exercise. She only has a total of 75 minutes to get to work, so she can't afford to get off the bus too early. How much time should she spend on the bus before getting off to walk?



38. **Politics** The three states with the most electoral votes are California, New York, and Texas. The graph shows the results of the popular votes in the 1996 presidential election. Bill Clinton received about 6,200,000 popular votes from the voters in New York and Texas, and Bob Dole received about 4,700,000 from the same voters. About how many people voted in the 1996 presidential election in New York? in Texas?



Mixed Review

39. Solve the system of equations $2x + y = 0$ and $5x + 3y = 1$ by using either substitution or elimination. (Lesson 3–2)

40. **Consumer Awareness** Mrs. Katz is planning a family vacation. She bought 8 rolls of film and 2 camera batteries for $23.00. The next day, her daughter went back and bought 6 more rolls of film and 2 batteries for her camera. This bill was $18.00. What is the price of a roll of film and a camera battery? (Lesson 3–2)

41. **SAT Practice** If two sides of a triangle have lengths of 30 and 60, which of the following *cannot* be the length of the third side?
A 30 B 31 C 40 D 60 E 61

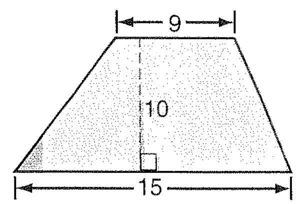
42. If $f(x) = [5x - 3]$, find $f\left(-\frac{1}{2}\right)$. (Lesson 2–6)

43. Write a prediction equation for the data in the following table. (Lesson 2–5)

x	−3	0	4	6	8
y	5	3	0	−3	−5

44. Solve $\frac{w-6}{3} \leq 6 - w$. (Lesson 1–6)

45. **Geometry** The formula for the area of a trapezoid is $A = \frac{1}{2}h(b_1 + b_2)$, where h is the height of the trapezoid and b_1 and b_2 are the lengths of the bases. Find the area of the trapezoid at the right. (Lesson 1–1)



For Extra Practice, see page 881.

Mathematics and Society

Quantum Computing

The article below appeared in *Science News* on May 14, 1994.

In the quantum world, a particle—undisturbed by any attempt to observe it—can be in myriad places at the same time. Thus, a single photon traveling through a crystal simultaneously follows all possible optical paths through the material. . . . Computer scientists have speculated that computers operating according to the rules of quantum mechanics can potentially take advantage of a similar multiplicity of paths to solve certain types of mathematical problems much more quickly than conventional computers can. Now, mathematician Peter W. Shor . . . has proved that, in principle, quantum computation can provide the shortcut needed to convert the factoring of large numbers from a time-consuming chore into an amazingly quick operation. . . . Quantum computers don't exist yet, and building them involves surmounting significant technological barriers. Nonetheless, some researchers are starting to produce designs . . . that may lead eventually to working models. ■

1. What are some advantages of greatly speeding up some computer calculations?
2. Can you think of an example in everyday life where moving numerous items along many paths is faster than using only one path?
3. If a quantum computer is built, we will learn a lot if it works successfully, and we will learn a lot if it doesn't work successfully. Explain.

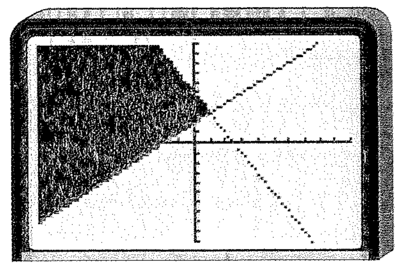
3–4A Graphing Technology Systems of Linear Inequalities

A Preview of Lesson 3–4

You can graph systems of linear inequalities with a graphing calculator by using the SHADE feature described in Lesson 2–7A. It shades above the first function entered and below the second function entered. Be sure to clear any equations currently stored in the Y= list of the calculator before you begin.

Example 1 **Graph the system of inequalities in the standard viewing window.**

$y \geq x + 2$
$y \leq -2x + 5$



The greater than or equal to symbol in $y \geq x + 2$ indicates the values on or above the line $y = x + 2$. Similarly, the less than or equal to symbol in $y \leq -2x + 5$ indicates values on or below the line $y = -2x + 5$. Therefore, the function $y = x + 2$ will be entered first, and $y = -2x + 5$ will be entered second.

Enter: ZOOM 6 2nd DRAW 7 X,T,θ,n + 2 , (–) 2 X,T,θ,n + 5) ENTER

The shaded area indicated points that satisfy the system of inequalities $y \geq x + 2$ and $y \geq -2x + 5$.

Before graphing again, you must clear the graphics screen.

Enter: 2nd DRAW 1 ENTER

EXERCISES

Use a graphing calculator to solve each system of inequalities. Sketch each graph on a sheet of paper.

1. $y \geq x$
   $y \leq 5$
2. $y \geq -2x + 4$
   $y \leq x - 1$
3. $y \leq 6x - 3$
   $y \geq 0.5x$
4. $y \leq 0.1x + 1$
   $y \geq -0.5x - 3$
5. $3x - 4y \leq 12$
   $2x + y \leq 10$
6. $y - 2 \leq x$
   $y \geq -3x + 5$
7. $y \geq 0$
   $y \leq -2x + 12$
8. $y \leq \frac{2}{3}x - 1$
   $y \geq -\frac{1}{5}x + 3$
9. $-5y \leq -2x$
   $2y \leq 3x - 8$

3-4 Graphing Systems of Inequalities

What YOU'LL LEARN

- To solve systems of inequalities by graphing.

Why IT'S IMPORTANT

You can use systems of inequalities to solve problems involving space science and higher education.

Real World APPLICATION

Space Science

When the National Aeronautics and Space Administration (NASA) chose the first astronauts in 1959, size was important since the space available in the Mercury capsule was very limited. NASA called for men who were between 5 feet 4 inches and 5 feet 11 inches tall, inclusively, and who were between 21 and 40 years of age.

This information can be represented by a **system of inequalities**. To solve such a system of inequalities, we need to find the ordered pairs that satisfy all the inequalities involved. One way to do this is to graph the inequalities on the same coordinate plane. The solution set is then represented by the intersection, or overlap, of the graphs.

***fabulous* FIRSTS**

Mae Carol Jemison (1956–)

Mae Carol Jemison was the first African-American woman astronaut. She was the computer engineer and physician aboard the space shuttle Endeavor, launched September 12, 1992.

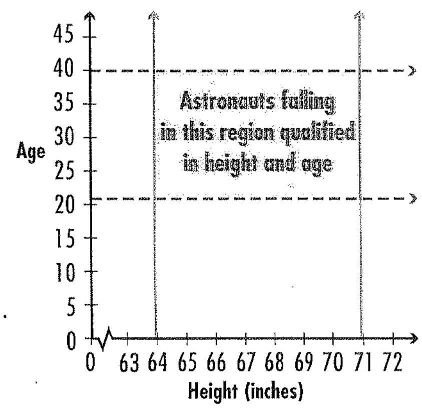
Since the heights of the astronauts are between 5 feet 4 inches (or 64 inches) and 5 feet 11 inches (or 71 inches), inclusively, we can write this information as the following two inequalities using h as the height in inches.

$h \geq 64$ and $h \leq 71$

This could also be expressed as $64 \leq h \leq 71$.

The acceptable ages, a, can be expressed as the following inequalities.

$a > 21$ and $a < 40$



The broken lines indicate that the boundaries are not part of the graphs.

This could also be expressed as $21 < a < 40$.

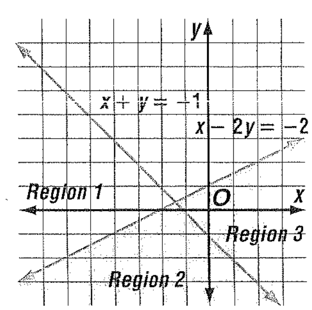
Graph both inequalities. Any point in the intersection of the two graphs is a solution to the system.

Example 1 **Solve the system of inequalities by graphing.**

$x - 2y \geq -2$
$x + y \leq -1$

$x - 2y \geq -2$ represents Regions 2 and 3.
$x + y \leq -1$ represents Regions 1 and 2.

The intersection of these regions is Region 2, which is the solution of the system of inequalities.

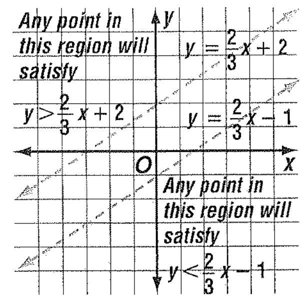


It is possible that two regions do *not* intersect. In such cases, we say the solution is the empty set, shown as $\varnothing$, and no solution exists.

Example 2 **Solve the system of inequalities by graphing.**

$y > \frac{2}{3}x + 2$

$y < \frac{2}{3}x - 1$



The two solutions have no points in common. The solution set is $\varnothing$.

As you recall, an absolute value inequality can be restated as two inequalities using an *and* or an *or*. So, an absolute value inequality can be graphed like a system of inequalities.

Example 3 **Solve the system of inequalities by graphing.**

$|x| < 3$

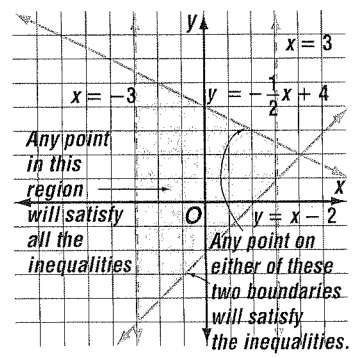
$y \geq x - 2$

$y \leq -\frac{1}{2}x + 4$

The inequality $|x| < 3$ can be rewritten as $x > -3$ and $x < 3$.

Graph all of the inequalities on the same grid and look for the region or regions that are common to all.

Check to be sure the proper region is shaded.



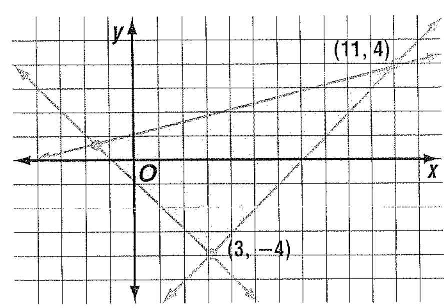
Example 4 INTEGRATION **Geometry**

The system of inequalities $x - y \leq 7$, $3x - 11y \geq -11$, and $x + y + 1 \geq 0$ form a triangle and its interior.

a. Graph the system and describe the triangle.

b. Name the coordinates of the vertices of the triangle.

a. The lines are graphed at the right. The triangle is a right triangle.



b. The coordinates (11, 4) and (3, −4) can be determined from the graph. To find the coordinates of the third vertex, solve the system of equations $3x - 11y = -11$ and $x + y + 1 = 0$.

$x + y + 1 = 0$ *Solve for x.*

$x = -y - 1$

(continued on the next page)

Substitute $-y - 1$ for x.

$$3x - 11y = -11$$
$$3(-y - 1) - 11y = -11$$
$$-3y - 3 - 11y = -11$$
$$-14y = -8$$
$$y = \frac{8}{14} \text{ or } \frac{4}{7}$$

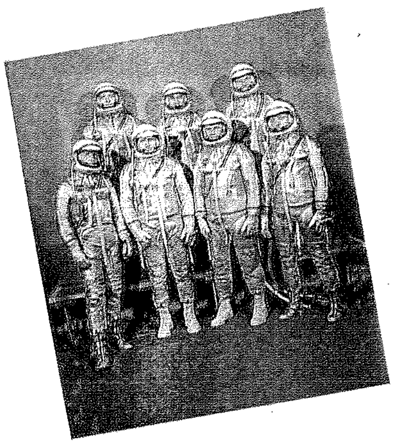
Evaluate when $y = \frac{4}{7}$.

$$x = -y - 1$$
$$= -\left(\frac{4}{7}\right) - 1$$
$$= -\frac{11}{7}$$

The coordinates of the vertices of the triangle are (11, 4), (3, −4), and $\left(-\frac{11}{7}, \frac{4}{7}\right)$.

CHECK FOR UNDERSTANDING

Communicating Mathematics



Study the lesson. Then complete the following.

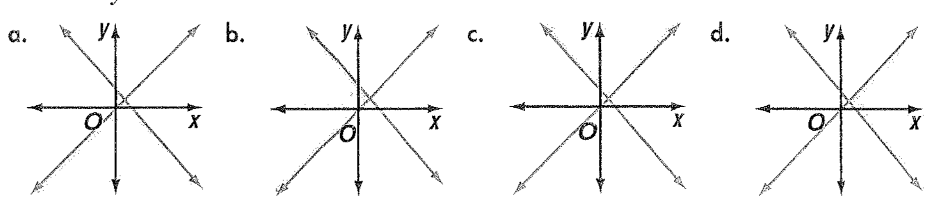
1. How do you determine whether a point is a solution to a system of inequalities?
2. **Describe** the difference between the graphs of the following two systems of inequalities.
   a. $2x + 3y > 6$ or $2x + 3y < 3$ b. $2x + 3y > 6$ and $2x + 3y < 3$
3. **Describe** how the graphs of $|x| \geq 2$ and $|x| \leq 2$ differ.
4. **Research** the names, ages, and heights of the original seven astronauts. Graph this information on a graph similar to the one on page 148. Describe your results in a paragraph.
5. **Write** a system of two inequalities that will have no intersection.

MATH JOURNAL

6. **Assess Yourself** Write a paragraph describing what you like and/or dislike about graphing systems of inequalities.

Guided Practice

7. Which graph illustrates the solution to the system of inequalities $x + y \geq 1$ and $x - y \leq 0$.



Solve each system of inequalities by graphing.

8. $x \leq 1$
   $y > 3$
9. $y \geq 2x - 2$
   $y \leq -x + 2$
10. $x - 3y \leq -3$
    $2x + 3y < 12$
11. $x - 3y \geq -9$
    $4x - y \leq 4$
    $x + 2y \geq -2$
12. **Geometry** Graph the system of inequalities $x \geq 0$, $y \geq 0$, and $x + y \leq 6$. Find the area of the region defined by the system of inequalities.

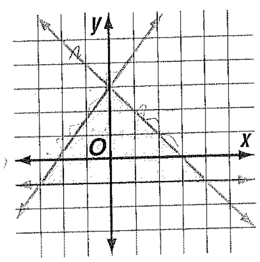
EXERCISES

Practice

Solve each system of inequalities by graphing.

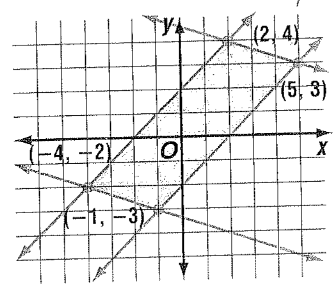
13. $x < 2$
    $y \geq 1$
14. $x - y \leq 3$
    $x + y \geq 2$
15. $x + y > 2$
    $y > 3$
16. $y - x \leq 3$
    $y \geq x + 2$
17. $y \geq x - 3$
    $y \geq -x + 1$
18. $y < -x - 3$
    $x > y - 2$
19. $y \leq 2x - 3$
    $y \leq \frac{1}{2}x + 1$
20. $y > \frac{2}{3}x - 1$
    $y \leq -\frac{3}{4}x + 2$
21. $|x| < 3$
    $|y| > 2$
22. $|x + 1| \leq 2$
    $x - 2y \geq 1$
23. $3x + 2y \leq 6$
    $4x - y < 2$
24. $4x - 3y \geq 7$
    $y < 2$
25. $x - 3y > 2$
    $2x - y < 4$
    $3x + 4y > 0$
26. $5x - y < 0$
    $4x + 3y > 6$
    $x - 3y > 3$
27. $y < 2x + 1$
    $y > 2x - 2$
    $3x + y > 8$

28. **Geometry** Study the graph at the right.
    a. Write the system of inequalities whose solution forms the triangular region shown in the graph.
    b. Replace $y \geq -1$ with an inequality that would result in a triangular region with four times the area of the first region.



29. **Geometry** Write a system of inequalities that would form a polygonal region having at least three sides.

30. **Geometry** Study the region at the right.
    a. What is the most descriptive name for the quadrilateral?
    b. Write the system of inequalities whose solution forms the region.
    c. Find the area of the region.



Critical Thinking

31. **Geometry** Find the area of the region defined by $|x| + |y| \leq 5$ and $|x| + |y| \geq 2$.

Applications and Problem Solving

Real World

32. **Higher Education** According to *Lovejoy's College Guide,* the middle 50% of the freshmen entering the University of Notre Dame have a score between 540 and 650, inclusively, on the verbal portion of the SAT. The middle 50% of these same freshmen have between 620 and 720, inclusively, on the math portion of the SAT. Write a system of inequalities that describes the SAT scores of the middle 50% of Notre Dame freshmen. Then, graph the system of inequalities.

33. **Time Management** Paloma attends Jones High School. Each school day, she spends 7 hours in school and usually some time after school studying and doing homework. She tries to sleep exactly 8 hours out of every 24-hour period. Let s represent the time Paloma spends on school and studying, and let a represent the time she spends on activities other than school, studying, and sleep. Write a system of inequalities that describes the way Paloma manages her time during a typical school day. Then graph the system of inequalities.

Mixed Review

34. Use Cramer's rule to solve the system of equations $x - 4y = 1$ and $2x + 3y = 13$. (Lesson 3–3)

Solve each system of equations. Use either substitution or elimination. (Lesson 3–2)

35. $4a - 3b = -4$
    $3a - 2b = -4$

36. $2r + s = 1$
    $r - s = 8$

37. Graph the system of equations $x + 5y = 10$ and $x + 5y = 15$. State its solution. (Lesson 3–1)

38. **ACT Practice** Ellis tried to compute the average of his 8 test scores. He mistakenly divided the correct total (T) of his scores by 7. The result was 12 more than what it should have been. Which equation would determine the value of T?

    A $8T - 12 = 7T$ B $\frac{T}{7} = \frac{T + 12}{8}$ C $\frac{T}{7} + 12 = \frac{T}{8}$

    D $\frac{T + 12}{7} = \frac{T}{8}$ E $\frac{T}{7} - 12 = \frac{T}{8}$

39. Salina earned \$8 more than Tyler selling newspaper subscriptions. At the end of the day, Salina was given \$1 as a sales bonus, and Tyler spent \$3 on snacks. When they returned home, they had a total of \$12. How much money did Tyler earn selling newspaper subscriptions? (Lesson 1–4)

For Extra Practice, see page 881.

SELF TEST

Solve the system of equations by graphing. (Lesson 3–1)

1. $2x - 5y = 14$
   $x - y = 1$

2. $4x - 3y = -9$
   $x + 2y = -5$

Solve each system of equations by using substitution or elimination. (Lesson 3–2)

3. $y = 3x$
   $x + 21 = -2y$

4. $4a + 3b = -2$
   $5a + 7b = 17$

5. Find the value of the determinant $\begin{vmatrix} -5 & -2 \\ -3 & 11 \end{vmatrix}$. (Lesson 3–3)

Use Cramer's rule to solve each system of equations. (Lesson 3–3)

6. $3x - 7y = 2$
   $6x - 13y = 4$

7. $6x + 5y = -7$
   $2x - 3y = 7$

8. **Travel** A hotel at the Downhill Ski Resort advertises two package deals. One package offers three nights at the hotel and a two-day lift ticket for \$245. The other offers five nights at the hotel and a three-day lift ticket for \$400. For these specials, what is the cost of staying at the hotel one night and what is the cost of a one-day lift ticket? (Lesson 3–3)

Solve each system of inequalities by graphing. (Lesson 3–4)

9. $x + y > 2$
   $x - 2y \leq -1$

10. $y < 2$
    $y \geq 2x$
    $y \geq x + 1$

3–5 Linear Programming

***What* YOU'LL LEARN**

- To find the maximum and minimum values of a function over a region by using linear programming techniques, and
- to solve problems by solving a simpler problem.

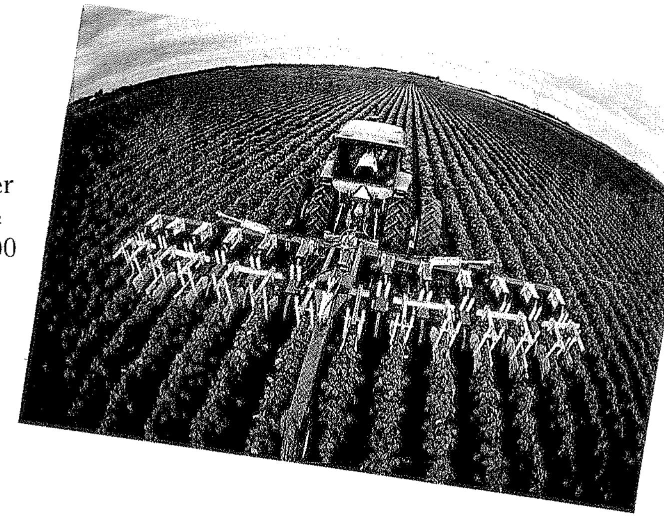
***Why* IT'S IMPORTANT**

You can use linear programming to solve problems involving agricultural and manufacturing.

Real World APPLICATION

Agriculture

Harris Grunden has 20 days in which to plant corn and soybeans. The corn can be planted at a rate of 60 acres per day and the soybeans at a rate of 70 acres per day. He has 1300 acres available for planting these two crops. Write inequalities to show the possible ways he can plant the available acres.



Let c represent the acres planted in corn and let s represent the acres planted in soybeans. Since Harris cannot plant negative acres of corn or soybeans, c and s must be nonnegative numbers.

$c \geq 0$ and $s \geq 0$

Since corn can be planted at a rate of 60 acres per day, Harris can plant no more than 20×60 or 1200 acres of corn.

$c \leq 1200$ *Thus,* $0 \leq c \leq 1200$.

Since soybeans can be planted at a rate of 70 acres per day, Harris can plant no more than 20×70 or 1400 acres of soybeans.

$s \leq 1400$ *Thus,* $0 \leq s \leq 1400$.

Harris cannot plant more than 1300 acres of corn and soybeans.

$c + s \leq 1300$

Since Harris has 20 days to plant the corn and soybeans, the number of days spent planting these crops must be less than or equal to 20.

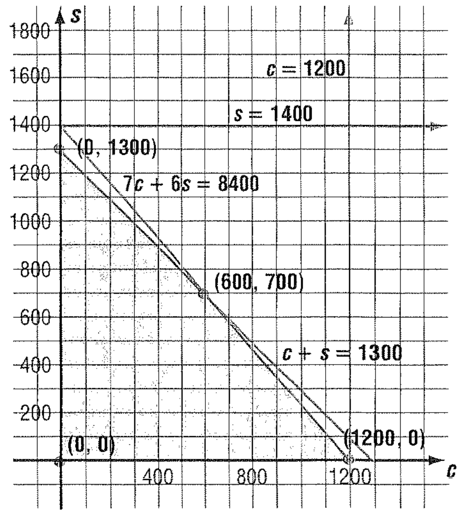
$\frac{c}{60} + \frac{s}{70} \leq 20$ $\frac{c}{60}$ *represents the time needed to plant the corn.*
$\frac{s}{70}$ *represents the time needed to plant the soybeans.*

$7c + 6s \leq 8400$ *Multiply each side by 420 to eliminate fractions.*

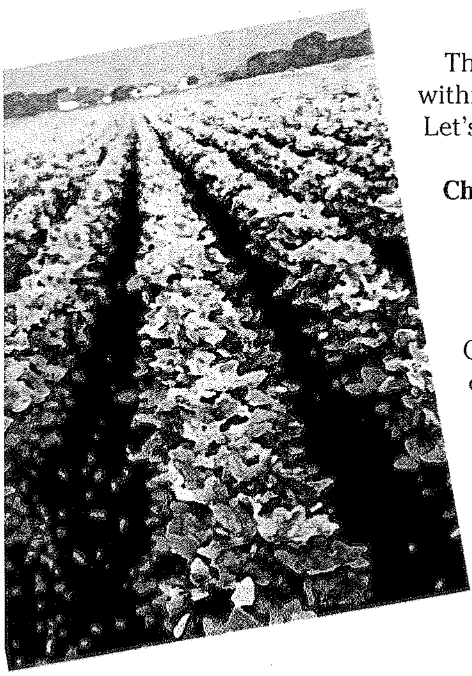
If we graph these inequalities, all of the points in their intersection are possible ways Harris can plant the available acres. The inequalities are called the **constraints**. The area of intersection of the graphs, in which every constraint is met, is called the **feasible region** of the planting.

Let's graph the constraints.

$0 \le c \le 1200$
$0 \le s \le 1400$
$c + s \le 1300$
$7c + 6s \le 8400$



The first two constraints indicate that the graph is in the first quadrant.



The feasible region contains all possible solutions. If we choose any point within the region, its ordered pair should be a solution to each inequality. Let's try (600, 400).

Check: $0 \le 600 \le 1200$
$0 \le 400 \le 1400$
$600 + 400 \le 1400$
$7 \cdot 600 + 6 \cdot 400 \le 8400$ *All are true.*

Choose a point outside the region and test it in each inequality. Is its ordered pair a solution to all of the inequalities?

Harris needs to decide how many acres of each crop to plant. There are many options, as we have seen by graphing the constraints. Of course Harris would like to make as much of a profit as possible. If the profit on corn is \$30 per acre and the profit on soybeans is \$26 per acre, how much of each should he plant to maximize his profit? Harris uses a spreadsheet to organize his results.

The notation f(c, s) is used to represent a function with two variables c and s.

The profit can be defined by the function $f(c, s) = 30c + 26s$. Mathematicians have shown that the maximum and minimum values of a function, $f(c, s)$, occur at the vertices of the feasible region. The coordinates of the vertices of a feasible region can be found by reading a graph or by solving systems of equations. The points we need to try have coordinates (0, 1300), (600, 700), (1200, 0), and (0, 0).

TECHNOLOGY Tip

You can use the method in Lesson 3-1A to find the coordinates of the vertices of the feasible region.

(c, s)	30c + 26s	Profit f(c, s)
(0, 1300)	30(0) + 26(1300)	\$33,800
(600, 700)	30(600) + 26(700)	36,200
(1200, 0)	30(1200) + 26(0)	36,000
(0, 0)	30(0) + 26(0)	0

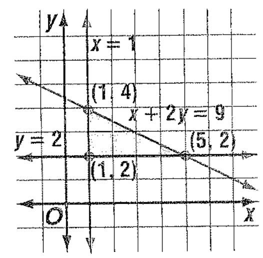
Check some other values within the feasible region to convince yourself that we have found the maximum profit for the 1300 acres. According to our results, Harris should plant 600 acres of corn and 700 acres of soybeans for a profit of \$36,200.

The process we have just used is called **linear programming**. This procedure is used to find the maximum or minimum value of a linear function subject to given conditions on the variables, called constraints. When a system of inequalities produces a convex polygonal region as a solution, the maximum or minimum value of a related function will occur at a vertex of the region.

Example 1 **Find the maximum and minimum values of $f(x, y) = 2x - 3y$ for the polygonal region determined by the system of inequalities.**

$x \geq 1$
$y \geq 2$
$x + 2y \leq 9$

First we must find the vertices of the feasible region. Graph the inequalities.



The polygon formed is a triangle with vertices at (1, 2), (5, 2), and (1, 4).

Use a chart to find the maximum and minimum values of the function.

(x, y)	2x − 3y	f(x, y)
(1, 2)	2(1) − 3(2)	−4
(5, 2)	2(5) − 3(2)	4
(1, 4)	2(1) − 3(4)	−10

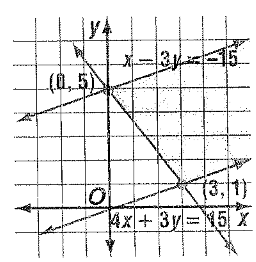
The maximum value is 4 at (5, 2). The minimum value is −10 at (1, 4).

Sometimes a polygonal region is not formed. In this case, the function is said to be **unbounded**.

Example 2 **Graph the following constraints. Then find the maximum and minimum values of the function $f(x, y) = 5x + 2y$.**

$x - 3y \leq 0$
$x - 3y \geq -15$
$4x + 3y \geq 15$

Graph the system of inequalities. There are only two points of intersection, (0, 5) and (3, 1).



(x, y)	5x + 2y	f(x, y)
(0, 5)	5(0) + 2(5)	10
(3, 1)	5(3) + 2(1)	17

The minimum is 10 at (0, 5).

Although $f(3, 1)$ is 17, it is not the maximum value since there are other points in the feasible region that produce greater values. For example, $f(4, 2) = 24$ and $f(300, 101) = 1702$. It appears that $f(x, y) = 5x + 2y$ has no maximum value when using the given constraints. Thus, the region is unbounded.

When using linear programming, we do not check every ordered pair in the feasible region to find the maximum and minimum values of the function. Instead we solve a simpler problem by checking only the values at the vertices. **Solve a simpler problem** is one of many problem-solving strategies that you can use to solve problems.

Example 3 **Find the sum of the whole numbers 1 to 1000, inclusive.**

PROBLEM SOLVING
Solve a Simpler Problem

We could add all of the numbers directly, but even with a calculator that would be time-consuming and tedious. Let's look at the sum of the whole numbers 1 to 10 to see if we can find a faster way.

$$\begin{aligned} S &= 1 + 2 + 3 + \ldots + 10 \\ (+)\ S &= 10 + 9 + 8 + \ldots + 1 \\ \hline 2S &= 11 + 11 + 11 + \ldots + 11 \\ 2S &= 10 \cdot 11 \\ 2S &= 110 \\ S &= 55 \end{aligned}$$

S represents the sum of the whole numbers.

Now, extend this concept to the original problem.

$$\begin{aligned} S &= 1 + 2 + 3 + \ldots + 1000 \\ (+)\ S &= 1000 + 999 + 998 + \ldots + 1 \\ \hline 2S &= 1001 + 1001 + 1001 + \ldots + 1001 \\ 2S &= 1000 \cdot 1001 \\ 2S &= 1{,}001{,}000 \\ S &= 500{,}500 \end{aligned}$$

The sum of the numbers of the right side of the equals sign has 1000 addends of 1001.

Therefore, the product of 1000 and 1001 equals twice the sum.

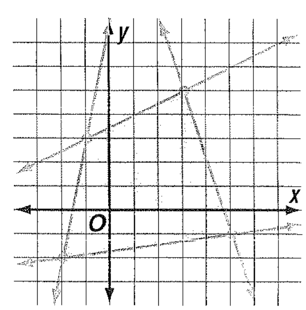
The sum of the whole numbers from 1 to 1000 is 500,500.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. A feasible region is defined by the graph at the right.



   a. **Name** the points at which a maximum or minimum value of a function could occur for the feasible region.
   b. **Find** the maximum and minimum value of the function $f(x, y) = 2x + 3y$.

2. **Define** linear programming in your own words.

3. **Describe** a problem that might be solved by the strategy of solving a simpler problem.

Guided Practice

A feasible region has vertices at (−3, 2), (4, 1), (2, 6), and (1, −2). Find the maximum and minimum values of each function.

4. $f(x, y) = x + 2y$

5. $f(x, y) = 4x - y$

Graph each system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the given function for this region.

6. $y \geq 1$
$x \leq 6$
$y \leq 2x + 1$
$f(x, y) = x + y$

7. $4y \leq x + 8$
$x + y \geq 2$
$y \geq 2x - 5$
$f(x, y) = 4x + 3y$

8. $y \geq 2$
$1 \leq x \leq 5$
$y \leq x + 3$
$f(x, y) = 3x - 2y$

9. **Solve a Simpler Problem** Carl Friedrich Gauss (1777–1855) of Germany was the greatest mathematician of his time. When he was in elementary school, his teacher wanted to keep the students busy by asking them to add the numbers from 1 to 100, inclusive. Within seconds, Gauss declared that the answer was 5050.
   a. How did Gauss add the numbers so quickly?
   b. What is the sum of the whole numbers from 1 to 2000, inclusive?

EXERCISES

Practice

A feasible region has vertices at (−1, 3), (3, 5), (4, −1), and (−1, −2). Find the maximum and minimum values of each function.

10. $f(x, y) = x - y$

11. $f(x, y) = 3x + 2y$

12. $f(x, y) = 3y - x$

13. $f(x, y) = -2x - y$

Graph each system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the given function for this region.

14. $2x + 3y \geq 6$
$3x - 2y \geq -4$
$5x + y \leq 15$
$f(x, y) = x + 3y$

15. $x \geq 1$
$y \geq 0$
$2x + y \leq 6$
$f(x, y) = 3x + y$

16. $x + y \geq 4$
$3x - 2y \leq 12$
$x - 4y \geq -16$
$f(x, y) = x - 2y$

17. $y \leq 2x + 1$
$1 \leq y \leq 3$
$y \leq -0.5x + 6$
$f(x, y) = 3x + y$

18. $y \leq x + 6$
$y + 2x \geq 6$
$2 \leq x \leq 6$
$f(x, y) = -x + 3y$

19. $x + y \geq 2$
$2y \geq 3x - 6$
$4y \leq x + 8$
$f(x, y) = 3y + x$

20. $x - 3y \geq -7$
$5x + y \leq 13$
$x + 6y \geq -9$
$3x - 2y \geq -7$
$f(x, y) = x - y$

21. $x \geq 2$
$x \leq 4$
$y \geq 1$
$x - 2y \geq -4$
$f(x, y) = x - 3y$

22. $x \geq 0$
$y \geq 0$
$x + 2y \leq 6$
$2y - x \leq 2$
$x + y \leq 5$
$f(x, y) = 3x - 5y$

Graphing Calculator

Use a graphing calculator to find the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the given function for this region.

23. $0 \leq x \leq 5$
$y \geq 0$
$-x + y \leq 2$
$x + y \leq 6$
$f(x, y) = 5x - 3y$

24. $x \leq 3$
$y \leq 5$
$x + y \geq 1$
$x \geq 0$
$y \geq 0$
$f(x, y) = 2x + 8y + 10$

25. $y \leq 1$
$y \geq -2$
$5x \leq -2$
$1.2x - y \geq -2.9$
$f(x, y) = 4x + 2y$

Critical Thinking

26. The vertices of a feasible region are $A(1, 2)$, $B(5, 2)$, and $C(1, 4)$. Write a function that satisfies each condition.
    a. A is the maximum and B is the minimum.
    b. C is the maximum and B is the minimum.
    c. B is the maximum and A is the minimum.
    d. A is the maximum and C is the minimum.
    e. A is the minimum and both B and C are maximums.

Applications and Problem Solving

Real World

27. **Employment** Rosalyn works no more than 20 hours a week during the school year. She is paid \$10 an hour for tutoring geometry students and \$7 an hour for delivering pizzas for Pizza King. She wants to spend at least 3 hours but no more than 8 hours a week tutoring. Find Rosalyn's maximum earnings.

28. **Manufacturing** The Northern Wisconsin Paper Mill can convert wood pulp to either notebook paper or newsprint. The mill can produce at most 200 units of paper a day. At least 10 units of notebook paper and 80 units of newspaper are required daily by regular customers. If the profit on a unit of notebook paper is \$500 and the profit on a unit of newsprint is \$350, how many units of each type of paper should the manager have the mill produce each day to maximize profits?

29. **Agriculture** Refer to the application at the beginning of the lesson. Suppose Harris has an opportunity to plant 50 acres of his father's land in corn and soybeans. Do you think Harris should farm the land? Explain.

30. **Solve a Simpler Problem** A team is eliminated from the Central Indiana Women's Basketball Tournament when it loses a game. If there are 30 teams playing in the tournament, how many games will need to be played to determine a champion?

31. **Solve a Simpler Problem** Opa Azul is a marketing research executive for a soft drink company. The research team has arranged to perform a survey in 16 different local shopping malls. To ensure that competing soft drink companies will not learn of the survey results, Opa has arranged for telephone lines to be set up so that each survey station has a direct line to each of the other stations. How many telephone lines does Opa need to have installed?

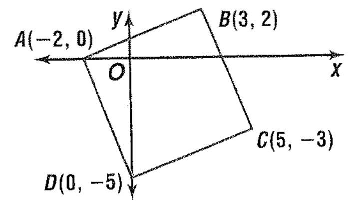
Mixed Review

32. **Geometry** ~~Consider~~ Graph the system of inequalities $x - 2y \leq -1$, $5x - 2y \geq -21$, $x - 2y \geq -9$, and $3x + 2y \leq 13$. (Lesson 3–4)
   a. What is the most descriptive name for the region formed by this system?
   b. Name the coordinates of the vertices.

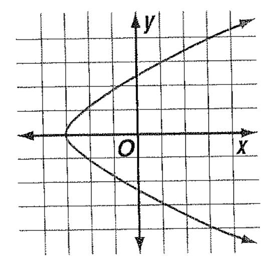
33. Use Cramer's rule to solve the system of equations. (Lesson 3–3)
$3x + 2y = 9$
$2x - 3y = 19$

34. **SAT Practice** What is the area of square $ABCD$?

A 25 B $4\sqrt{29}$ C 29
D $25 + \sqrt{2}$ E 49



35. Is the relation at the right a function? (Lesson 2–1)



36. **Business** Reliable Rentals rents cars for \$12.95 per day plus 15¢ per mile. Luis Romero works for a company that limits expenses for car rentals to \$90 per day. What is the maximum number of miles that Mr. Romero can drive each day? (Lesson 1–6)

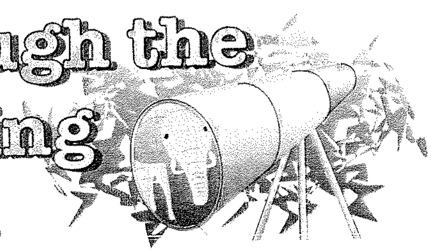
For Extra Practice, see page 882.

37. Simplify $4(12 - 5b) - 3(4b - 2)$. (Lesson 1–2)

WORKING ON THE
In·ves·ti·ga·tion

Refer to the Investigation on pages 60–61.

Through the Looking Glass

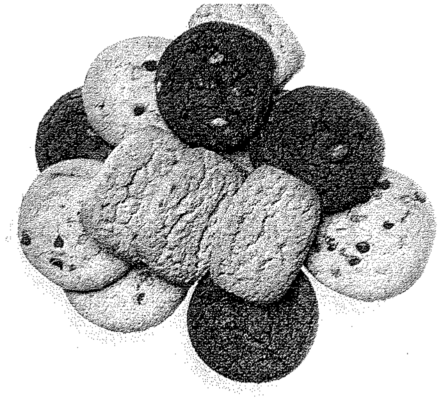


A proportion is an equality in which $\frac{a}{b} = \frac{c}{d}$. A proportion exists that relates the dimensions of the scope, the height of the animal's image through the tube, and the distance you are from the animal.

1. Look for this relationship and compare your results from the different scopes as worked on the Investigation in Lesson 3–2.
2. Use your data to write a proportion for each scope.
3. Graph these equations on the same coordinate plane. Write any observations you make about these graphs.

Add the results of your work to your Investigation Folder.

3-6 Applications of Linear Programming



What YOU'LL LEARN

- To solve problems involving maximum and minimum values by using linear programming techniques.

Why IT'S IMPORTANT

You can use linear programming to solve problems involving manufacturing and business.

Real World APPLICATION

Business

Mrs. Fernandez's Cookie Factory makes different-sized packages of cookies that contain a combination of chocolate chip and peanut butter cookies. Mrs. Fernandez places at least three of each type of cookie in each of the combination packages. The largest package, the "Baker's Dozen," contains thirteen cookies. It costs Mrs. Fernandez 19¢ to make a chocolate chip cookie and 13¢ to make a peanut butter cookie. She sells them for 44¢ and 39¢, respectively. How should Mrs. Fernandez package the cookies to maximize profit? Which packaging will yield the least profit?

Linear programming can be used to solve many types of problems like the one above. These problems have certain restrictions placed on the variables, and some function of the variables must be maximized or minimized. The steps used to solve a problem using linear programming are listed below.

Linear Programming Procedure	
	1. Define the variables.
	2. Write a system of inequalities.
	3. Graph the system of inequalities.
	4. Find the coordinates of the vertices of the feasible region.
	5. Write an expression to be maximized or minimized.
	6. Substitute the coordinates of the vertices into the expression.
	7. Select the greatest or least result. Answer the problem.

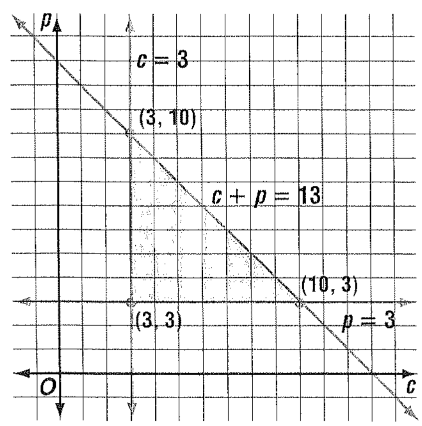
Use this procedure to solve the problem given above.

FYI

The largest cookie ever made was a chocolate chip cookie made in Arcadia, California, on October 15, 1993. It was a rectangle 35 feet by 28 feet 7 inches, and it contained more than 3 million chocolate chips.

Let c represent the number of chocolate chip cookies, and let p represent the number of peanut butter cookies in a package. Write the system of inequalities and graph.

$$c \geq 3$$
$$p \geq 3$$
$$c + p \leq 13$$



The vertices of the feasible region are at (3, 3), (3, 10), and (10, 3).

Then write an equation for the profit.

	Selling Price		*Baking Cost*		*Profit*
Chocolate Chip:	44¢	−	19¢	=	25¢
Peanut Butter:	39¢	−	13¢	=	26¢

The profit function is $f(c, p) = 25c + 26p$.

Make a chart to find the maximum and minimum profit.

(c, p)	$25c + 26p$	$f(c, p)$
(3, 3)	25(3) + 26(3)	153
(3, 10)	25(3) + 26(10)	335
(10, 3)	25(10) + 26(3)	328

Using the chart, we can see that packaging 3 chocolate chip cookies with 10 peanut butter cookies yields a maximum profit, \$3.35. Packaging 3 chocolate chip cookies with 3 peanut butter cookies yields a minimum profit, \$1.53.

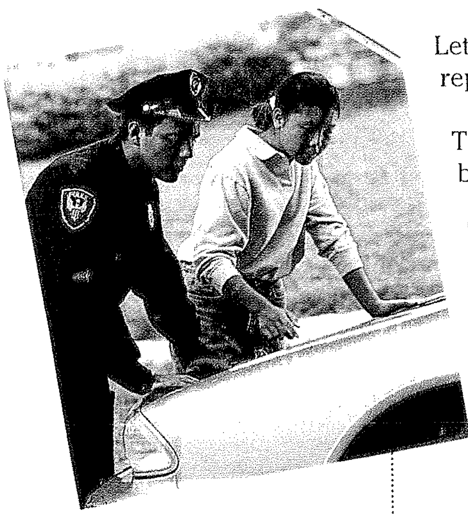
Students who use test-taking strategies can improve their scores. Making the best use of the time available is one important strategy.

Example

Real World APPLICATION

Education

Dolores Acosta arrives at school late because her car broke down, and therefore, has only 45 minutes to complete a history exam. The exam has 2 open-ended questions and 30 multiple-choice questions. Each correct open-ended question is worth 20 points, and each multiple-choice question is worth 2 points. She knows that it usually takes her 15 minutes to answer an open-ended question and only one minute to answer a multiple-choice question. Assume that for each question Dolores answers, she receives full credit. How many of each type of question should she answer to receive the maximum possible points?



Let e represent the number of open-ended questions answered, and let m represent the number of multiple-choice questions answered.

The number of open-ended questions must be between 0 and 2, inclusive.

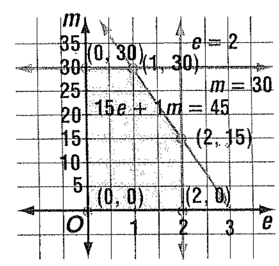
$0 \le e \le 2$

The number of multiple-choice questions must be between 0 and 30, inclusive.

$0 \le m \le 30$

The total time spent on the test must be less than or equal to 45 minutes.

$15e + 1m \le 45$



Graph the system. The vertices of the feasible region are at (0, 0), (2, 0), (2, 15), (1, 30), and (0, 30).

The function that describes the number of points earned is $f(e, m) = 20e + 2m$.

(e, m)	$20e + 2m$	$f(e, m)$
(0, 0)	20(0) + 2(0)	0
(2, 0)	20(2) + 2(0)	40
(2, 15)	20(2) + 2(15)	70
(1, 30)	20(1) + 2(30)	80
(0, 30)	20(0) + 2(30)	60

Dolores should answer 1 open-ended question and 30 multiple-choice questions to get a maximum score of 80.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. **You Decide** Pan says that a function always has a minimum value for a given region. Lula says that sometimes a function does not have a minimum value. Who is correct and why?
2. **Explain** why coordinates of the vertices of the feasible region produce the maximum and minimum values in a linear programming situation.

MATH JOURNAL

3. **Write** a paragraph explaining in your own words how to solve a linear programming problem.

Guided Practice

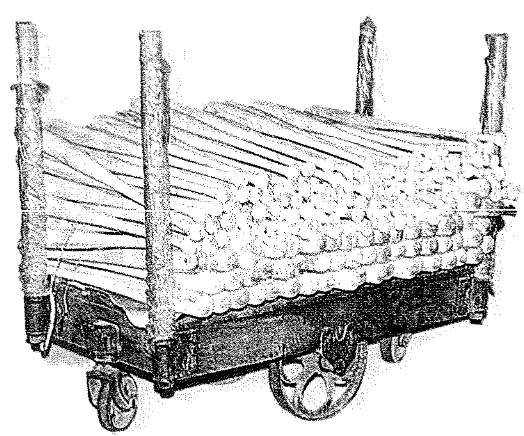
4. NaKisha Heyman has just finished writing a research paper. She has hired a typist who will type the paper using a word processor. The typist charges \$3.50 per page if no charts or graphs are used and \$8.00 per page if a chart or graph appears on the page. NaKisha knows there will be at most 40 pages having no charts or graphs. There will be no more than 16 pages with charts or graphs, and the paper will be 50 pages or less.
   a. Write inequalities that limit the number of plain pages to be typed.
   b. Write inequalities that limit the number of pages with charts or graphs to be typed.
   c. Write an equality that expresses the total number of pages to be prepared.
   d. Draw the graph showing the feasible region.
   e. List the coordinates of all the vertices of the feasible region.
   f. Write an expression for the cost to have the paper typed.
   g. Which vertex produces the greatest cost?
   h. What is the greatest possible cost to have the paper typed?

EXERCISES

Applications and Problem Solving

Real World

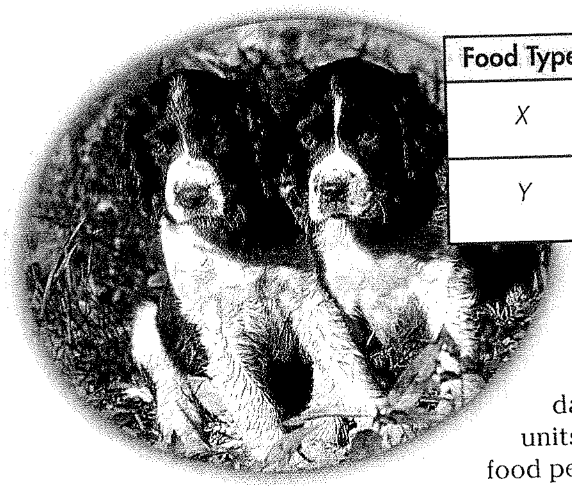
5. **Manufacturing** Superbats, Inc., manufactures two different quality wood baseball bats, the Wallbanger and the Dingbat. The Wallbanger takes 8 hours to trim and turn on a lathe and 2 hours to finish it. It has a profit of \$17. The Dingbat takes 5 hours to trim and turn on a lathe and 5 hours to finish, but its profit is \$29. The total time per day available for trimming and lathing is 80 hours and for finishing is 50 hours.
   a. If w represents the number of Wallbangers produced per day and d represents the number of Dingbats produced per day, write a system of inequalities to represent the number of Wallbanger and Dingbats that can be produced per day.
   b. Draw the graph showing the feasible region.
   c. Write an expression for the profit per day.
   d. How many of each type of bat should be produced to have the maximum profit? What is the maximum profit?



6. **Business** The available parking area of a parking lot is 600 square meters. A car requires 6 square meters of space, and a bus requires 30 square meters of space. The attendant can handle no more than 60 vehicles.

   a. Let c represent the number of cars, and let b represent the number of buses. Write a system of inequalities to represent the number of cars and buses that can be parked on the lot.
   b. If the parking fees are \$2.50 for cars and \$7.50 for buses, how many of each type of vehicle should the attendant accept to maximize income? What is the maximum income?
   c. The parking fees for special events are \$4.00 for cars and \$8.00 for buses. How many of each vehicle should the attendant accept during a special event to maximize income? What is the maximum income?

7. **Veterinary Medicine** The table below shows the amounts of nutrient A and nutrient B in two types of dog food, X and Y.



Food Type	Amount of Ingredient A	Amount of Ingredient B
X	1 unit per pound	$\frac{1}{2}$ unit per pound
Y	$\frac{1}{3}$ unit per pound	1 unit per pound

The dogs in Kay's K-9 Kennel must get at least 40 pounds of food per day. The food may be a mixture of foods X and Y. The daily diet must include at least 20 units of nutrient A and at least 30 units of nutrient B. The dogs must not get more than 100 pounds of food per day.

   a. Food X costs \$0.80 per pound and food Y costs \$0.40 per pound. What is the least possible cost per day for feeding the dogs?
   b. If the price of food X is raised to \$1.00 per pound, and the price of food Y stays the same, should Kay change the combination of foods she is using? Explain why or why not.

8. **Manufacturing** One of the dolls that Dolls R Us manufactures is Talking Tommy. Another doll without the talking mechanism is called Silent Sally. In one hour, the company can produce 8 Talking Tommy dolls or 20 Silent Sally dolls. Because of the demand, the company knows that it must produce at least twice as many Talking Tommy dolls as Silent Sally dolls. The company spends no more than 48 hours per week making these two dolls. The profit on each Talking Tommy is \$3.00, and the profit on each Silent Sally is \$7.50.

   a. How many of each doll should be produced to maximize profit each week?
   b. What is the profit?

9. **Retail** A sales associate at a paint store plans to mix color A and color B. The sales associate has exactly 32 units of blue dye and 54 units of red dye. Each gallon of color A requires 4 units of blue dye and one unit of red dye. Each gallon of color B requires one unit of blue dye and 6 units of red dye.

   a. Let a represent the number of gallons of color A, and let b represent the number of gallons of color B. Write the inequalities that represent the number of gallons of paint that can be mixed.
   b. Find the maximum number of gallons, $a + b$, that can be mixed.

10. Manufacturing TeeVee Electronics, Inc., makes console and wide-screen televisions. The equipment in the factory allows for making at most 450 console televisions and 200 wide-screen televisions in one month. The chart below shows the cost of making each type of television, as well as the profit for each type.

Television	Cost per Unit	Profit per Unit
Console	\$600	\$125
Wide Screen	\$900	\$200

During the month of November, the company can spend \$360,000 to make these televisions. To maximize profits, how many of each type should they make?

11. Education Carol Sommers has 50 minutes to take an English test that has 20 multiple-choice questions and 20 short-answer questions. She knows she can answer a multiple-choice question in $1\frac{1}{2}$ minutes and a short-answer question in 2 minutes. Each correct multiple-choice answer receives 2 points, and each correct short-answer receives 3 points. Assume that any question Carol answers, she gets correct.

a. What is the maximum possible score she can receive?

b. What advice might you give Carol to improve her score?

Critical Thinking

12. Consider the feasible region defined by the system of inequalities below.

$$0 \le x \le 5$$
$$0 \le y \le 6$$
$$x + 2y \le 13$$
$$2x + y \le 11$$

a. Suppose the profit function for the feasible region is $f(x, y) = 3x + 4y$. Graph the feasible region. On the same coordinate plane, graph the profit function for $f(x, y) = 32, 28, 24, 20,$ and 16. What does the graph tell you about the maximum point?

b. Suppose the profit function for the feasible region is $g(x, y) = 3x + 6y$. Graph the feasible region on another coordinate plane. Then add the graph of the profit function for $g(x, y) = 42, 36, 30,$ and 24. What does the graph tell you about the maximum point?

Mixed Review

13. Graph the system of inequalities $x \ge 0$, $y \ge 3$, $y \ge 2x + 1$ and $y \le -0.5x + 6$. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the function $f(x, y) = 3x - 2y$ for this region. (Lesson 3–5)

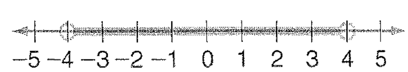
14. Geometry Write a system of inequalities that will form a region shaped like an isosceles right triangle. (Lesson 3–4)

15. SAT Practice Grid-in Points A, B, C, and D lie on a line in that order. If $AC = \frac{4}{3}AB$ and $BD = 6BC$, then what is $\frac{AB}{CD}$?

16. Graph $5y - 25x = -10$. (Lesson 2–2)

For Extra Practice, see page 882.

17. State an absolute value inequality for the graph at the right. (Lesson 1–7)



3–7 Solving Systems of Equations in Three Variables

***What* YOU'LL LEARN**

- To solve a system of three equations in three variables.

***Why* IT'S IMPORTANT**

You can use systems of equations to solve problems involving banking and consumer awareness.

Real World APPLICATION

Business

The Nutty Food Company sells trail mixes and other snacks by the pound. A clerk is filling a barrel with peanuts, raisins, and carob-coated pretzels. The manager wants the associate to make 80 pounds of this mixture and to sell it for \$3.35 per pound. The peanuts sell for \$3.20 per pound, the raisins sell for \$2.40 per pound, and the carob-coated pretzels sell for \$4.00 per pound. If the mixture has twice as many pounds of carob-coated pretzels as raisins, how many pounds of each ingredient should the clerk use?

Problems like this can be expressed using a system of equations in three variables. Let p represent the number of pounds of peanuts, let r represent the number of pounds of raisins, and let c represent the number of pounds of carob-coated pretzels. Then write a system of equations using the information given.

$p + r + c = 80$ *The clerk makes 80 pounds of the mixture.*

$3.20p + 2.40r + 4.00c = 268$ *The total selling price is 80 × \$3.35, or \$268.*

$c = 2r$ *There are twice as many pounds of carob-coated pretzels as raisins.*

Solving systems such as this one is similar to solving systems of equations in two variables.

Since $c = 2r$, substitute $2r$ for c in each of the first two equations.

$$p + r + (2r) = 80 \rightarrow p + 3r = 80$$
$$3.20p + 2.40r + 4.00(2r) = 268 \rightarrow 3.20p + 10.40r = 268$$

The result is two equations with the same two variables. Use elimination to solve for r.

$$p + 3r = 80 \xrightarrow{\text{Multiply by 3.20}} 3.20p + 9.60r = 256$$
$$3.20p + 10.40r = 268 \qquad (-)\ 3.20p + 10.40r = 268$$

Subtract to eliminate p. $-0.8r = -12$

Solve for r. $r = 15$

Since $r = 15$, substitute 15 for r in the equation $p + 3r = 80$.

$p + 3(15) = 80$

$p + 45 = 80$

$p = 35$ *Solve for p.*

Finally, substitute 15 for r in the original equation $c = 2r$.

$c = 2(15)$

$= 30$

The associate should use 35 pounds of peanuts, 15 pounds of raisins, and 30 pounds of carob-coated pretzels.

The solution of a system of equations in three variables x, y, and z, is called an **ordered triple** (x, y, z).

Example 1 **Solve the system of equations.**

$$x + 2y - 3z = 50$$
$$2x + y + 2z = 3$$
$$2x - 5y + 4z = -79$$

Use elimination to make a system of two equations in two variables.

$x + 2y - 3z = 50$ → Multiply by 2. → $2x + 4y - 6z = 100$

$2x + y + 2z = 3$ → Multiply by −1. → $(+)\ -2x - y - 2z = -3$

Add to eliminate x. $3y - 8z = 97$

$2x + y + 2z = 3$ → $2x + y + 2z = 3$

$2x - 5y + 4z = -79$ → Multiply by −1. → $(+)\ -2x + 5y - 4z = 79$

Add to eliminate x. $6y - 2z = 82$

The result is two equations with the same two variables. Use elimination to solve for z.

$3y - 8z = 97$ → Multiply by −2. → $-6y + 16z = -194$

$6y - 2z = 82$ → $(+)\ 6y - 2z = 82$

Add to eliminate y. $14z = -112$

Solve for z. $z = -8$

Substitute -8 for z in the equation $3y - 8z = 97$.

$$
\begin{aligned}
3y - 8(-8) &= 97 \\
3y + 64 &= 97 \\
3y &= 33 \\
y &= 11 \quad \textit{Solve for y.}
\end{aligned}
$$

Substitute 11 for y and -8 for z in the original equation $x + 2y - 3z = 50$.

$$
\begin{aligned}
x + 2(11) - 3(-8) &= 50 \\
x + 22 + 24 &= 50 \\
x + 46 &= 50 \\
x &= 4 \quad \textit{Solve for x.}
\end{aligned}
$$

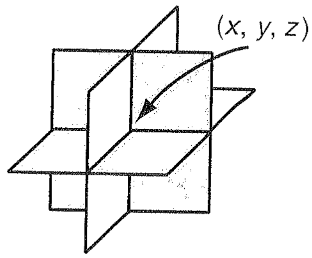
The solution is $(4, 11, -8)$.

LOOK BACK

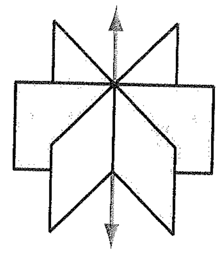
You can refer to Lesson 3-1 for information on the number of solutions for systems of two linear equations in two variables.

You know that a system of two linear equations in two variables does not always have a unique solution that is an ordered pair. Similarly, a system of three linear equations in three variables does not always have a unique solution that is an ordered triple. The graph of each equation in a system of three linear equations in three variables is a plane. The three planes can appear in various configurations.

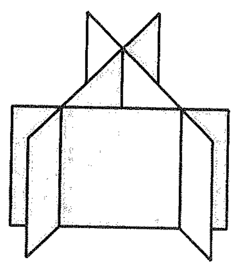
The three planes intersect at one point, so the system has a unique solution, an ordered triple (x, y, z).

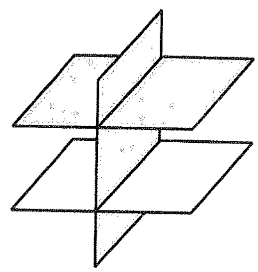


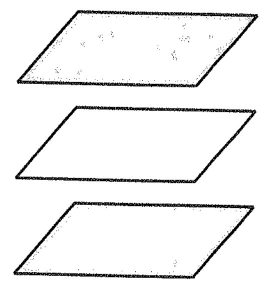
The three planes intersect in a line. There is an infinite number of solutions to the system.



Each of the figures below shows three planes that have *no* points in common. These systems of equations have *no* solutions.







If all three planes coincide, there are infinite number of solutions. Also if two planes coincide and intersect the third plane in a line, there are infinite number of solutions.

When graphing two equations in two variables, it is obvious when the system is inconsistent or dependent. However, it is usually *not* obvious when solving a system of three equations algebraically whether there is a unique solution, no solutions, or many solutions.

Example 2

Solve the system of equations.

$$\mathbf{3x - 6y + 3z = 33}$$
$$\mathbf{2x - 4y + 2z = 22}$$
$$\mathbf{4x + 2y - z = -6}$$

Eliminate x in the first two equations.

$3x - 6y + 3z = 33$ → Multiply by 2. → $6x - 12y + 6z = 66$

$2x - 4y + 2z = 22$ → Multiply by −3. → $(+)\ -6x + 12y - 6z = -66$

$$0 = 0$$

The equation $0 = 0$ is always true. This indicates that there are infinite number of solutions. In this case, the first two equations represent the same plane. This plane intersects the plane represented by the third equation. The coordinates of any point on the line of intersection are a solution to the system of equations.

Example 3 **Real World APPLICATION** **Education**

The three American universities with the greatest endowments are Harvard, Yale, and Princeton. Their combined endowments are \$12.09 billion. Together Yale and Princeton have \$0.53 billion more in endowments than Harvard. Princeton's endowments trail Harvard's by \$2.70 billion. What are the endowments of each of these universities?

Explore Read the problem and define the variables.
Let h represent Harvard's endowments, let y represent Yale's endowments, and let p represent Princeton's endowments.

Plan Write three equations.

$h + y + p = 12.09$	*The combined endowments are \$12.09 billion.*
$y + p = h + 0.53$	*Together Yale and Princeton have \$0.53 billion more than Harvard.*
$p = h - 2.70$	*Princeton's endowments trail Harvard's by \$2.70 billion.*

Solve Use the elimination method to solve for h.

$$\begin{aligned} h + y + p &= 12.09 &\rightarrow&\quad h + y + p = 12.09 \\ y + p &= h + 0.53 &\rightarrow&\quad (-)\ -h + y + p = 0.53 \\ & & &\quad 2h = 11.56 \\ & & &\quad h = 5.78 \end{aligned}$$

Substitute 5.78 for h in the equation $p = h - 2.70$.

$$\begin{aligned} p &= 5.78 - 2.70 \\ &= 3.08 \end{aligned}$$

Substitute 5.78 for h and 3.08 for p in the equation $h + y + p = 12.09$.

$$\begin{aligned} 5.78 + y + 3.08 &= 12.09 \\ y + 8.86 &= 12.09 \\ y &= 3.23 \quad \textit{Solve for } y. \end{aligned}$$

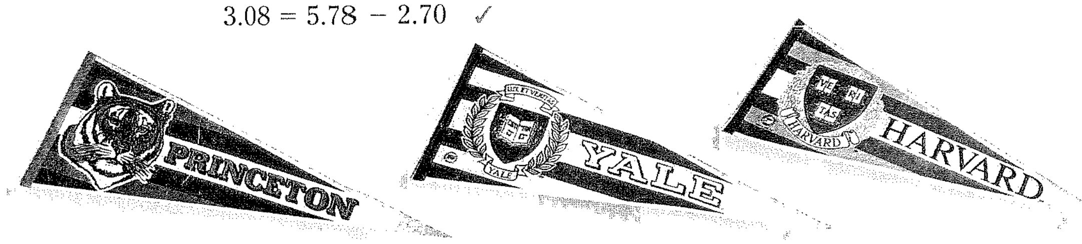
Harvard has \$5.78 billion, Yale has \$3.23 billion, and Princeton has \$3.08 billion.

Examine Check to see if all of the criteria are met.

The combined endowments are \$12.09 billion.
$5.78 + 3.23 + 3.08 = 12.09$ ✓

Yale and Princeton have \$0.53 billion more than Harvard.
$3.23 + 3.08 = 5.78 + 0.53$ ✓

Princeton's endowments trail Harvard's by \$2.70 billion.
$3.08 = 5.78 - 2.70$ ✓



FYI

Funds held by a university, hospital, or other institution are called *endowments*. Usually these funds are invested, and only the income from these investments is spent. In this way, the endowment can last forever.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Study the lesson. Then complete the following.

1. Refer to the application at the beginning of the lesson. Explain why substitution is the best first step in solving the system of equations.
2. **Describe** a situation that might occur when you solve a system of three equations in three variables that has each of the following number of solutions.
   a. none
   b. an infinite number

Guided Practice

For each system of equations, an ordered triple is given. Determine whether it is a solution of the system.

3. $3x - 7y + 2z = 43$
   $5x + 2y - 3z = -1$
   $2x + 5y - z = -12$; $(4, -3, 5)$

4. $5a - 3b + c = -3$
   $7a + 2b - 3c = -35$
   $a - 6b + 7c = 51$; $(-2, 0, 7)$

Solve each system of equations.

5. $4x - 3y + 5z = 43$
   $2x + y = 9$
   $3y - 2z = -9$

6. $6a - 2b = 18$
   $3b + 5c = -34$
   $a + 6c = -28$

7. $4x + 3y + 2z = 34$
   $2x + 4y + 3z = 45$
   $3x + 2y + 4z = 47$

8. $x + y + z = -1$
   $3x - 2y - 4z = 16$
   $2x - y + z = 19$

9. Write a system of three equations in three variables that has $(-6, 2, -5)$ as a solution.
10. Write a system of three equations in three variables in which $(-2, -3, 6)$ satisfies only two of the three equations.

EXERCISES

Practice

For each system of equations, an ordered triple is given. Determine whether it is a solution of the system.

11. $3a + 7b - 4c = -17$
    $2a - 8b - c = 8$
    $6a - b + 3c = 23$; $(2, -1, 4)$

12. $x + 3z = -5$
    $5x - 2y = -22$
    $5y - 6z = 36$; $(-2, 6, -1)$

Solve each system of equations.

13. $5r + 2s = 0$
    $-3t = 12$
    $6s + 5t = 10$

14. $2b - c = -13$
    $2a = 12$
    $3a + b = 13$

15. $x + y - z = -1$
    $x + y + z = 3$
    $3x - 2y - z = -4$

16. $b + c = 4$
    $2a + 4b - c = -3$
    $3b = -3$

17. $5x + 7y = -1$
    $-2y + 3z = 9$
    $7x - z = 27$

18. $r - s + 3t = -8$
    $2s - t = 15$
    $3r + 2t = -7$

19. $5a - b + 3c = 5$
    $2a + 7b - 2c = 5$
    $4a - 5b - 7c = -65$

20. $6x + 2y - 3z = -17$
    $7x - 5y + z = 72$
    $2x + 8y + 3z = -21$

Solve each system of equations.

21. $3x + 4y - 3z = 5$
$x + 6y + 2z = 3$
$6x + 2y + 3z = 4$

22. $4x + 7y - z = -10$
$6x - 3y + 6z = 3$
$2x + y + 8z = 9$

23. $2r + 3s + 4t = 3$
$5r - 9s + 6t = 1$
$\frac{1}{3}r - \frac{1}{2}s + \frac{1}{3}t = \frac{1}{12}$

24. $2x + y + z = 7$
$12x - 2y - 2z = 2$
$\frac{2x}{3} - y + \frac{z}{3} = -\frac{1}{3}$

25. The sum of three numbers is 12. The first is five times the second and the sum of the first and third is 9. Find the numbers.

26. The sum of three numbers is 20. The first number is the sum of the second and the third. The third number is three times the first. Find the numbers.

27. The sum of three numbers is 18. The first is eight times the sum of the second and third. The sum of the first number and the last number is 11. Find the numbers.

Critical Thinking

28. Now that you know how to solve a system of three equations in three variables, use what you know to solve the system of equations below.

$w + x + y + z = 2$
$2w - x - y + 2z = 7$
$2w + 3x + 2y - z = -2$
$3w - 2x - y - 3z = -2$

Real World

Applications and Problem Solving

CAREER CHOICES

A **stockbroker** provides financial advice about buying or selling stocks, bonds, or other financial products.

A bachelor's degree in business administration, economics, or finance is expected.

For more information, contact:

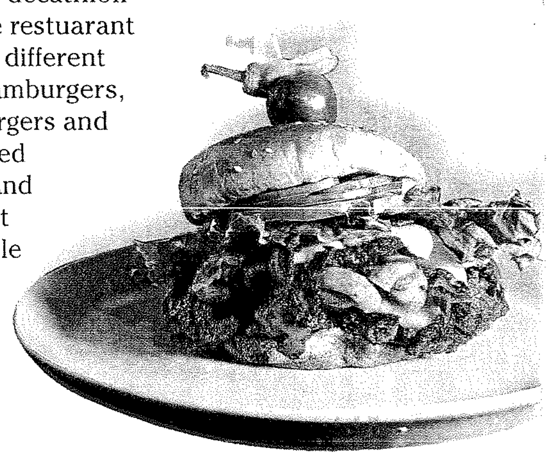
Securities Industry Association
120 Broadway
New York, NY 10271

29. **Banking** Maria Hernandez has $15,000 that she would like to invest in certificates of deposit. The bank has the following rates.

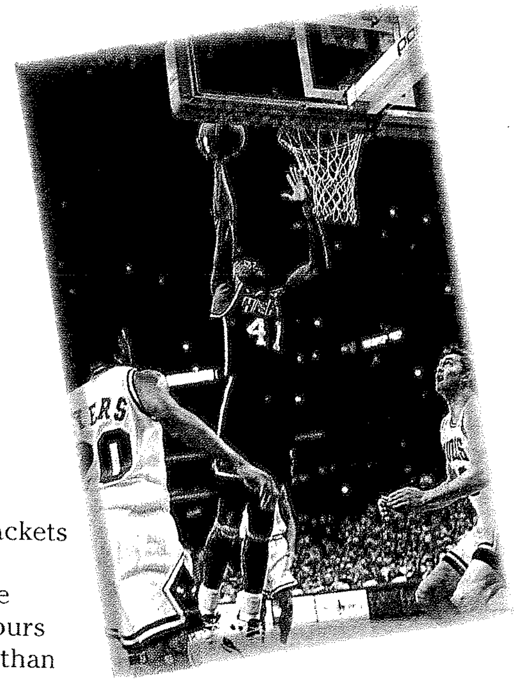
Number of Years	1	2	3
Rate	3.4%	5.0%	6.0%

She does not want to have all her money committed for more than one year, so she plans to invest some money at each rate. She wants her total interest for one year to be $800, so she will not be in a higher tax bracket. She decides to put $1000 more in a 2-year certificate than in a 1-year certificate and invest the rest in a 3-year certificate. How much should she invest in each type of certificate?

30. **Consumer Awareness** A fast-food restaurant offers three different types of hamburgers at three different prices. The types are the hamburger, the double cheeseburger, and the jumbo burger. The decathlon team at Kennedy High School went to the restuarant on three different occasions and ordered different burgers. The first time, they ordered 3 hamburgers, 5 double cheeseburgers, and 6 jumbo burgers and paid $25.24. The second time, they ordered 2 hamburgers, 7 double cheeseburgers, and 5 jumbo burgers and paid $25.68. The last time, they ordered 4 hamburgers, 4 double cheeseburgers, and 7 jumbo burgers and paid $26.59. The coach, who did not go along on the burger-buying trips, needs to know the price of each type of burger. What is the price of each kind of burger?



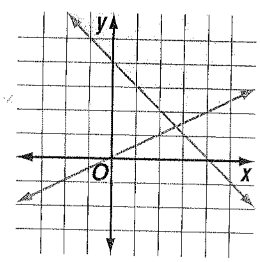
31. **Basketball** One night Glen Rice of the Miami Heat scored a total of 35 points against the Los Angeles Clippers. In basketball, it is possible to make a 3-point field goal, a 2-point field goal, or a 1-point free throw. He made as many 2-pointers as 3-pointers and free throws combined. He scored one point more with 2-pointers than he did with 3-pointers and free throws combined. How many of each did he score?



Mixed Review

32. **Manufacturing** Stitches Inc. can make at most 30 jean jackets and 20 leather jackets in a week. It takes a worker 10 hours to make a jean jacket and 20 hours to make a leather jacket. The total number of hours by all of the employees can be no more than 500 hours per week. (Lesson 3–6)
   a. If the profit on a jean jacket is the same as the profit on a leather jacket, how many of each should be made to maximize profit?
   b. How many of each should be made if the profit on a leather jacket is three times the profit on a jean jacket?

33. Which system of inequalities is represented by the graph at the right? (Lesson 3–4)
   a. $x + y \geq 4$ and $x \leq 2y$
   b. $x + y \geq 4$ and $x \geq 2y$
   c. $x + y \leq 4$ and $x \leq 2y$
   d. $x + y \leq 4$ and $x \geq 2y$



34. **SAT Practice** **Quantitative Comparison**

Column A	Column B
The percent of increase from 75 to 100	The percent of decrease from 100 to 75

   A if the quantity in Column A is greater
   B if the quantity in Column B is greater
   C if the two quantities are equal
   D if the relationship cannot be determined from the information given

35. **Economics** The Serves-You-Best Rental Car Company has two rental offers. The first offer charges the customer \$20 plus 25¢ a mile for a compact car. The second offer charges the customer \$35 plus 25¢ a mile for a luxury car. (Lesson 2–4)
   a. Write an equation to represent each offer, if the cars are rented for one day.
   b. What is the relationship between the graphs of these offers?
   c. If both cars are driven 750 miles, what is the cost difference between renting the compact car and the luxury car?

For **Extra Practice**, see page 882.

36. If $f(x) = x^2 + 3x$, find $f(5)$. (Lesson 2–1)

37. Solve $2\,|-x - 6| = -3x$. (Lesson 1–5)

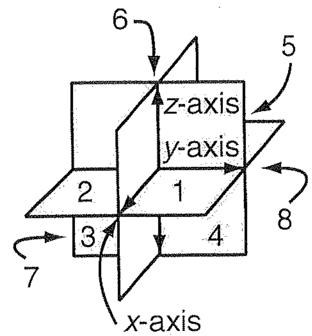
3–7B Graphing Equations in Three Variables

Materials: isometric dot paper

An Extension of Lesson 3–7

To draw the graph of an equation in three variables, it is necessary to add a third dimension to our coordinate system. The graph of an equation of the form $Ax + By + Cz = D$, where either A, B, C, or D can be equal to zero, is a plane.

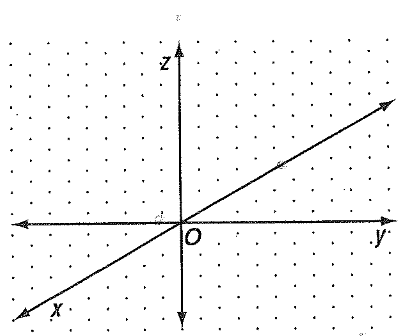
When graphing in space (three dimensions), space is separated into eight regions, called **octants.** Think of three coordinate planes intersecting at right angles as shown at the right. Any point lying on a coordinate plane is not in an octant.



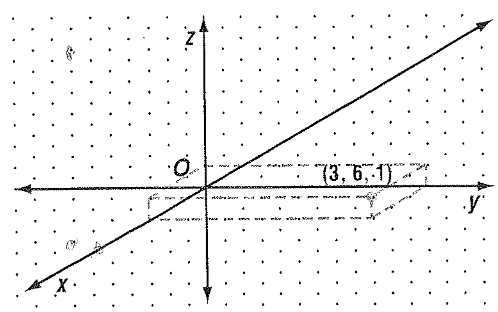
The octants are numbered as shown.

Activity 1 Use isometric dot paper to draw and label a three-dimensional axis system. Then graph the ordered triple (3, 6, 1).

Step 1 Draw the x-, y-, and z-axes as shown below.



Step 2 Locate 3 on the positive x-axis, 6 on the positive y-axis, and 1 on the positive z-axis. Complete a "box" by drawing lines parallel to the axes through each intercept. Draw the graph, which is a point, at (3, 6, 1).



This point is in octant 1, which is also called the *<u>first octant</u>.*

It is not necessary to show the entire "box" when you graph an ordered triple. The desired point will always be the corner farthest from the point of origin.

To graph a linear equation in three variables, first find the intercepts of the graph. Connect the intercepts on each axis. This forms a portion of a plane that lies in a single octant.

Activity 2 Graph $2x + 4y + 3z = 12$.

To find the x-intercept, let $y = 0$ and $z = 0$.

$2x = 12$

$x = 6$

To find the y-intercept, let $x = 0$ and $z = 0$.

$4y = 12$

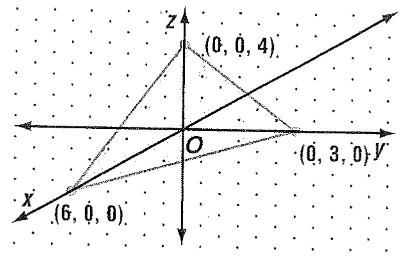
$y = 3$

To find the z-intercept, let $x = 0$ and $y = 0$.

$3z = 12$

$z = 4$

To indicate the plane, graph the intercepts, which have coordinates (6, 0, 0), (0, 3, 0), and (0, 0, 4), respectively, and then connect the points. Remember that the plane extends indefinitely.



Draw **Graph each ordered triple using isometric dot paper. Name the octant in which each point lies.**

1. (5, 2, 3)
2. (7, 5, −6)
3. (3, 0, 1)
4. (3, −7, 2)

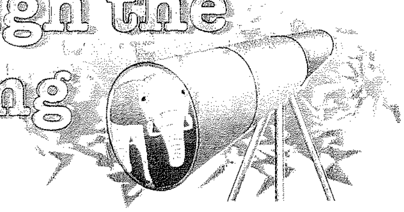
Graph each equation using isometric dot paper. Name the coordinates for the x-, y-, and z-intercepts.

5. $4x + y + 2z = 4$
6. $3x - 2y + 2z = 6$
7. $3x - y + 6z = 3$
8. $4x + 5y - 10z = 20$
9. $3z - 2x = 6$
10. $3x - 4y = -12$

Write **Write an equation of the plane given its x-, y-, and z-intercepts, respectively.**

11. 2, −2, 5
12. $\frac{1}{2}$, 3, −2
13. Describe the relationship between quadrants and octants.
14. Consider the graph of $x = 2$ in one, two, and three dimensions.
    - a. Describe the graph on a number line.
    - b. Describe the graph on a coordinate plane.
    - c. Describe the graph in a three-dimensional coordinate axis.
    - d. Compare the graphs in parts a, b, and c.

Through the Looking Glass



Refer to the Investigation on pages 60–61.

Through observations of species in their habitat, naturalists have been able to identify characteristics of the species, their migratory and breeding patterns, and the size of their population in any given area. These efforts have helped in preventing extinction of some species and better control for those species that are overpopulating some areas of Earth. Continued observations with more advanced equipment will make the naturalist's job easier in the future and offer an abundance of information that cannot be acquired from other long-distance observations.

Analyze

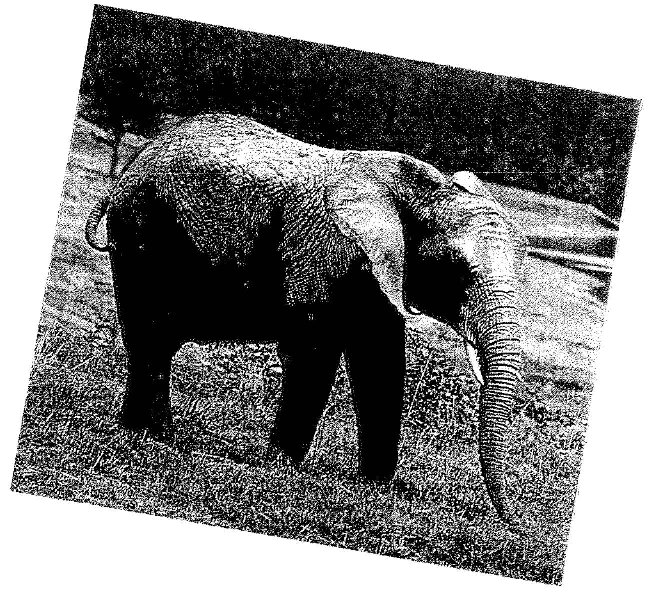
You have conducted experiments and organized your data in various ways. It is now time to analyze your findings and state your conclusions.

PORTFOLIO ASSESSMENT

You may want to keep your work on this Investigation in your portfolio.

1 Write an expression that relates the size of the image through your scope with the distance you are from the image for each scope. How does this relate to the dimensions of each scope?

2 Study the various graphs you made for each of the scopes. Write an explanation of how the scope can help you determine the actual size of an animal you are observing.



3 If you were trying to observe an object from 100 yards away, describe the tube that would enable you to see the entire object in your view.

4 Make a general statement about the dimensions of the tube in relation to the distance you are from an object when trying to view it in its entirety.

Write

Imagine that you are a tourist on safari on an African plain. There is an elephant standing a distance away across the river. The current of the river is swift and you cannot cross it. You have in your possession a tape measure, a pen, a calculator, and a piece of paper that can be rolled into various sizes of view tubes.

5 You would like to determine the height of the elephant. Write how you would explain to another tourist how you could accurately determine the elephant's height.

interNET CONNECTION

Data Collection and Comparison To share and compare your data with other students, visit:
www.algebra2.glencoe.com

CHAPTER 3 HIGHLIGHTS

NET CONNECTION

Chapter Review For additional lesson-by-lesson review, visit: **www.algebra2.glencoe.com**

VOCABULARY

After completing this chapter, you should be able to define each term, property, or phrase and give an example or two of each.

Algebra

consistent system (p. 127)
constraints (p. 153)
Cramer's rule (p. 141)
dependent system (p. 128)
determinant (p. 141)
element (p. 141)
elimination method (p. 133)
feasible region (p. 153)
inconsistent system (p. 128)
independent system (p. 127)
linear programming (p. 155)
octants (p. 172)
ordered triple (p. 166)
second-order determinant (p. 141)
substitution method (p. 133)
system of equations (p. 126)
system of inequalities (p. 148)
unbounded (p. 155)

Problem Solving

solve a simpler problem (p. 156)

UNDERSTANDING AND USING THE VOCABULARY

Choose the letter of the term that best matches each phrase.

1. a square array of numbers or variables having a numerical value
2. a system of equations that has an infinite number of solutions
3. the region of intersection of graphs of inequalities, where every constraint is met
4. a method of solving equations in which one equation is solved for one variable in terms of the other variable
5. a system of equations that has at least one solution
6. a method of solving equations in which one variable is eliminated when the two equations are combined
7. the solution of a system of equations in three variables (x, y, z)
8. a method for finding the maximum or the minimum value of a function with two variables.
9. the numbers or variables written within a determinant
10. a system of equations that has exactly one solution
11. a function in which no maximum value exists

a. consistent system
b. dependent system
c. determinant
d. elements
e. elimination method
f. feasible region
g. independent system
h. linear programming
i. ordered triple
j. substitution method
k. unbounded

OBJECTIVES AND EXAMPLES

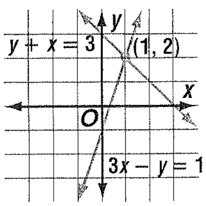
Upon completing this chapter, you should be able to:

- solve systems of equations by graphing (Lesson 3–1)

Solve the system of equations.

$y + x = 3$
$3x - y = 1$

Graph each equation. The intersection of the graphs is the solution.



The solution is (1, 2).

REVIEW EXERCISES

Use these exercises to review and prepare for the chapter test.

Graph each system of equations and state its solution. Also, state whether the system is *consistent and independent, consistent and dependent,* or *inconsistent.*

12. $3x + 2y = 12$, $x - 2y = 4$
13. $8x - 10y = 7$, $4x - 5y = 7$
14. $y - 2x = 8$, $y = \frac{1}{2}x - 4$
15. $20y + 13x = 10$, $0.65x + y = 0.5$

- use the substitution and elimination methods to solve systems of equations (Lesson 3–2)

Solve the system of equations.

$x = 4y + 7$
$x = -y - 3$

$$-y - 3 = 4y + 7 \qquad x = 4(-2) + 7$$
$$-10 = 5y \qquad = -8 + 7$$
$$-2 = y \qquad = -1$$

The solution is $(-1, -2)$.

Solve each system of equations. Use either substitution or elimination.

16. $x + y = 4$, $x - y = 8.5$
17. $2x + 3y = -6$, $3x + 2y = 25$
18. $7y - 2x = 10$, $-3y + x = -3$
19. $-6y - 2x = 0$, $11y + 3x = 4$
20. $3x - 5y = -13$, $4x + 2y = 0$
21. $c + d = 5$, $2c - d = 4$

- solve systems of equations by using Cramer's rule (Lesson 3–3)

Solve the system of equations.

$4x + 7y = -1$
$2x + y = 7$

$$x = \frac{\begin{vmatrix} -1 & 7 \\ 7 & 1 \end{vmatrix}}{\begin{vmatrix} 4 & 7 \\ 2 & 1 \end{vmatrix}} = \frac{-1(1) - 7(7)}{4(1) - 7(2)} \text{ or } 5$$

$$y = \frac{\begin{vmatrix} 4 & -1 \\ 2 & 7 \end{vmatrix}}{\begin{vmatrix} 4 & 7 \\ 2 & 1 \end{vmatrix}} = \frac{4(7) - (-1)(2)}{4(1) - 7(2)} \text{ or } -3$$

The solution is $(5, -3)$.

Use Cramer's rule to solve each system of equations.

22. $2x - 3y = 4$, $x + 5y = 2$
23. $7x + 3y = 5$, $2x + 4y = 3$
24. $2x - y = 7$, $x + 3y = 7$
25. $u + 11 = 8v$, $8(u - v) = 3$
26. $f - 2g = -1$, $2f + 3g = -16$
27. $m - n = 0$, $4m + 10n = -6$

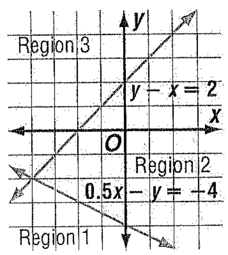
OBJECTIVES AND EXAMPLES

- solve systems of inequalities by graphing (Lesson 3–4)

Solve the system of inequalities by graphing.
$y - x \leq 2$
$0.5x + y \geq -4$

$y - x \leq 2$ represents Regions 1 and 2.
$0.5x + y \geq -4$ represents Regions 2 and 3.

The intersection is Region 2, which is the solution of this system of inequalities.



REVIEW EXERCISES

Solve each system of inequalities by graphing.

28. $y \leq 4$
$y > -3$

29. $y > 3$
$x \leq 1$

30. $y < x + 1$
$x > 5$

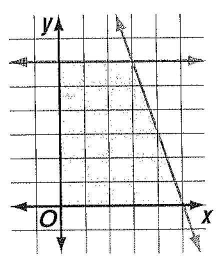
31. $x + y \geq 3$
$x \leq 0$

32. $y \leq x + 4$
$2y \geq x - 3$

33. $y < 2$
$y \geq -7$
$y \geq 2x$
$y \leq x + 1$

- find the maximum and minimum values of a function over a region using linear programming techniques (Lesson 3–5)

$x \geq 0$
$y \geq 0$
$3x + y \leq 15$
$y \leq 6$
$f(x, y) = 3x + y$



vertices: (0, 0), (5, 0),
(3, 6), (0, 6)

(x, y)	3x + y	f(x, y)
(0, 0)	3(0) + 0	0
(5, 0)	3(5) + 0	15
(3, 6)	3(3) + 6	15
(0, 6)	3(0) + 6	6

maximum value = 15, minimum value = 0

Graph each system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the given function.

34. $f(x, y) = -2x + y$
$x \geq -5$
$x \leq 4$
$y \geq -1$
$y \leq 3$

35. $f(x, y) = 3x + 2y$
$x \geq 0$
$y \geq 0$
$x + 3y \leq 15$
$4x + y \leq 16$

- solve problems involving maximum and minimum values by using linear programming techniques (Lesson 3–6)

The available parking area of a parking lot is 600 m^2. A car requires 6 m^2 of space, and a bus requires 30 m^2 of space. The attendant can handle no more than 60 vehicles. If a car is charged \$3.00 to park and a bus is charged \$8.00, how many of each should the attendant accept to maximize income? 50 cars, 10 buses

36. **Community Service** A theater at which a drug abuse program is being presented seats 150 people. The proceeds will be donated to a local drug information center. Admission is \$2 for adults and \$1 for students. Every two adults must bring at least one student. How many adults and students should attend in order to raise the maximum amount of money?

OBJECTIVES AND EXAMPLES

- solve a system of three equations in three variables (Lesson 3–7)

Solve the system of equations.

$2x + y - z = 2$
$x + 3y + 2z = 1$
$x + y + z = 2$

$$\begin{aligned} -2x - y + z &= -2 \\ (+)\ 2x + 6y + 4z &= 2 \\ \hline 5y + 5z &= 0 \\ y + z &= 0 \\ y &= -z \end{aligned}$$

$$\begin{aligned} 2(2) + (-z) - z &= 2 \\ 4 - 2z &= 2 \\ -2z &= -2 \\ z &= 1 \end{aligned}$$

$$\begin{aligned} x + (-z) + z &= 2 \\ x &= 2 \end{aligned}$$

$$\begin{aligned} 2 + y + 1 &= 2 \\ y &= -1 \end{aligned}$$

The solution is $(2, -1, 1)$.

REVIEW EXERCISES

Solve each system of equations.

37. $x + 4y - z = 6$
$3x + 2y + 3z = 16$
$2x - y + z = 3$

38. $2a + b - c = 5$
$a - b + 3c = 9$
$3a - 6c = 6$

39. $e + f = 4$
$2d + 4e - f = -3$
$3e = -3$

APPLICATIONS AND PROBLEM SOLVING

40. Donkey Basketball Your school has contracted with a professional animal trainer to host a donkey basketball game at the school. The school has guaranteed an attendance of at least 1000 people and \$4800 in total ticket sales. The tickets are \$4 for students \$6 for nonstudents, of which the animal trainer receives \$3 from students and \$4 from nonstudents. What is the minimum amount of money the animal trainer could receive? What is the maximum amount of money the animal trainer could receive? (Lesson 3–6)

41. Lunch Costs Melissa, Wes, and Daryl went to Fred's Burgers to get food for their friends at school. Melissa spent \$6.35 on two burgers, one order of french fries, and two colas. Wes ordered 1 burger, 2 orders of french fries, and 2 colas. His bill was \$5.45. Daryl's order of 3 burgers, 3 orders of french fries, and 3 colas totaled \$11.01. Find the price of each item. (Lesson 3–7)

42. State Fair A dairy makes three types of cheese—cheddar, Monterey Jack and Swiss—and sells the cheese in three booths at the state fair. At the beginning of one day, the first booth received x pounds of each type of cheese. The second booth received y pounds of each type of cheese, and the third booth received z pounds of each type of cheese. By the end of the day, the dairy had sold 131 pounds of cheddar, 291 pounds of Monterey Jack, and 232 pounds of Swiss. The table below shows the percent of the cheese delivered in the morning that was sold at each booth. How many pounds of cheddar cheese did each booth receive in the morning? (Lesson 3–7)

Type	Booth 1	Booth 2	Booth 3
Cheddar	40%	30%	10%
Monterey Jack	40%	90%	80%
Swiss	30%	70%	70%

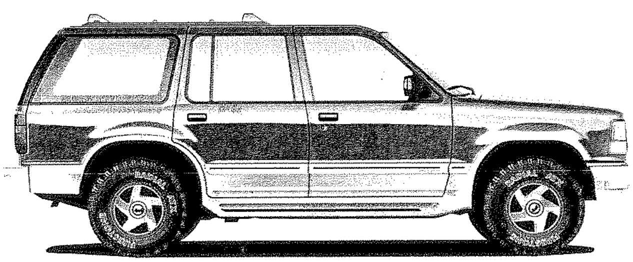
A practice test for Chapter 3 is provided on page 914.

ALTERNATIVE ASSESSMENT

COOPERATIVE LEARNING PROJECT

You have just been promoted to the position of national purchasing agent for a large rental car company. Having worked as a manager for several years in one of their local offices, you have become familiar with many different makes and models of automobiles. The executive officers of the company have narrowed the field to two models having the following characteristics.

Characteristics	Model A	Model B
Purchase price per unit	$10,000	$15,000
Number of units needed	5000	5000
Total cash outlay	$50,000,000	$75,000,000
Projected revenue per year	$36,500,000	$36,500,000
Projected expense per year	$21,900,000	$18,250,000
Projected utilization	85%	85%
Useful life per unit	3 years	4 years
Resale value per unit	$1,500	$2,000



The executive officers have asked you to create a graph of the information on the chart and make a recommendation. Here are some things you might want to show on your graph.

- total cash outlay
- purchase price of each unit
- profit line
- when each model has paid for itself
- when the two models generate the same profit

You have three recommendations to choose from: Model A, Model B or a combination of the two. Tell which recommendation you would choose and why.

What other factors might influence your decision?

THINKING CRITICALLY

- Write a system of equations in three variables that has one unique solution. Explain why there is only one solution.
- Write a system of equations in three variables that has no solution. Explain why there is no solution.
- Write a system of equations in three variables that has an infinite number of solutions. Explain why this is the case.

PORTFOLIO SUGGESTIONS

Select an item from this chapter that you feel shows your best work and place it in your portfolio. Explain why you selected it.

LONG-TERM PROJECT

In·ves·ti·ga·tion

3-2-1-Blast-Off!

MATERIALS NEEDED

- shoe box
- paper cups
- wooden craft sticks
- ruler with groove down the middle
- table tennis ball
- plastic spoon
- golf tee
- masking tape
- string
- rubber bands (various sizes)
- paper clips
- measuring tape

You work for a scientific research company that has just received a contract for designing, building, and testing a prototype launching system. Currently, the client is using inefficient designs supplied by previous companies. The client has requested rush status for this operation and is requiring a shoot-off to demonstrate the accuracy and features of the system.

Your research company must design, build, and test the launch system. It must be calibrated by shooting a number of test shots. To calibrate the launcher, shoot several test shots, mark the results, and adjust the launcher until it will hit a designated target. The test data will show the accuracy of the system. Then a detailed individual report must be written.

The client has furnished the following raw materials to be used to construct the launching system.

- launcher component kit (shoe box, paper cups, and craft sticks)
- linear scale (ruler with groove down the middle)
- projectile (table tennis ball)
- cradles (plastic spoon and golf tee)
- adhesive lamination (masking tape)
- rope (string)
- power supply (rubber bands of various size)
- bars (paper clips)
- test range calibration device (measuring tape)